Generalized Exponents of Tournament Matrices

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Abstract. In [1], we introduced the generalized exponent for primitive martices. In this paper the generalized exponents of tournament matrices are derived.

1. Introduction

The directed graph Γ defined by a (0,1) matrix M_n consists of n vertices $1, 2, \ldots,$ n such that an arc (ij) goes from i to j if and only if the (i, j) entry of M_n is one.

A tournament T_n is a directed graph Γ such that each pair of distinct vertices i and j is joined by exactly one of the arcs (ij) or (ji) and no vertex is joined to itself by an arc. A tournament matrix M_n is a matrix that defines a tournament T_n.

According to [1], $\exp_T(i, j)$:= The smallest integer p such that there is a walk of length from i to j for each integer $t \geq p$ $(1 \leq i, j \leq n)$.

The tournament T_n is called primitive provided all of the number $\exp_T(i, j)$ are finite, and the number

$$\exp(T) := \max_{ij}\{\exp_T(ij)\}$$

is called the exponent of T_n.

It is well known (see [2]) that a tournament T_n is primitive if and only if $n \geq 4$ and T is irreducible (i.e. strongly connected). Let

$$\exp(n) := \max_T\{\exp(T)\}$$

where the maximum is taken over all the primitive tournaments T with n vertices. It is well known ([2]) that

$$\exp(n) = n + 2.$$

Let the exponent of vertex i defined by

$$\exp_T(i) := \max_j\{\exp_T(i, j)\} \qquad (i = 1, 2, \ldots, n).$$

Thus $\exp_T(i)$ is the smallest integer p such that there is a walk of length p (and thus of every length larger than p) from i to each vertex j of T. We choose to order the vertices of T is such a way that

$$\exp_T(1) \leq \exp_T(2) \leq \cdots \leq \exp_T(n).$$

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We define

$$\exp(n,k) := \max_T\{\exp_T(k)\}, \qquad (k = 1,2,\ldots,n)$$

where the maximum is taken over all primitive tournaments T with n vertices. It follows that $\exp(n,n) = \exp(n)$, the largest exponent of a primitive tournament with n vertices.

The number $\exp_T(i)$ has an interpretation in terms of a memoryless communication system associated with T (see [1]). In fact, $\exp_T(k)$ is the smallest power of M_n for which there are k rows with no zero entry.

In this paper we show that if a tournament T with n vertices is stronly connected and $n > 6$ then

$$\exp(n,k) = k + 2,$$

for any integer k with $1 \leq k \leq n$. Thus some new propertices are derived about tournaments.

2. Preliminaries

Let $T = (V, E)$ be a tournament whose set of vertices is V and whose set of arcs is E.

Let x and y be vertices of T. A path from x to y is a sequence $x = x_0 x_1 \ldots x_k = y$ of vertices with $k \geq 0$ such that $(x_i x_{i+1})$ is an arc for $i = 0, 1, \ldots, k-1$. A vertex K is called an r-king if for every vertex i in T, there is a path from K to i of length r (or shortly r-path). By the definition of a tournament it is obvious that $r \geq 3$.

For $i \in V(T)$

$$N^+(i) : = \{j \mid (i,j) \in E, j \in V\}$$
$$N^-(i) : = \{j \mid (j,i) \in E, j \in V\}.$$

Clearly, $N^+(i) \cup N^-(i) \cup \{i\} = V(T)$ for each $i \in V(T)$, and

$$|N^+(i)| + |N^-(i)| = n - 1,$$

where $|S|$ denote the cardinality of the set S.

Some basic results concerning tournaments are the following well-known theorems.(see, e.g. [3]).

Theorem A. (Camion) *Every strong tournament has a spanning circuit (Hamiltonian).*

Theorem B. (Benhocine) *Every tournament has either a Hamilton circuit or a spanning path.*

Theorem C. *Each vertex of a strong tournament with n vertices is contained in a circuit of length k, for $k = 3, 4, \ldots, n$.*

Theorem D. *A tournament T_n is primitive if and only if $n \geq 4$ and T is irreducible (i.e. T_n is a strong tournament).*

3. The Main Result

Lemma 1. *If T_n, where $n \geq 6$, is an irreducible tournament and K is a vertex with maximum outdegree in T_n, then there exists a 3-path from K to every vertex of $N^-(K)$.*

Proof: Since T_n is irreducible, $N^-(K) \neq \emptyset$. Let $|N^+(K)| = r$. Then $r \geq \binom{n}{2}/n = (n-1)/2 \geq 3$ $(n \geq 6)$.

For any vertex $j \in N^-(K)$, by maximality of outdegree $d^+(K)$ there exists at least one vertex $i \in N^+(K)$ such that (ij).

Let the subtournament induced by $N^+(K)$ be T_r.

Case 1. If there exists a vertex $x \in N^+(K)$ such that (xi), then there is a path of length 3 from K to j, $Kxij$.

Case 2. If for each vertex $x \in N^+(K) - \{i\}$, there is a (ix), then $d^+_{T_r}(i)$, the outdegree of i in T_r, is $r-1$. It follows that one of the following exists:

Subcase 2.1. If there is a vertex $y \in N^-(K)$ such that (yj), then by the maximality of $d^+(K)$ there exists a vertex $x \in N^+(K)$ such that (xy). Thus there is a 3-path from K to j, $Kxyj$.

Subcase 2.2. If $|N^-(K)| \geq 2$ and for each vertex $y \in N^-(K) - \{j\}$, there is a (jy), then there must be $x \in N^+(K) - \{i\}$, such that (xj). Thus there is a-path $Kixj$.

If $|N^-(K)| = 1$ i.e. $N^-(K) = \{j\}$, then there must exist a $x \in N^+(K) - \{i\}$ such that (xj). (If not, each vertex of $N^+(K) - \{i\}$ can not reach j. This is contrary to the connectivity.) Hence there is a 3-path $Kixj$. This completes the proof of the lemma. ∎

Lemma 2. *Let T_n, where $n \geq 6$, be an irreducible tournament. Let K be a vertex with maximum outdegree in T_n and $d^+(K) = r$. If the subtournament T induced by $N^+(K)$ is strongly connected, then there is a 3-path from K to every vertex of $N^+(K)$.*

Proof: By Theorem A T_r has a Hamilton circuit, say $x_1 x_2 \ldots x_r x_1$, $x_i \in N^+(K)$, $i = 1, 2, \ldots, r$, $r \geq 3$. Thus for each x_i, $i = 1, 2, \ldots, r$, there exists a 3-path from K to x_i as follows.

$$\begin{aligned} &K x_{i-2} x_{i-1} x_i && \text{if } 3 \leq i \leq r, \\ &K x_{r-1} x_r x_1 && \text{if } i = 1, \\ &K x_r x_1 x_2 && \text{if } i = 2. \end{aligned}$$

∎

According to Lemmas 1 and 2 and Theorem C, we obtain

Theorem 1. *T_n, $n \geq 6$, is an irreducible tournament. Let K be a vertex with maximum outdegree in T_n and T_r be the subtournament induced by $N^+(K)$. If T_r is irreducible, then K is a 3-king of T_n.*

Lemma 3. *T_n, where $n \geq 6$, is an irreducible tournament. Let K be a vertex with maximum outdegree in T_n. If $d^+(K) = n-2$ and the subtournament T_{n-2} induced by $N^+(K)$ is reducible, then T_n has a 3-king.*

Proof: Let $N^-(K) = \{w\}$.

By Theorems A and B, we see that T_{n-2} has a spanning path, say $x_1 x_2 \ldots x_{n-2}$ and $(x_1 x_{n-2})$.

Clearly, for x_i, $i = 3, 4, \ldots, n-2$, there is a path of length 3 from K to x_i, $K x_{i-2} x_{i-1} x_i$. We need only show that there is a 3-path from K to x_i, $i = 1, 2$.

By the connectivity of T_n, there must exist a path

$$x_{n-2} \ldots x_t x_j w, \tag{1}$$

where $x_j \in N^+(K)$. If not, x_{n-2} cannot reach vertex w. This is contrary to the connectivity.

<u>Case 1.</u> If there is a $x_i \in \{x_3, \ldots, x_{n-2}\}$ such that $(x_i x_1)$, then there exist following paths of length 3

$$K x_i x_1 x_2 \quad \text{from } K \text{ to } x_2,$$
$$K x_{i-1} x_i x_1 \quad \text{from } K \text{ to } x_1.$$

<u>Case 2.</u> For each of $x_3, \ldots, x_{n-2}$ there is a $x_1 x_i$, $i = 3, \ldots, n-2$ (i.e. $i = 2, 3, \ldots, n-2$).

<u>Subcase 2.1.</u> If $(w x_1), (w x_2)$, then by path (1) there are

$$K x_j w x_1 \quad \text{from } K \text{ to } x_1,$$
$$K x_j w x_2 \quad \text{from } K \text{ to } x_2.$$

<u>Subcase 2.2.</u> If $(w x_1), (x_2 w)$ and if there is at least one $x_i \in \{x_4, x_5, \ldots, x_{n-2}\}$ such that $(x_i x_2)$, then there are paths of length 3, $K x_2 w x_1$ and $K x_{i-1} x_i x_2$.

In preceding cases, there are 3-paths from K to each vertex of $N^+(K)$. By Lemma 1, there is a 3-path from K to w. By Theorem C, there is a 3-path from K to K. Namely, K is a 3-king of T_n.

<u>Subcase 2.3.</u> If $(w x_1), (x_2 w)$ and $(x_2 x_i)$ for $i = 3, \ldots, n-2$, then x_2 is a 3-king of T_n, because of the following 3-path.

$$x_2 w K x_i \quad i = 1, 2, \ldots, n-2.$$
$$x_2 x_j w K \quad (j \neq 2, \text{ see path (1)}).$$
$$x_2 x_{n-3} x_{n-2} w \quad \text{if path (1) is } (x_{n-2} w).$$

$x_2 x_t x_j w$ $(t \neq 2, j$, but it is possible for $t = n-2)$ if length of path (1) is larger than 1.

Subcase 2.4. If $(x_1 w)$, then x_1 is a 3-king of T_n, because of

$$\begin{array}{ll} x_1 w K x_i & i = 1,2,\ldots,n-2 \\ x_1 x_j w K & (j \neq 1) \\ x_1 x_t x_j w & \text{(similar to the proof of subcase 2.3).} \end{array}$$

This completes the proof of Lemma 3. ∎

Lemma 4. *T_n, $n > 6$, is an irreducible tournament. Let K be a vertex with maximum outdegree in T_n. If $d^+(K) = n-3$ and the subtournament T_{n-3} induced by $N^+(K)$ is reducible, then T_n has a 3-king.*

Proof: Let $N^-(K) = \{u_1, u_2\}$. Suppose $(u_1 u_2)$ generality. Since T_{n-3} is not strongly connected, T_{n-3} has a spanning path, say $x_1 x_2 \ldots x_{n-3}$ and $x_1 x_{n-3}$.

As the proof of Lemma 3, we see that there are 3-paths from K to x_i, $i = 3,4,\ldots,n-3$. Now we need only show that there is a 3-path from K to x_i, $i = 1,2$.

Case 1. If there exists a $x_i \in \{x_3,\ldots,x_{n-3}\}$ such that $(x_i x_1)$, then there are the following 3-paths.

$$K x_i x_1 x_2$$
$$K x_{i-1} x_i x_1 .$$

Case 2. If for every x_i, $i = 2,\ldots,n-3$, $(x_1 x_i)$, by the maximality of $d^+(K)$, $\exists u_i \in N^-(K)$, $i \in \{1,2\}$, such that $(u_i x_1)$. Next, there is at least a $x_j \in N^+(K) - \{x_1\}$ such that $(x_j u_i)$. Thus there is a 3-path from K to x_1, $K x_j u_i x_1$.

Subcase 2.1. If $\exists x_i \in \{x_4,\ldots,x_{n-3}\}$ such that $(x_i x_2)$, then there is a 3-path from K to x_2, $K x_{i-1} x_i x_2$. Thus, by Theorem C and Lemma 1, K is a 3-king.

Subcase 2.2. If for every x_i, $i = 3,4,\ldots,n-3$, $(x_2 x_i)$, then we consider the following cases.

Subcase 2.2.1. If $\exists u_i \in N^-(K)$, $i = 1$ or 2, $(u_i x_2)$, then as in the proof of case 2 above, there is a $x_j \in N^+(K)$, $j \neq 2$, such that $(x_j u_i)$. Thus there is a 3-path $K x_j u_i x_2$. As before, K is a 3-king.

Subcase 2.2.2. If $(x_2 u_1)$, $(x_2 u_2)$, then by the connectivity of T_n, there is a path from x_{n-3} to u_i, $i = 1$ or 2, as follows.

$$x_{n-3} \ldots x_t x_j u_i \quad i = 1 \text{ or } 2, \quad j = 3,\ldots,n-3. \tag{2}$$

Let's consider the following cases about the adjacency between x_j and u_i, $i = 1$ or 2.

Since there is either $(x_j u_1)$ or $(x_j u_2)$, one of the following exists:

(1) $(x_j u_1)$ and $(x_j u_2)$.
Since $n > 6$, $|N^+(K)| > 3$. Hence $\exists x_t, t \in \{3, \ldots, n-3\} \backslash \{j\}$, (see path (2)) such that $x_2 x_t x_j u_1$ (Notice that $(x_2 x_i), i = 3, 4, \ldots, n-2$). And $x_2 u_1 u_2 K, x_2 x_j u_1 u_2, x_2 u_1 K x_i, i = 1, 2, \ldots, n-3$. Thus x_2 is a 3-king.

(2) $(x_j u_2)$ and $(u_1 x_j)$.
Notice that $u_1 K x_2 u_1, u_1 K x_2 u_2, u_1 u_2 K x_i, i = 1, \ldots, n-3, u_1 x_j u_2 K$. Thus u_1 is a 3-king.

(3) $(u_2 x_j)$ and $(x_j u_1)$.
We have $x_2 x_j u_1 u_2$, $x_2 u_1 u_2 K$, $x_2 u_1 K x_i$ $i = 1, \ldots, n-3$, $x_2 x_t x_j u_1$, where $t \in \{3, 4, \ldots, n-3\} \backslash \{j\}$ (see path (2)). Thus x_2 is a 3-king.

The proof is complete. ∎

Lemma 5. *T_n, $n > 6$, is an irreducible tournament. Let K be a vertex with maximum outdegree in T_n. If $d^+(K) = r < n-3$ and T_r induced by $N^+(K)$ is reducible, then T_n has a 3-king.*

Proof: We will show that K is a 3-king of T_n.

Let the spanning path of T_r be $x_1 x_2 \ldots x_r$ where $(x_1 x_r)$. According to the proof of Lemmas 3, 4, there are 3-path from K to $x_i, i = 3, 4, \ldots, r$. We show that there is a 3-path from K to $x_i, i = 1, 2$.

Case 1. $\exists x_i \in \{3, \ldots, r-1\}$ such that $(x_i x_1)$.

As in the proof of Lemma 3, we see there are 3-paths from K to $x_i, i = 1, 2$.

Case 2. For every $x_i, i = 2, 3, \ldots, r, (x_1 x_i)$.

Since $|N^-(K)| > 2$, by maximality of $d^+(K)$, there is a $u_i \in N^-(K)$ such that $(u_i x_1)$ and a $x_i x_1, x_i \neq x_1, x_i \in N^+(K)$ such that $(x_i u_i)$. Thus there is a 3-path $K x_i u_i x_1$.

If $\exists x_i \in \{x_4, \ldots, x_r\}$ such that $(x_i x_2)$, then as in the proof of Lemmas 3, 4, there is a 3-path from to x_2. If for every $x_i, i = 3, \ldots, r, (x_2 x_i)$, then $d^+_{T_r}(x_2) \geq r-2$. Since $|N^-(K)| \geq 3$, by the maximality of $d^+(K)$, there exists a $u_i \in N^-(K)$ such that $(u_i x_2)$ and $\exists x_i \in N^+(K) - \{x_2\}$, such that $(x_i u_i)$. Thus there is also a 3-path $K x_i u_i x_2$.

Hence there is a 3-path from K to every $x_i \in N^+(K)$.

According to Lemma 1 and Theorem C, we conclude that K is a 3-king of T_n. The proof is complete. ∎

Now we can give the main results.

Theorem 2. *If T_n, $n > 6$, is an irreducible tournament, then T_n has a 3-king.*

Proof: By Lemmas 1,2,3,4,5, the proof is complete. ∎

Corollary 2.1. *If T_n, $n > 6$, is an irreducible tournament, then T_n has a vertex from which there is a path (walk) of length r, $r = 3, 4, \ldots$ to each vertex of T_n.*

Proof: Let A_n be a tournament matrix corresponding to T_n. By Theorem 2, A^3 has a row, say the kth row, with no zero entry. Since A_n is irreducible, A_n has no column of all zeros. Thus A^r, $r \geq 3$, has kth row with no zeros. Thus A^r, $r \geq 3$, no zeros in its kth row. ∎

According to Corollary 2.1, we have

Theorem 3. *If T_n, $n > 6$, is an irreducible tournament, then* $\exp_{T_n}(1) = 3$.

In [1], we have proved

Lemma 7. *If Γ is a primitive digraph of order n, then*

$$\exp_\Gamma(k) \leq \exp_\Gamma(k-1) + 1, \qquad 2 \leq k \leq n.$$

Hence we obtain

Theorem 4. *If T_n, $n > 6$, is an irreducible tournament, then*

$$\exp_T(k) = k + 2 \qquad 1 \leq k \leq n.$$

Proof: By Lemma 7 and Theorem 3

$$\begin{aligned} \exp_T(k) &\leq \exp_T(1) + (k-1) \\ &= 3 + (k-1) \\ &= k + 2. \end{aligned}$$

Consider the tournament T_n defined on the vertices $1, 2, \ldots, n$ as follows: The arcs (ij) if $1 \leq i \leq n$. This tournament contains a simple cycle of length n so it is irreducible and hence primitive.

For $1 \leq j \leq n$, there are walks of length r, $r = j+2, j+3, \ldots$ from vertex j to $1, j \geq 1$, but there is no walk of length $j+1$. (If the reader sketches the tournament T_n the reasons for this and subsequent statements should become apparent).

$$\begin{aligned} \exp_T(k) &= \max\{\exp_T(k,j) \mid j = 1, 2, \ldots, n\} \\ &= \exp_T(k, 1) \\ &= k + 2, \end{aligned}$$

where T is T_n. ∎

It follows that each irreducible tournament T_n has at least k vertices from each of which there is a path of length $k+2$ to every vertex of T_n.

Corollary 4.1. ([3]) *If T_n, $n > 6$, is an irreducible tournament with primitive exponent e, then* $e \leq \exp_{T_n}(n) = n + 2$.

Now that there are exactly 1, 6, 35 non-isomorphic irreducible tournaments T_4, T_5, T_6 respectively (see e.g. [3]). It is not difficult to verify (or prove) the following theorem directly.

Theorem 5. *If T_n, $3 \leq n \leq 6$, is an irreducible tournament, then*

$$\exp_{T_4}(1) = 6$$
$$\exp_{T_5}(1) = 4$$
$$\exp_{T_6}(1) = 4.$$

There is essentially only one irreducible tournament T_i, $i = 4, 5, 6$ with $\exp_{T_i}(1)$ given as above. They are as follows:

$$T_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad T_5 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad T_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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