Gracefully Labeling Prisms

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Abstract

We provide graceful labelings for prisms $C_{2m} \times P_n$ with even cycles, for all $n \geq 2$, and prisms $C_{2m+1} \times P_n$ with odd cycles when $3 \leq n \leq 12$. Further, we verify that the windmill graph $K_4^{(m)}$ is graceful for $m \leq 22$, and that the square of a path P_n^2 is graceful for $n \leq 32$.

1 Introduction

Let $G = (V, E)$ be a simple graph with $|V|$ nodes and $|E|$ edges. Let $f : V \rightarrow 0, 1, \ldots, |E|$ be an injective map function. Define an induced function $g : E \rightarrow 1, 2, \ldots, |E|$ by setting $g(p, q) = |f(p) - f(q)|$ for all $(p, q) \in E$. If g maps E onto $1, 2, \ldots, |E|$, then f is said to be a graceful labeling [6] or β-valuation [9] of G. A graph is *graceful* if it has a graceful labeling.

The prism $P_{m,n}$ $(n \geq 2)$ is defined as the Cartesian product $C_m \times P_n$ of a cycle of length m and a path with n vertices. Frucht and Gallian [3] have shown that $C_m \times P_2$ is always graceful, and Jungreis and Reid [8] found graceful labelings of $C_{2m} \times P_{2n}$ and $C_{4m} \times P_{2n+1}$. In this paper, we complete the analysis showing all prisms with even cycles are graceful. For prisms with odd cycles, we show $C_{2m+1} \times P_n$ is graceful for $n \leq 12$.

Let v_{ij}, where $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, denote the mn vertices of the prism $C_m \times P_n$, and let $f(i, j) \rightarrow \{0, 1, 2, \ldots, (2n-1)m\}$ denote a graceful labeling of $C_m \times P_n$. We use the notation $\delta(x) = x(mod\ 2)$ [4] to represent the parity of a non-negative integer and further denote v_{ij} as an odd or even vertex depending on the parity of $i + j$.

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2 Prisms with Even Length Cycles

Our labelings for prisms are based on the graceful labelings of the grid graphs $P_m \times P_n$ [1]. A graceful labeling for grid graphs is described by

$$f(i,j) = (in - \lfloor\frac{i}{2}\rfloor + \lfloor\frac{j}{2}\rfloor)\delta(i+j+1)$$

$$+ \left(2mn - (m+n) - in + \lfloor\frac{i+1}{2}\rfloor - \lfloor\frac{j}{2}\rfloor\right) \delta(i+j)$$

All of our labelings for prisms in this paper start from a labeling for grid graphs, and are based on the following idea. The prism $C_m \times P_n$ contains exactly n more edges than the grid $P_m \times P_n$, and hence the maximum vertex number for prisms and grids are $(2n-1)m$ and $(2n-1)m - n$, respectively. In the grid labeling given above, the edge labels of the rightmost $\lfloor m/2 \rfloor$ columns range from 1 to $(2n-1)m/2 - n$ inclusive. Adding n to the labels of all odd vertices in the leftmost $\lceil m/2 \rceil$ columns of the grid raises the edge labels they define so they range from $(2n-1)m/2 + n + 1$ to $(2n-1)m$ inclusive. The missing edges labels from $(2n-1)m/2 - n + 1$ to $(2n-1)m/2 + n$ remain to be defined between the first and the last columns, and the centermost two columns. The edge labels 1 to $6n-3$ are defined in the last three columns of the grid graph, and hence are independent of m. Therefore, by modifying the labels of the last one, two, or three columns, we gain additional freedom to accommodate the missing edges.

Theorem 1 *The prism graph $C_m \times P_n$ is graceful if $m \equiv 0 \pmod 2$*

Proof : Jungreis and Reid [8] proved $C_m \times P_n$ is graceful when $m \equiv 0 (mod\, 4)$, ie. $C_{4p} \times P_n$ $(p \geq 1, n \geq 2)$ We define an alternate labeling:

$$f(i,j) = (in - \lfloor\frac{i}{2}\rfloor + \lfloor\frac{j}{2}\rfloor)\delta(i+j+1)$$

$$+ \left((2n-1)m - in + \lfloor\frac{i+1}{2}\rfloor - \lfloor\frac{j}{2}\rfloor - n\lfloor\frac{2i}{m}\rfloor\right) \delta(i+j)$$

When $m \equiv 2 (mod\, 4)$, there are two cases depending on the parity of n. For $n \equiv 0 (mod\, 2)$, i.e. $C_{4p+2} \times P_{2q} (p \geq 1, q \geq 1)$, Jungreis and Reid gave the following labeling:

$$f(i,j) = (in - \lfloor \frac{i}{2} \rfloor + \lfloor \frac{j}{2} \rfloor)\delta(i+j+1)$$
$$+ \left((2n-1)m - in + \lfloor \frac{i+1}{2} \rfloor - \lfloor \frac{j}{2} \rfloor - n\lfloor \frac{2i}{m} \rfloor\right) \delta(i+j)$$
$$- \lfloor \frac{i}{m-1} \rfloor$$

Let $H = (2n-1)m/2$. For $n \equiv 1 (mod\, 2)$, ie. $C_{4p+2} \times P_{2q+1} (p \geq 1, q \geq 1)$, we define the following labeling:

$$f(i,j) = (in - \lfloor \frac{i}{2} \rfloor + \lfloor \frac{j}{2} \rfloor + n\lfloor \frac{2i}{m} \rfloor)\delta(i+j+1)$$
$$+ \left((2n-1)m - in + \lfloor \frac{i+1}{2} \rfloor - \lfloor \frac{j}{2} \rfloor\right) \delta(i+j),$$
$$for\ i \leq m-3$$

$$f(m-2,0) = H - n + 2$$

$$f(m-2,1) = H + 2n$$

$$f(m-2,j) = \left(H + 2n - \lfloor \frac{j+1}{2} \rfloor\right) \delta(j)$$
$$+ \left(H + 4n + \lfloor \frac{j}{2} - 2 \rfloor\right) \delta(j+1), \quad for\ j \geq 2.$$

$$f(m-1,0) = H - n + 1$$

$$f(m-1,n-1) = H + \frac{3(n-1)}{2}$$

$$f(m-1,j) = \left(H + \lfloor \frac{j-1}{2} \rfloor\right) \delta(j) + \left(H + n - \lfloor \frac{j}{2} + 1 \rfloor\right) \delta(j+1),$$
$$for\ 1 \leq j \leq n-2.$$

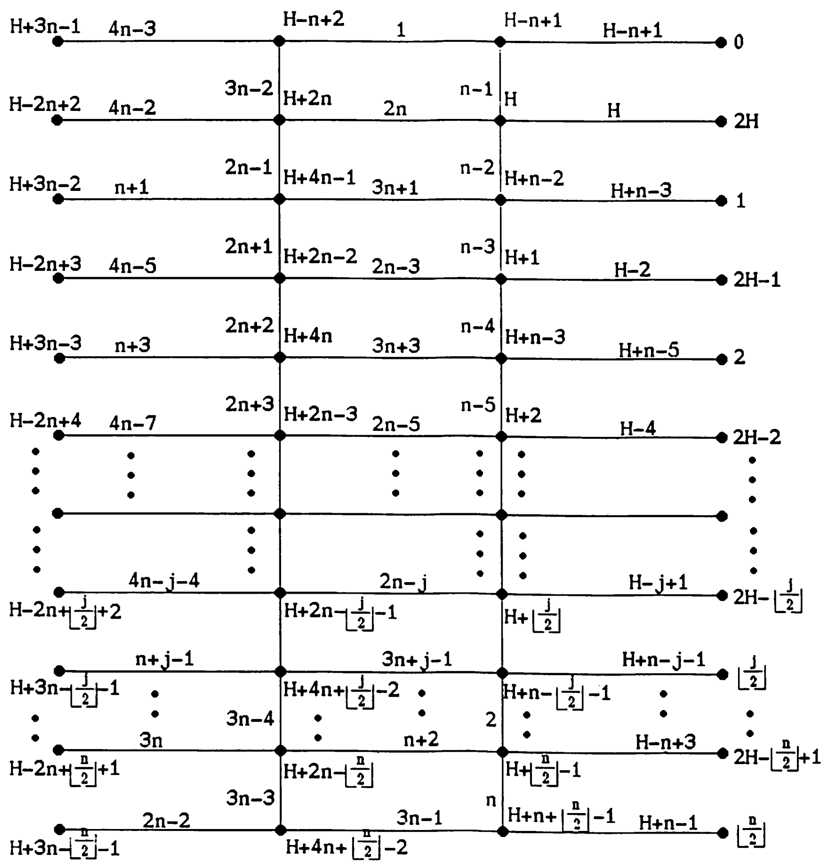

Figure 1: The last two columns of $C_{4p+2} \times P_{2q+1}$, with edge and vertex labels.

0	48	5	43	10	38	18	33	24	23
50	3	45	8	40	16	35	21	31	25
1	47	6	42	11	37	19	32	36	28

0	162	17	145	34	128	60	111	78	77
170	9	153	26	136	52	119	69	103	85
1	161	18	144	35	127	61	110	120	92
169	10	152	27	135	53	118	70	101	86
2	160	19	143	36	126	62	109	121	91
168	11	151	28	134	54	117	71	100	87
3	159	20	142	37	125	63	108	122	90
167	12	150	29	133	55	116	72	99	88
4	158	21	141	38	124	64	107	123	97

Figure 2: Graceful labelings of $C_{10} \times P_3$ and $C_{10} \times P_9$

The last two columns define edge labels from 1 to $4n-2$, so the edge labels in the rightmost $\lfloor m/2 \rfloor$ columns range from 1 to $(2n-1)m/2 - n$. The edge labels in the leftmost $\lceil m/2 \rceil$ columns range from $(2n-1)m/2 + n+1$ to $(2n-1)m$. Between the first and last column are defined the edges $H-n+1, H-n+3, \ldots, H-2, H$, where $H-n+1$ is defined on the first row and the others on even rows, and $H+2, H+4, \ldots, H+n-3, H+n-1$, where $H+n-1$ is defined on the last row and the others on odd rows. The edges between the centermost two columns are $H-n+2, H-n+4, \ldots, H+n-2, H+n$. So all $(2n-1)m$ edges have distinct labels. Figure 1 illustrates the last two columns of this construction, and Figure 2 provides some examples. ∎

3 Prisms with Odd Length Cycles

For the case of $C_{2k+1} \times P_n$, we use an identical labeling as in the even case, for the first $n-3$ columns. For $3 \leq n \leq 12$, we have identified appropriate labelings of the last three columns by computer search. Unfortunately, we failed to identify a general pattern underlying our construction, but it appears for any specific value of n this method gives a solution for arbitrary odd cycle lengths $m = 2k + 1$.

Theorem 2 *The prism graph $C_m \times P_n$ is graceful if $m \equiv 1 \pmod 2$ a $2 \leq n \leq 12$*

Proof : Frucht and Gallian [4] gave a graceful labeling when $n = 2$. Here we give graceful labelings for $3 \leq n \leq 12$. In each case, the following formula labels the first $m - 3$ columns for sufficiently large prisms:

$$f(i,j) = (in - \lfloor \frac{i}{2} \rfloor + \lfloor \frac{j}{2} \rfloor)\delta(i + j + 1)$$

$$+ \left((2n - 1)m - in + \lfloor \frac{i+1}{2} \rfloor - \lfloor \frac{j}{2} \rfloor - n\lfloor \frac{2i+1}{m} \rfloor\right) \delta(i + j)$$

In each case, the labelings of the last three columns will be defined by a table. The $(m - 3)$rd column in the tables is starred if the entries are identical to the default labeling.

Case 1. $n = 3$ and $m \equiv 1 \pmod 2$, *i.e.* $C_{2p+1} \times P_3$

For $m = 3$ and $m = 5$, see Figure 3 for graceful labelings. For $m \geq 7$ and $i \geq m - 3$, use the following tables to define the last three columns:

$m \equiv 1 \pmod 4$, *i.e.* $m = 4p + 1$, $p \geq 2$

i / j	$m-3$	$m-2$	$m-1$
0	$10p - 5$	$10p + 6$	$10p + 5$
1	$10p + 7$	$10p + 11$	$10p - 2$
2	$10p - 1$	$10p + 2$	$10p + 4$

$m \equiv 3 \pmod 4$, *i.e.* $m = 4p + 3$, $p \geq 1$

i / j	$m-3$	$m-2$	$m-1$
0	$10p + 1$	$10p + 9$	$10p + 13$
1	$10p + 11$	$10p + 8$	$10p + 6$
2	$10p + 2$	$10p - 3$	$10p + 12$

To demonstrate correctness, note that the last three columns define edge labels from 1 to 15, so the edge labels in the rightmost $\lfloor m/2 \rfloor$ columns range from 1 to $10p + 2$. The edge labels in the leftmost $\lceil m/2 \rceil$ columns range from $10p + 9$ to $20p + 5$. The edges between the first and the last columns are $10p+3$, $10p+5$, and $10p+7$. The edges between the centermost two columns are $10p+4$, $10p+6$, and $10p+8$, so all $20p+5$ edges have distinct labels from 1 to $20p+5$. A similar analysis holds for $4 \leq n \leq 12$ and has been omitted for the remaining cases.

Case 2. $n=4$ and $m \equiv 1 \ (mod\ 2)$, *i.e.* $C_{2p+1} \times P_4$

For $m = 3$, see Figure 3 for a graceful labeling. For $m \geq 5$ and $i \geq m-3$, use the following tables to define the last three columns:

$m \equiv 1 \ (mod\ 4)$, *i.e.* $m = 4p+1$, $p \geq 1$

j \ i	$m-3$	$m-2$	$m-1$
0	$14p-7$	$14p-11$	$14p+7$
1	$14p+10$	$14p+3$	$14p+2$
2	$14p-6$	$14p$	$14p+11$
3	$14p+6$	$14p+8$	$14p-2$

$m \equiv 3 \ (mod\ 4)$, *i.e.* $m = 4p+3$, $p \geq 1$

j \ i	$m-3$*	$m-2$	$m-1$
0	$14p$	$14p+7$	$14p+18$
1	$14p+17$	$14p+19$	$14p+5$
2	$14p+1$	$14p+11$	$14p+14$
3	$14p+16$	$14p+15$	$14p+9$

Case 3. $n=5$ and $m \equiv 1 \ (mod\ 2)$, *i.e.* $C_{2p+1} \times P_5$

For $m = 3$, see Figure 3 for a graceful labeling. For $m \geq 5$ and $i \geq m-3$, use the following tables to define the last three columns:

$m \equiv 1 \ (mod\ 4)$, *i.e.* $m = 4p+1$, $p \geq 1$

j \ i	$m-3$*	$m-2$	$m-1$
0	$18p-9$	$18p+9$	$18p+11$
1	$18p+13$	$18p+19$	$18p+2$
2	$18p-8$	$18p+3$	$18p+10$
3	$18p+12$	$18p-2$	$18p-5$
4	$18p-7$	$18p-6$	$18p+7$

$m \equiv 3 \ (mod\, 4)$, $i.e.\ m = 4p+3,\ p \geq 1$

j \ i	$m-3^*$	$m-2$	$m-1$
0	$18p$	$18p+16$	$18p+23$
1	$18p+22$	$18p+4$	$18p+6$
2	$18p+1$	$18p+5$	$18p+20$
3	$18p+21$	$18p+18$	$18p+9$
4	$18p+2$	$18p+12$	$18p+17$

Case 4. $n = 6$ and $m \equiv 1 \ (mod\, 2)$, $i.e.$ $C_{2p+1} \times P_6$

For $m = 3$ and $m = 5$, see Figure 4 for graceful labelings. For $m \geq 7$ and $i \geq m-3$, use the following tables to define the last three columns:

$m \equiv 1 \ (mod\, 4)$, $i.e.\ m = 4p+1,\ p \geq 2$

j \ i	$m-3$	$m-2$	$m-1$
0	$22p-11$	$22p-7$	$22p+14$
1	$22p+16$	$22p-13$	$22p-5$
2	$22p-10$	$22p+10$	$22p+12$
3	$22p+15$	$22p-3$	$22p-2$
4	$22p+5$	$22p-19$	$22p+9$
5	$22p-17$	$22p-12$	$22p$

$m \equiv 3 \ (mod\, 4)$, $i.e.\ m = 4p+3,\ p \geq 2$

j \ i	$m-3^*$	$m-2$	$m-1$
0	$22p$	$22p+16$	$22p+28$
1	$22p+27$	$22p+20$	$22p+9$
2	$22p+1$	$22p+21$	$22p+18$
3	$22p+26$	$22p+34$	$22p+13$
4	$22p+2$	$22p+17$	$22p+23$
5	$22p+25$	$22p+3$	$22p+5$

Case 5. $n = 7$ and $m \equiv 1 \ (mod\, 2)$, $i.e.$ $C_{2p+1} \times P_7$

For $m = 3$, see Figure 5 for a graceful labeling. For $m \geq 5$ and $i \geq m-3$, use the following tables to define the last three columns:

$m \equiv 1 \ (mod\,4), \ i.e. \ m = 4p+1, \ p \geq 1$

i / j	$m-3^*$	$m-2$	$m-1$
0	$26p-13$	$26p+13$	$26p+7$
1	$26p+19$	$26p-1$	$26p-6$
2	$26p-12$	$26p+9$	$26p+12$
3	$26p+18$	$26p+2$	$26p+3$
4	$26p-11$	$26p+4$	$26p+15$
5	$26p+17$	$26p+21$	$26p-4$
6	$26p-10$	$26p-2$	$26p+20$

$m \equiv 3 \ (mod\,4), \ i.e. \ m = 4p+3, \ p \geq 1$

i / j	$m-3^*$	$m-2$	$m-1$
0	$26p$	$26p+21$	$26p+25$
1	$26p+32$	$26p+12$	$26p+6$
2	$26p+1$	$26p-1$	$26p+22$
3	$26p+31$	$26p+14$	$26p+11$
4	$26p+2$	$26p+28$	$26p+33$
5	$26p+30$	$26p+20$	$26p+8$
6	$26p+3$	$26p+27$	$26p+26$

Case 6. $n = 8$ and $m \equiv 1 \ (mod\,2)$, $i.e.$ $C_{2p+1} \times P_8$

For $m = 3$, see Figure 5 for a graceful labeling. For $m \geq 5$ and $i \geq m-3$, use the following tables to define the last three columns:

$m \equiv 1 \ (mod\,4), \ i.e. \ m = 4p+1, \ p \geq 1$

i / j	$m-3^*$	$m-2$	$m-1$
0	$30p-15$	$30p+14$	$30p+16$
1	$30p+22$	$30p+3$	$30p+6$
2	$30p-14$	$30p-6$	$30p+12$
3	$30p+21$	$30p+8$	$30p-8$
4	$30p-13$	$30p-9$	$30p+17$
5	$30p+20$	$30p-10$	$30p-5$
6	$30p-12$	$30p+11$	$30p+23$
7	$30p+19$	$30p+26$	$30p-1$

$m \equiv 3 \ (mod\, 4), \ i.e. \ m = 4p + 3, \ p \geq 1$

i / j	$m-3*$	$m-2$	$m-1$
0	$30p$	$30p+30$	$30p+38$
1	$30p+37$	$30p+8$	$30p+20$
2	$30p+1$	$30p+29$	$30p+33$
3	$30p+36$	$30p+9$	$30p+10$
4	$30p+2$	$30p+16$	$30p+25$
5	$30p+35$	$30p+19$	$30p+14$
6	$30p+3$	$30p+13$	$30p+39$
7	$30p+34$	$30p+32$	$30p+15$

Case 7. $n = 9$ and $m \equiv 1 \ (mod\, 2)$, $i.e.$ $C_{2p+1} \times P_9$

For $m = 3$, see Figure 5 for a graceful labeling. For $m \geq 5$ and $i \geq m-3$, use the following tables to define the last three columns:

$m \equiv 1 \ (mod\, 4), \ i.e. \ m = 4p + 1, \ p \geq 1$

i / j	$m-3*$	$m-2$	$m-1$
0	$34p-17$	$34p+17$	$34p+11$
1	$34p+25$	$34p-8$	$34p-4$
2	$34p-16$	$34p+6$	$34p+16$
3	$34p+24$	$34p-5$	$34p-7$
4	$34p-15$	$34p+12$	$34p+21$
5	$34p+23$	$34p-1$	$34p+2$
6	$34p-14$	$34p-2$	$34p+28$
7	$34p+22$	$34p-10$	$34p-3$
8	$34p-13$	$34p+8$	$34p+13$

$m \equiv 3 \ (mod\, 4), \ i.e. \ m = 4p + 3, \ p \geq 1$

i / j	$m-3*$	$m-2$	$m-1$
0	$34p$	$34p+34$	$34p+43$
1	$34p+42$	$34p+9$	$34p+14$
2	$34p+1$	$34p+15$	$34p+30$
3	$34p+41$	$34p+23$	$34p+19$
4	$34p+2$	$34p-1$	$34p+29$
5	$34p+40$	$34p+27$	$34p+10$
6	$34p+3$	$34p+5$	$34p+36$
7	$34p+39$	$34p+12$	$34p+13$
8	$34p+4$	$34p+24$	$34p+45$

Case 8. $n = 10$ and $m \equiv 1 \ (mod\, 2)$, $i.e.$ $C_{2p+1} \times P_{10}$

For $m = 3$, see Figure 6 for a graceful labeling. For $m \geq 5$ and $i \geq m-3$, use the following tables to define the last three columns:

$m \equiv 1 \ (mod\,4)$, $i.e.\ m = 4p+1,\ p \geq 1$

i / j	$m-3^*$	$m-2$	$m-1$
0	$38p-19$	$38p+19$	$38p+22$
1	$38p+28$	$38p-9$	$38p-7$
2	$38p-18$	$38p+8$	$38p+29$
3	$38p+27$	$38p-5$	$38p-6$
4	$38p-17$	$38p+1$	$38p+13$
5	$38p+26$	$38p+10$	$38p+2$
6	$38p-16$	$38p+6$	$38p+16$
7	$38p+25$	$38p+30$	$38p-4$
8	$38p-15$	$38p$	$38p+23$
9	$38p+24$	$38p+31$	$38p-2$

$m \equiv 3 \ (mod\,4)$, $i.e.\ m = 4p+3,\ p \geq 1$

i / j	$m-3^*$	$m-2$	$m-1$
0	$38p$	$38p+38$	$38p+42$
1	$38p+47$	$38p+10$	$38p+17$
2	$38p+1$	$38p+37$	$38p+36$
3	$38p+46$	$38p+14$	$38p+12$
4	$38p+2$	$38p+11$	$38p+33$
5	$38p+45$	$38p+31$	$38p+18$
6	$38p+3$	$38p+20$	$38p+51$
7	$38p+44$	$38p+26$	$38p+21$
8	$38p+4$	$38p+16$	$38p+50$
9	$38p+43$	$38p+8$	$38p+24$

Case 9. $n = 11$ and $m \equiv 1 \ (mod\,2)$, $i.e.$ $C_{2p+1} \times P_{11}$

For $m = 3$, see Figure 6 for a graceful labeling. For $m \geq 5$ and $i \geq m-3$, use the following tables to define the last three columns:

$m \equiv 1 \ (mod\,4)$, $i.e.\ m = 4p + 1,\ p \geq 1$

i / j	$m-3^*$	$m-2$	$m-1$
0	$42p-21$	$42p+21$	$42p+11$
1	$42p+31$	$42p-10$	$42p+4$
2	$42p-20$	$42p+20$	$42p+26$
3	$42p+30$	$42p+1$	$42p-11$
4	$42p-19$	$42p+9$	$42p+25$
5	$42p+29$	$42p-6$	$42p-8$
6	$42p-18$	$42p+7$	$42p+16$
7	$42p+28$	$42p+33$	$42p-1$
8	$42p-17$	$42p+22$	$42p+19$
9	$42p+27$	$42p+23$	$42p-4$
10	$42p-16$	$42p+2$	$42p+34$

$m \equiv 3 \ (mod\,4)$, $i.e.\ m = 4p + 3,\ p \geq 1$

i / j	$m-3^*$	$m-2$	$m-1$
0	$42p$	$42p+42$	$42p+53$
1	$42p+52$	$42p+11$	$42p+14$
2	$42p+1$	$42p+41$	$42p+40$
3	$42p+51$	$42p+43$	$42p+15$
4	$42p+2$	$42p+38$	$42p+47$
5	$42p+50$	$42p+65$	$42p+28$
6	$42p+3$	$42p+36$	$42p+46$
7	$42p+49$	$42p+32$	$42p+25$
8	$42p+4$	$42p+10$	$42p+45$
9	$42p+48$	$42p+34$	$42p+22$
10	$42p+5$	$42p+18$	$42p+56$

Case 10. $n = 12$ and $m \equiv 1 \ (mod\,2)$, $i.e.$ $C_{2p+1} \times P_{12}$

For $m = 3$, see Figure 6 for a graceful labeling. For $m \geq 5$ and $i \geq m-3$, use the following tables to define the last three columns:

0	13	11
15	3	6
1	9	2

0	23	5	13	12
25	3	20	7	14
1	22	6	2	11

0	7	16
21	4	17
1	19	11
20	8	6

Figure 3: Labelings of $C_3 \times P_3$, $C_5 \times P_3$, and $C_3 \times P_4$

$m \equiv 1\ (mod\, 4)$, $i.e.\ m = 4p + 1,\ p \geq 1$

i / j	$m-3$*	$m-2$	$m-1$
0	$46p-23$	$46p+23$	$46p+13$
1	$46p+34$	$46p-11$	$46p-5$
2	$46p-22$	$46p+22$	$46p+35$
3	$46p+33$	$46p-10$	$46p+7$
4	$46p-21$	$46p-14$	$46p+28$
5	$46p+32$	$46p+17$	$46p-9$
6	$46p-20$	$46p-8$	$46p+27$
7	$46p+31$	$46p+12$	$46p+3$
8	$46p-19$	$46p+11$	$46p+25$
9	$46p+30$	$46p+19$	$46p-4$
10	$46p-18$	$46p+21$	$46p+37$
11	$46p+29$	$46p+26$	$46p-1$

$m \equiv 3\ (mod\, 4)$, $i.e.\ m = 4p + 3,\ p \geq 1$

i / j	$m-3$*	$m-2$	$m-1$
0	$46p$	$46p+46$	$46p+58$
1	$46p+57$	$46p+12$	$46p+21$
2	$46p+1$	$46p+45$	$46p+51$
3	$46p+56$	$46p+13$	$46p+16$
4	$46p+2$	$46p-3$	$46p+39$
5	$46p+55$	$46p+15$	$46p+22$
6	$46p+3$	$46p+29$	$46p+42$
7	$46p+54$	$46p+27$	$46p+31$
8	$46p+4$	$46p+35$	$46p+60$
9	$46p+53$	$46p+14$	$46p+24$
10	$46p+5$	$46p+33$	$46p+48$
11	$46p+52$	$46p+11$	$46p+10$

∎

0	18	6
27	5	22
1	24	21
26	15	11
2	23	9

0	22	3
33	6	12
1	21	25
32	4	14
2	27	28
31	19	26

0	50	11	30	27
55	6	44	21	47
1	49	12	14	26
54	7	43	15	32
2	48	13	28	18
53	8	42	24	29

Figure 4: Labelings of $C_3 \times P_5$, $C_3 \times P_6$ and $C_5 \times P_6$

0	26	15
39	7	36
1	25	32
38	8	10
2	35	22
37	12	21
3	17	11

0	30	12
45	8	11
1	29	25
44	9	36
2	40	31
43	7	33
3	26	20
42	10	35

0	34	26
51	9	38
1	33	37
50	10	47
2	32	41
49	11	22
3	31	36
48	46	5
4	39	21

Figure 5: Labelings of $C_3 \times P_7$, $C_3 \times P_8$, and $C_3 \times P_9$

0	38	18
57	10	24
1	37	40
56	11	51
2	36	32
55	12	47
3	35	34
54	6	13
4	48	50
53	31	41

0	42	44
63	11	45
1	41	25
62	12	15
2	40	23
61	13	46
3	39	35
60	14	21
4	57	16
59	8	53
5	27	18

0	46	56
69	12	62
1	45	53
68	13	32
2	44	47
67	14	51
3	43	25
66	15	27
4	42	28
65	7	48
5	54	59
64	41	16

Figure 6: Labelings of $C_3 \times P_{10}$, $C_3 \times P_{11}$ and $C_3 \times P_{12}$

4 Windmill Graphs and P_n^2

By computer search, we have provided evidence to support conjectures that two additional classes of graphs are graceful. Bermond [2] has conjectured that the *windmill graph* $K_4^{(m)}$ is graceful for all $m \geq 4$, where the windmill graph is defined by joining K_1 to each vertex of m vertex disjoint instances of K_3. Windmills were known to be graceful for $m \leq 8$, ([4] p. 496), but the following table provides labelings for $4 \leq m \leq 22$. In each labeling, the central vertex is labeled 0.

T. Grace [7] conjectured that P_n^2 is graceful, where P_n^2 is the square of the path on n vertices. Previously, P_n^2 was known to be graceful when $n \leq 12$ ([4] pp. 497-498), but we provide labelings for $n \leq 32$.

5 Conclusion

We have proven that all prisms with even cycles are graceful, and provided graceful labelings for prisms defined by odd cycles and short paths. Further, we have extended previous bounds on the number of graceful windmills and squared paths.

It is frustrating that these special cases do not appear to lead to more general results. The gracefulness of prisms with odd cycles, windmills, and squared paths remain open problems.

$K_4^{(4)}$	(0,22,24,5)	(0,16,10,23)	(0,20,12,21)	(0,15,4,18)	
$K_4^{(5)}$	(0,29,30,7)	(0,26,8,28)	(0,24,27,13)	(0,19,25,10)	(0,17,21,5)
$K_4^{(6)}$	(0,35,36,2)	(0,27,32,10)	(0,24,31,11)	(0,26,30,12)	(0,23,29,8)
	(0,25,28,9)				
$K_4^{(7)}$	(0,41,42,2)	(0,32,38,17)	(0,27,37,18)	(0,29,36,13)	(0,31,35,11)
	(0,26,34,12)	(0,30,33,5)			
$K_4^{(8)}$	(0,47,48,2)	(0,37,8,44)	(0,34,20,43)	(0,30,17,42)	(0,31,41,15)
	(0,22,40,19)	(0,35,39,11)	(0,33,38,6)		
$K_4^{(9)}$	(0,53,54,2)	(0,42,50,9)	(0,35,49,15)	(0,38,48,11)	(0,29,47,26)
	(0,40,46,16)	(0,28,45,23)	(0,31,44,19)	(0,39,43,7)	
$K_4^{(10)}$	(0,59,60,2)	(0,47,56,10)	(0,44,55,17)	(0,40,54,15)	(0,45,53,12)
	(0,34,52,30)	(0,31,51,28)	(0,29,50,24)	(0,43,49,7)	(0,35,48,16)
$K_4^{(11)}$	(0,66,65,2)	(0,62,52,11)	(0,61,49,23)	(0,60,42,39)	(0,59,50,16)
	(0,58,45,14)	(0,57,35,30)	(0,56,36,32)	(0,55,48,15)	(0,54,46,17)
	(0,53,47,28)				
$K_4^{(12)}$	(0,72,71,2)	(0,68,57,12)	(0,67,54,14)	(0,66,50,18)	(0,65,55,19)
	(0,64,43,38)	(0,63,34,30)	(0,62,47,23)	(0,61,44,41)	(0,60,52,25)
	(0,59,37,31)	(0,58,51,9)			
$K_4^{(13)}$	(0,78,77,2)	(0,74,62,13)	(0,73,59,15)	(0,72,53,20)	(0,71,60,17)
	(0,70,42,36)	(0,69,47,37)	(0,68,50,45)	(0,67,51,21)	(0,66,57,26)
	(0,65,41,38)	(0,64,39,35)	(0,63,56,8)		
$K_4^{(14)}$	(0,84,83,2)	(0,80,67,14)	(0,79,64,16)	(0,78,58,22)	(0,77,65,27)
	(0,76,57,52)	(0,75,47,41)	(0,74,42,35)	(0,73,62,29)	(0,72,55,18)
	(0,71,61,31)	(0,70,49,45)	(0,69,46,43)	(0,68,60,9)	
$K_4^{(15)}$	(0,90,89,2)	(0,86,72,15)	(0,85,69,17)	(0,84,66,62)	(0,83,70,24)
	(0,82,53,48)	(0,81,60,23)	(0,80,61,25)	(0,79,67,28)	(0,78,47,40)
	(0,77,50,42)	(0,76,65,20)	(0,75,49,43)	(0,74,44,41)	(0,73,64,10)
$K_4^{(16)}$	(0,96,95,2)	(0,92,77,16)	(0,91,74,18)	(0,90,71,24)	(0,89,75,21)
	(0,88,62,55)	(0,87,60,57)	(0,86,64,23)	(0,85,72,20)	(0,84,59,49)
	(0,83,44,40)	(0,82,70,32)	(0,81,53,45)	(0,80,51,46)	(0,79,48,42)
	(0,78,69,11)				
$K_4^{(17)}$	(0,102,101,2)	(0,98,82,17)	(0,97,79,19)	(0,96,76,21)	(0,95,80,26)
	(0,94,70,66)	(0,93,64,56)	(0,92,67,61)	(0,91,77,42)	(0,90,68,32)
	(0,89,59,50)	(0,88,48,45)	(0,87,53,46)	(0,86,74,23)	(0,85,52,47)
	(0,84,71,27)	(0,83,73,11)			
$K_4^{(18)}$	(0,108,107,2)	(0,104,87,18)	(0,103,84,20)	(0,102,81,22)	(0,101,85,27)
	(0,100,75,72)	(0,99,73,68)	(0,98,66,60)	(0,97,82,30)	(0,96,63,55)
	(0,95,71,34)	(0,94,54,50)	(0,93,79,23)	(0,92,57,47)	(0,91,78,29)
	(0,90,51,42)	(0,89,53,46)	(0,88,77,12)		
$K_4^{(19)}$	(0,114,113,2)	(0,110,92,19)	(0,109,89,21)	(0,108,86,23)	(0,107,90,28)
	(0,106,77,71)	(0,105,80,27)	(0,104,74,66)	(0,103,87,31)	(0,102,76,69)
	(0,101,67,58)	(0,100,64,61)	(0,99,84,24)	(0,98,57,44)	(0,97,83,32)
	(0,96,59,49)	(0,95,55,50)	(0,94,52,48)	(0,93,82,12)	
$K_4^{(20)}$	(0,120,119,2)	(0,116,97,20)	(0,115,94,22)	(0,114,91,24)	(0,113,95,29)
	(0,112,82,76)	(0,111,85,80)	(0,110,73,70)	(0,109,92,28)	(0,108,61,52)
	(0,107,75,33)	(0,106,63,55)	(0,105,89,27)	(0,104,79,69)	(0,103,65,58)
	(0,102,88,34)	(0,101,57,53)	(0,100,87,41)	(0,99,60,49)	(0,98,86,15)
$K_4^{(21)}$	(0,126,125,2)	(0,122,102,21)	(0,121,99,23)	(0,120,96,25)	(0,119,100,30)
	(0,118,87,79)	(0,117,90,29)	(0,116,84,34)	(0,115,97,43)	(0,114,77,68)
	(0,113,85,80)	(0,112,65,52)	(0,111,69,58)	(0,110,94,35)	(0,109,83,73)
	(0,108,63,57)	(0,107,93,15)	(0,106,66,62)	(0,105,67,64)	(0,104,56,49)
	(0,103,91,17)				
$K_4^{(22)}$	(0,132,131,2)	(0,128,107,22)	(0,127,104,24)	(0,126,101,26)	(0,125,105,31)
	(0,124,92,86)	(0,123,95,30)	(0,122,89,81)	(0,121,102,34)	(0,120,70,59)
	(0,119,90,83)	(0,118,76,73)	(0,117,99,35)	(0,116,77,67)	(0,115,88,37)
	(0,114,71,66)	(0,113,69,60)	(0,112,72,57)	(0,111,98,14)	(0,110,58,54)
	(0,109,63,47)	(0,108,96,17)			

Table 1: Graceful Labelings for Windmill Graphs $K_4^{(m)}$, $4 \leq m \leq 22$.

P_n^2	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}	L_{15}	L_{16}
	L_{17}	L_{18}	L_{19}	L_{20}	L_{21}	L_{22}	L_{23}	L_{24}	L_{25}	L_{26}	L_{27}	L_{28}	L_{29}	L_{30}	L_{31}	L_{32}
P_3^2	0	3	2													
P_4^2	0	5	4	2												
P_5^2	0	7	6	3	1											
P_6^2	0	4	9	7	1	8										
P_7^2	0	3	11	10	1	5	7									
P_8^2	0	2	13	5	1	10	11	4								
P_9^2	0	15	14	6	1	3	13	9	2							
P_{10}^2	0	17	16	11	1	4	15	6	2	14						
P_{11}^2	0	19	18	5	1	17	12	2	9	11	3					
P_{12}^2	0	21	20	15	1	4	19	12	2	14	18	5				
P_{13}^2	0	23	22	9	1	12	21	6	2	18	20	13	3			
P_{14}^2	0	25	24	6	1	17	23	9	2	19	22	7	20	16		
P_{15}^2	0	27	26	15	1	6	25	22	2	9	24	7	3	13	21	
P_{16}^2	0	29	28	8	1	23	27	15	2	12	26	21	3	5	22	16
P_{17}^2	0	31	30	23	1	13	29	16	2	11	28	22	3	7	27	25
	4															
P_{18}^2	0	33	32	9	1	18	31	21	2	14	30	25	3	23	29	8
	4	22														
P_{19}^2	0	35	34	26	1	14	33	12	2	16	32	27	3	9	31	24
	4	7	30													
P_{20}^2	0	37	36	10	1	22	35	27	2	16	34	19	3	13	33	11
	4	28	32	9												
P_{21}^2	0	39	38	15	1	11	37	30	2	14	36	23	3	18	35	29
	4	31	34	13	5											
P_{22}^2	0	41	40	9	1	11	39	32	2	18	38	27	3	22	37	10
	4	33	30	8	12	25										
P_{23}^2	0	43	42	14	1	32	41	29	2	25	40	23	3	17	39	9
	4	28	38	12	5	37	26									
P_{24}^2	0	45	44	11	1	26	43	29	2	34	42	21	3	19	41	13
	4	33	40	9	5	35	24	12								
P_{25}^2	0	47	46	13	1	24	45	28	2	31	44	12	3	27	43	29
	4	35	42	15	5	33	41	11	6							
P_{26}^2	0	49	48	13	1	17	47	36	2	22	46	24	3	32	45	35
	4	29	44	12	5	38	43	15	6	33						
P_{27}^2	0	51	50	39	1	22	49	20	2	33	48	25	3	28	47	14
	4	40	46	12	5	21	45	41	6	11	43					
P_{28}^2	0	53	52	14	1	36	51	39	2	30	50	34	3	20	49	28
	4	23	48	41	5	15	47	17	6	40	46	13				
P_{29}^2	0	55	54	15	1	17	53	22	2	40	52	18	3	26	51	23
	4	33	50	29	5	16	49	43	6	13	48	39	7			
P_{30}^2	0	57	56	14	1	17	55	43	2	20	54	40	3	13	53	21
	4	23	52	30	5	36	51	15	6	39	50	11	46	41		
P_{31}^2	0	59	58	16	1	25	57	43	2	23	56	40	3	30	55	33
	4	35	54	28	5	41	53	45	6	17	52	47	7	13	51	
P_{32}^2	0	61	60	15	1	19	59	46	2	21	58	37	3	14	57	40
	4	32	56	34	5	25	55	13	6	44	54	39	7	42	48	9

Table 2: Graceful Labelings for P_n^2, $3 \leq n \leq 32$.

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