The Class of t-sc Graphs and Their Stable Complementing Permutations

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Abstract. In this paper, we present a new generalization of the self-complementary graphs, called the t-sc graphs. Various properties of this class of graphs are studied generalizing earlier results on self-complementary graphs. Certain existential results on t-sc graphs are presented, followed by the construction of some infinite classes of t-sc graphs. Finally, the notion of t-sc graphs is linked with the notion of factorization. This leads to a generalization of r-partite self-complementary graphs.

1. Introduction and definitions.

The class of self-complementary graphs has been extensively studied by many people, among others by C.R.J. Clapham [2], R.A. Gibbs [8], S.B. Rao [10], G. Ringel [11], and H. Sachs [12], and many problems have been solved for this class of graphs, such as the hamiltonian problem and the characterization of potentially and forcibly self-complementary degree sequences (see the references given in [10]). This interesting class has also been generalized into the class of multipartite self-complementary graphs by T. Gangopadhyay and S.P. Rao Hebbare [5]. Several important notions such as path-lengths, range of diameters have already been studied for the generalized class (see [6], [7]).

In the present paper a new generalization of the self-complementary graphs, the class of t-sc graphs, is presented and various properties of this class of graphs are studied — generalizing earlier results of Ringel [11] and Sachs [12]. In Section 2 of this paper, we study some structural properties of stable complementing permutations. In Section 3, we study certain existential results on t-sc graphs. In Section 4, we construct infinite classes of t-sc graphs having a stable complementing permutation. In conclusion, we define the notion of t-rpsc graphs which constitutes a generalization of r-partite self-complementary graphs, extensively studied in ([5], [6]). For all undefined terms we refer to Harary [9].

Given an integer t, the t-tuple $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ is called a t-sc graph if there exists a complete graph G such that:

i) each G_i is a spanning subgraph of G;
ii) $E(G)$ is the disjoint union of $E(G_1), E(G_2), \ldots, E(G_t)$;
iii) $G_1, G_2, \ldots, G_t$ are all isomorphic graphs.

A t-sc graph $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ is called connected if G_1 is connected.

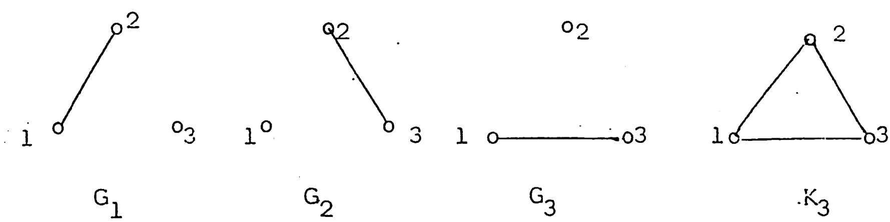
Let $(G_1, G_2, \ldots, G_t)$ be a t-sc graph. Let σ_i be an isomorphism from G_i to G_{i+1}, $1 \leq i \leq t-1$, and let σ_t be an isomorphism from G_t to G_1. Then the

t-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ is called a *complementing permutation class* (cpc) for $(G_1, G_2, \ldots, G_t)$.

Let π be a cycle of σ_i. We denote by $|\pi|$ the length of π, that is, the number of points of G_i contained in π. We say π is a fixed point if $|\pi| = 1$.

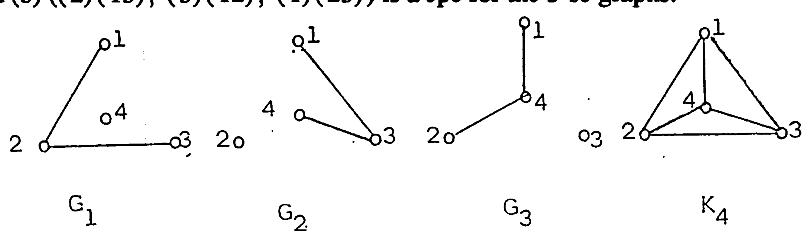
Clearly, if $t = 2$ then $G_2 = G_1^C$ and G_1 is a self-complementary graph in the usual sense. Also, if $(\sigma_1, \sigma_2,)$ is a cpc for (G_1, G_2) then σ_1 is a complementing permutation for the self-complementary graph G_1, in the usual sense of the term.

Let $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ be a cpc for a t-sc graph. If $\sigma_1 = \sigma_2 = \ldots \sigma_t = \sigma$ (say) then σ is called a *stable complementing permutation* (scp) for $(G_1, G_2, \ldots, G_t)$.



(a) $\sigma_1 = \sigma_2 = \sigma_3 = (123)$ (b) $\sigma_1 = (2)(13)$, $\sigma_2 = (3)(12)$, $\sigma_3 = (1)(23)$.
Figure 1

Figure 1 shows the only 3-sc graph on 3 points. Clearly, (a) $\sigma = (123)$ is an scp and (b) $((2)(13), (3)(12), (1)(23))$ is a cpc for the 3-sc graphs.

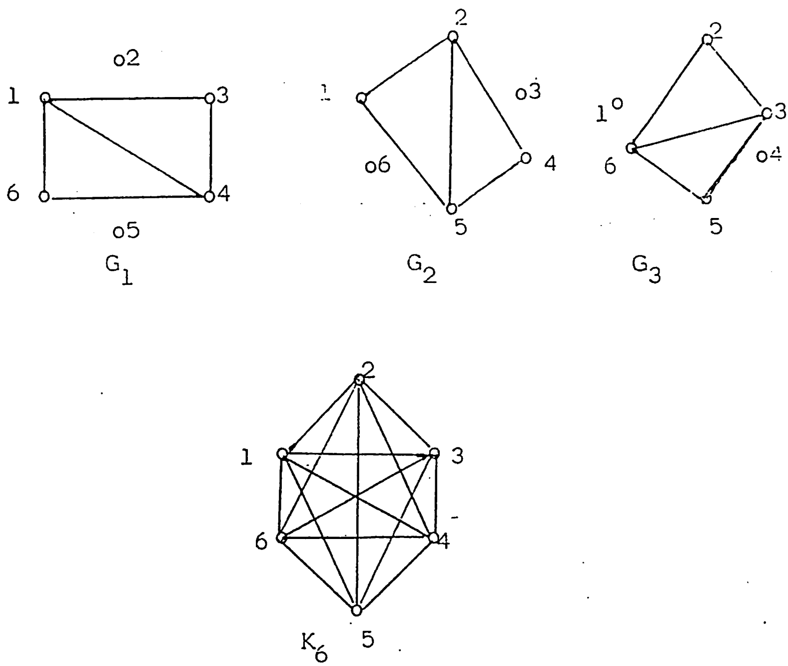


$\sigma_1 = \sigma_2 = \sigma_3 = (1)(234)$.
Figure 2

Figure 2 shows a 3-sc graph on 4 points with $\sigma = (1)(234)$ as an scp.
Figure 3 depicts a 3-sc graph on 6 points with an scp $\sigma = (123456)$ and a cpc $((126)(345), (123)(456), (156)(234))$.
Figure 4 depicts a 6-sc graph on 4 points with a cpc $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$ where each σ_i is as given in the figure.

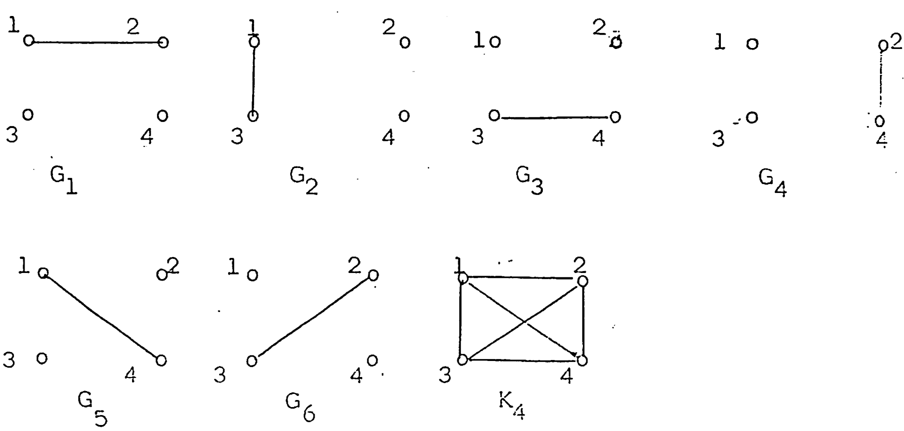
The notion of t-sc graphs is intimately linked with the notion of factorization. For instance, if $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ is a t-sc graph with the property that G_1 is regular with degree d, then $\mathcal{G}$ constitutes a d-factorization of K_n where $n = |V(G_1)|$.

This relationship is strongly reflected in Section 3, where we repeatedly invoke the following.



(a) $\sigma = (123456)$ (b) $\sigma_1 = (126)(345)$, $\sigma_2 = (123)(456)$, $\sigma_3 = (156)(234)$.

Figure 3



$\sigma_1 = (1)(23)(4) = \sigma_3$, $\sigma_2 = (3)(14)(2)$, $\sigma_4 = (4)(12)(3)$,
$\sigma_5 = (12)(34)$, $\sigma_6 = (2)(13)(4)$.

Figure 4

Theorem 1.1. (See Harary [9]). *The graph K_{2n+1} can be factored into n spanning cycles.*

Proof: Let $V(K_{2n+1}) = \{u_1, u_2, \ldots, u_{2n+1}\}$. We construct n paths P_i on the points $u_1, u_2, \ldots, u_{2n}$ as follows:

$$P_i = u_i u_{i-1} u_{i+1} u_{i-2} \ldots u_{i+n-1} u_{i-n}.$$

Thus, the jth point of P_i is u_k, where $k = i + (-1)^{j+1}[j/2]$ and all subscripts are taken as the integers $1, 2, \ldots, 2n \pmod{2n}$. The spanning cycle C_i is then constructed by joining u_{2n+1} to the end points of P_i.

We also make use of Beineke's [1] construction in which K_{2n} is factored into n hamiltonian paths. ∎

2. Fundamental properties of stable complementing permutations.

In this section, we study the properties of an scp of a t-sc graph and establish analogues of some well-known theorems on self-complementary graphs.

Lemma 2.1. *Let $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ be a cpc for $(G_1, G_2, \ldots, G_t)$. Then for each $i \in \{1, 2, \ldots, t\}$, $\sigma_i \sigma_{i+1} \ldots \sigma_{i-1}$ is an autormorphism for G_i.*

Proof:

$$\begin{aligned}
uv \in E(G_i) &\Leftrightarrow \sigma_i(u)\sigma_i(v) \in E(G_{i+1}) \\
&\Leftrightarrow \sigma_i\sigma_{i+1}(u)\sigma_i\sigma_{i+1}(v) \in E(G_{i+2}) \\
&\Leftrightarrow \ldots \Leftrightarrow \sigma_i\sigma_{i+1}\ldots\sigma_{t-1}(u)\sigma_i\sigma_{i+1}\ldots\sigma_{t-1}(v) \in E(G_t) \\
&\Leftrightarrow \sigma_i\ldots\sigma_{t-1}\sigma_t(u)\sigma_i\ldots\sigma_{t-1}\sigma_t(v) \in E(G_1) \\
&\Leftrightarrow \sigma_i\sigma_{i+1}\ldots\sigma_t\sigma_1(u)\sigma_i\sigma_{i+1}\ldots\sigma_t\sigma_1(v) \in E(G_2) \\
&\Leftrightarrow \sigma_i\sigma_{i+1}\ldots\sigma_t\sigma_1\sigma_2\ldots\sigma_{i-1}(u)\sigma_i\sigma_{i+1}\ldots\sigma_t\sigma_1\sigma_2\ldots\sigma_{i-1}(v) \in E(G_1).
\end{aligned}$$

This proves the Lemma. ∎

Lemma 2.2. *Let σ be an scp for $(G_1, \ldots, G_t)$. Then σ^t is an automorphism for each G_i, $i = 1, 2, \ldots, t$.*

Proof: This follows by substituting $\sigma_i = \sigma\ \forall i = 1, 2, \ldots, t$. ∎

The existence of an scp is a very desirable property for a t-sc graph. The following Lemma gives a sufficient condition for the existence of an scp.

Lemma 2.3. *Let $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ be a cpc for a t-sc graph $(G_1, G_2, \ldots, G_t)$. If $\sigma_1 = \sigma_2 = \ldots = \sigma_{t-1} = \sigma$ (say), then σ is an scp for $(G_1, G_2, \ldots, G_t)$.*

Proof: It is enough to show that σ is an isomorphism from G_t to G_1. This follows since $uv \in E(G_t)$. $\Leftrightarrow$ $uv \notin E(G_i)$ for each $i = 1, 2, \ldots, t-1$. $\Leftrightarrow$ $\sigma(u)\ \sigma(v) \notin E(G_i)$ for each $i = 2, 3, \ldots, t$. $\Leftrightarrow$ $\sigma(u)\ \sigma(v) \in E(G_1)$. ∎

Corollary 2.4. *Let G be a sc graph and σ a complementing permutation of G. Then σ is an scp for the 2-sc graphs $(G, \overline{G})$.*

The following corollary now directly follows from Lemma 2.2 and Corollary 2.4.

Corollary 2.5. *Let G be a sc graph with a complementing permutation σ. Then σ^2 is an automorphism of G.*

Lemma 2.6. *Let σ be an scp for a t-sc graph $(G_1, G_2, \ldots, G_t)$. Then σ has at most one fixed point.*

Proof: Let $\pi_1 = (u), \pi_2 = (v)$ be two fixed points in σ. Let $uv \in E(G_i)$. Then $\sigma(u)\ \sigma(v) \in E(G_{i+1})$. But $\sigma(u) = \pi_1(u) = u$ and $\sigma(v) = \pi_2(v) = v$. So $uv \in E(G_{i+1})$. Thus, $uv \in E(G_i) \cap E(G_{i+1})$, a contradiction. This proves the Lemma. ∎

Theorem 2.7. *If a t-sc graph $(G_1, G_2, \ldots, G_t)$ on n points has an scp σ then $n \equiv 0$ or $1 \pmod{t}$. If $n \equiv 0 \pmod{t}$ then all cycles π of σ have $|\pi| \equiv 0 \pmod{t}$. If $n \equiv 1 \pmod{t}$ then σ has exacly 1 cycle of length 1, all other cycles π having $|\pi| \equiv 0 \pmod{t}$.*

Proof: Let $\pi = (v_1, v_2, \ldots, v_{kt+r})$ be a cycle of σ with $r < t$ and $kt + r > 1$. Clearly, $v_1 v_2 \in E(G_i)$ for some i. Without loss of generality, let $v_1 v_2 \in E(G_1)$. Then since σ is an scp we have

$$\begin{aligned} v_1 v_2 \in E(G_1) &\Rightarrow v_2 v_3 \in E(G_2) \\ &\Rightarrow \ldots \Rightarrow v_t v_{t+1} \in E(G_t) \Rightarrow v_{t+1} v_{t+2} \in E(G_1) \\ &\Rightarrow v_{t+2} v_{t+3} \in E(G_2) \Rightarrow \ldots \Rightarrow v_{2t} v_{2t+1} \in E(G_t) \\ &\quad \ldots \quad \ldots \quad \ldots \\ &\Rightarrow v_{(k-1)t+1} v_{(k-1)t+2} \in E(G_1) \Rightarrow v_{(k-1)t+2} v_{(k-1)t+3} \in E(G_2) \\ &\Rightarrow \ldots \Rightarrow v_{kt} v_{kt+1} \in E(G_t) \Rightarrow v_{kt+1} v_{kt+2} \in E(G_1) \\ &\Rightarrow v_{kt+2} v_{kt+3} \in E(G_2) \Rightarrow \ldots \Rightarrow v_{kt+r-1} v_{kt+r} \in E(G_{r-1}) \\ &\Rightarrow v_{kt+r} v_1 \in E(G_r) \Rightarrow v_1 v_2 \in E(G_{r+1}). \end{aligned}$$

Thus, it follows that $r + 1 = 1$, that is, $r = 0$. Thus, every cycle π of σ with $|\pi| > 1$ has length $\equiv 0 \pmod{t}$.

Using Lemma 2.6 we now obtain that either (a) every cycle π of σ has $|\pi| \equiv 0 \pmod{t}$ or (b) σ has exactly one fixed point and every other cycle π of σ has $|\pi| \equiv 0 \pmod{t}$. It now easily follows that if (a) is true then $n \equiv 0 \pmod{t}$ and if (b) is true then $n \equiv 1 \pmod{t}$. This proves the theorem. ∎

Lemma 2.8. *Let p be a prime number such that for some $r \geq 1$, p^r divides t. If for $n > 1$, $(G_1, G_2, \ldots, G_t)$ is t-sc on n-points then $n \equiv 0$ or $1 \pmod{p^r}$. In particular, if $p = 2$ then $n \equiv 0$ or $1 \pmod{2^{r+1}}$.*

Proof: This follows since $n(n-1)/2t$, being the number of edges in G_1, has to be an integer and p divides n if and only if p does not divide $n - 1$. ∎

Corollary 2.9. *If p is a prime number and $(G_1, G_2, \ldots, G_p)$ is p-sc on n points then $n \equiv 0$ or $1 \pmod{p}$.*

Corollary 2.9 is stronger than Theorem 2.7 in that it does not need the existence of an scp. In Figure 4, we have already exhibited a 6-sc graph on 4 points. This is possible since the graph does not have an scp. Thus, the example also demonstrates that not every t-sc graph has an scp.

Corollary 2.10. *Let $r \geq 1$ and 2^r be a factor of t. If σ is an scp for the t-sc graph $(G_1, G_2, \ldots, G_t)$ and π is a cycle of σ then $|\pi| \equiv 0$ or $1 \pmod{2}^{r+1}$, unless $|\pi| = 1$.*

Proof: Let H_i be the subgraph of G_i induced by the points of π. Then $(H_1, H_2, \ldots, H_t)$ is t-sc on $|\pi|$ points. So by Lemma 2.8 $|\pi| \equiv 0$ or $1 \pmod{2^{r+1}}$. ▮

The following is a theorem on self-complementary graphs.

Corollary 2.11. *(Ringel [11], Sachs [12]). Let G be self-complementary and σ a complementing permutation of G. Then either $|V(G)| \equiv 0 \pmod{4}$ and all the cycles of σ have length of $\equiv 0 \pmod{4}$, or $|V(G)| \equiv 1 \pmod{4}$ and all but one cycle of σ have lengths $\equiv 0 \pmod{4}$, the remaining cycle having length one.*

Proof: Let π be a cycle of σ. Since σ is an scp of the 2-sc graph $(G, \overline{G})$, by Corollary 2.10, either $|\pi| = 1$ or $|\pi| \equiv 0$ or $1 \pmod{4}$. By Lemma 2.6 there can be at most one fixed point, proving the corollary. ▮

We conclude this section with a demonstration as to how, given a cpc $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ for the t-sc graph $(G_1, G_2, \ldots, G_t)$ we can generate other cpcs from it.

Lemma 2.12. *Let $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ be a cpc for the t-sc graph $(G_1, G_2, \ldots, G_t)$. Then there exists an integer $r \geq 1$ such that for all $i = 1, 2, \ldots, t$ and $s < r$, $(\sigma_i, \sigma_{i+1} \ldots \sigma_t \sigma_1 \ldots \sigma_{i-1})^r =$ identity and $(\sigma_i, \sigma_{i+1} \ldots \sigma_t \sigma_1 \ldots \sigma_{i-1})^s \neq$ identity (where suffixes are taken modulo t).*

Proof: Let r be the smallest integer ≥ 1 such that $(\sigma_1 \sigma_2 \ldots \sigma_t)^r =$ identity. Let $i > 1$ and let s be the smallest integer ≥ 1 such that $(\sigma_i, \sigma_{i+1} \ldots \sigma_t \sigma_1 \ldots \sigma_{i-1})^s =$ identity. We shall prove that $s = r$. Now,

$$\begin{aligned}\sigma_i, \sigma_{i+1} \ldots \sigma_t &= \sigma_i, \sigma_{i+1} \ldots \sigma_t (\sigma_1 \ldots \sigma_t)^r \\ &= (\sigma_i, \sigma_{i+1} \ldots \sigma_t \sigma_1 \ldots \sigma_{i-1})^r \quad \sigma_i \sigma_{i+1} \ldots \sigma_t.\end{aligned}$$

So, $(\sigma_i, \sigma_{i+1} \ldots \sigma_t \sigma_1 \ldots \sigma_{i-1})^r =$ identity $= (\sigma_i, \sigma_{i+1} \ldots \sigma_t \sigma_1 \ldots \sigma_{i-1})^s$. Thus, by definition of s, it follows $r \geq s$. Again,

$$\begin{aligned}\sigma_i, \sigma_{i+1} \ldots \sigma_t &= (\sigma_i, \sigma_{i+1} \ldots \sigma_t \sigma_1 \ldots \sigma_{i-1})^s \sigma_i \sigma_{i+1} \ldots \sigma_t \\ &= \sigma_i \sigma_{i+1} \ldots \sigma_t (\sigma_1 \sigma_2 \ldots \sigma_t)^s.\end{aligned}$$

So $(\sigma_1 \sigma_2 \ldots \sigma_t \sigma_1 \ldots \sigma_{i-1})^s =$ identity $= (\sigma_2 \sigma_2 \ldots \sigma_t)^r$.

Now by definition of s, it follows that $s \geq r$. It now follows that $r = s$.

Theorem 2.13. *Let $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ be a cpc for the t-sc graph $(G_1, G_2, \ldots, G_t)$. Let r be as in Lemma 2.12. Then for all s, $1 \leq s \leq r-1$, the permutations $(\sigma_i\sigma_{i+1}\ldots\sigma_t\sigma_1\sigma_2\ldots\sigma_{i-1})^s\ \sigma_i$ constitute $r-1$ distinct isomorphisms from G_i to G_{i+1}.*

Proof: By Lemma 2.1, $\sigma_i\sigma_{i+1}\ldots\sigma_t\sigma_1\sigma_2\ldots\sigma_{i-1}$ is an automorphism of G_i. So $(\sigma_i\sigma_{i+1}\ldots\sigma_{i-1})^s\ \sigma_i$ is an isomorphism from G_i to G_{i+1}. Suppose now for s, t, $1 \leq s < t \leq r-1$

$$(\sigma_i\sigma_{i+1}\ldots\sigma_{i-1})^s \quad \sigma_i = (\sigma_i\sigma_{i+1}\ldots\sigma_{i-1})^t\ \sigma_i.$$

Then $(\sigma_i\sigma_{i+1}\ldots\sigma_{i-1})^{t-s}$ = identity.

But $t - s < r$. So by definition of r, $t = s$. This proves the theorem. ∎

As an illustration for Theorem 2.13, we consider the graph in Figure 3. If σ_1, σ_2, σ_3, are as in 3(b), then $\sigma_1\ \sigma_2\ \sigma_3\ \sigma_1 = \sigma_2\ \sigma_3\ \sigma_1\ \sigma_2 = \sigma_3\ \sigma_1\ \sigma_2\ \sigma_3 = (153)(264)$. Thus, $(153)(264)$ is an scp of (G_1, G_2, G_3). Also since $(\sigma_1\ \sigma_2\ \sigma_3)^2$ = identity, $r = 2$.

3. Existence of t-sc graphs for every inteqer t.

We begin with a construction of t-sc graphs for every integer t.

Theorem 3.1. *For every integer t, there is a t-sc graph on $2t$ points with an scp σ consisting of a single cycle.*

Proof: The proof uses the construction given in Beineke [1]. Let $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ where G_i is the path

$$u_iu_{i-1}\,u_{i+1}\,U_{i-2}\,u_{i+2}\,u_{i-3}\ldots u_{i+t-1}\,u_{i-t}$$

constructed on the points $u_1, u_2, \ldots, u_{2t}$.

Then, clearly, $\sigma = (u_1 u_2 \ldots u_{2t})$ is an scp for $\mathcal{G}$, proving the theorem. ∎

For odd integers t, our construction of a t-sc graph requires only t points as shown below in

Theorem 3.2. *For every odd integer t, there is a t-sc graph on t points.*

Proof: Let $t = 2n+1$. Consider K_{2n+1}. It has $n(2n+1)$ edges. Let $V(K_{2n+1}) = \{u_1, u_2, \ldots, u_{2n+1}\}$. By Theorem 1.1, it follows that K_{2n+1} is the union of n spanning cycles. Let these cycles be $C_1, C_2, \ldots, C_n$. Then as in the proof of Theorem 1.1 C_i contains the edge u_iu_{i-1}, $i = 1, 2, \ldots, n$. Define

$$D_1 = C_1 - u_1u_{2n}$$
$$D_i = C_i - u_iu_{i-1}, \qquad i = 2, 3, \ldots, n.$$

Then each D_i can be split into two paths of length n each, say P_{i1} and P_{i2}. Now let G_i be the graph with

$$\begin{aligned} V(G_i) &= V(K_{2n+1}) && i = 1,2,\dots,t \\ E(G_i) &= E(P_{i1}) && i = 1,2,\dots,n \\ &= E(P_{i-n2}) && i = n+1,\ n+2,\dots,t-1 \\ &= \{u_{2n}u_1\} \cup \left(\bigcup_{j=2}^{n}\{u_{j-1}u_j\}\right) && i = t. \end{aligned}$$

Note that $E(G_t)$ is the path $u_{2n}\ u_1 u_2 \dots u_n$ which is also a path of length n.

Clearly, $G_1, G_2, \dots, G_t$ are all isomorphic graphs. So $\mathcal{G} = (G_1, G_2, \dots, G_t)$ is a t-sc graph on t points. This proves the theorem. ∎

We illustrate the construction described in Theorem 3.2 in the figure below for $t = 7$.

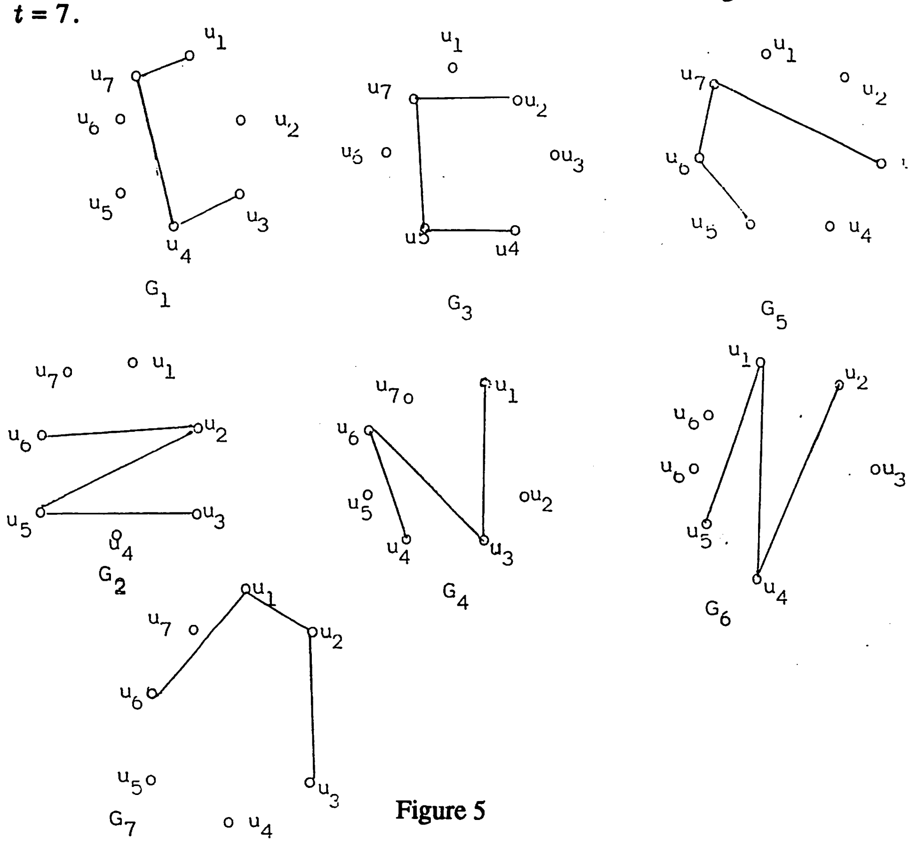


Figure 5

The next theorem tells us that if t is a power of an odd prime number then the construction given in Theorem 3.2 gives us a minimal t-sc graph.

Theorem 3.3. *Let $t = p^r$, where p is an odd prime. Then every t-sc graph has at least t points.*

Proof: Suppose a t-sc graph has n points. By Lemma 2.8 it follows that $n \equiv 0$ or $1 \pmod{p)^r}$. So $n \geq p^r = t$. ∎

Theorem 3.4. *If t is even then no t-sc graph on t points exists.*

Proof: Let $\mathcal{G}$ be a t-sc graph on t points. Then $t(t-1)/2t$ has to be an integer, implying $t-1$ is even, a contradiction. Hence, the theorem. ∎

The next theorem tells us that if t is a power of 2 then the construction in Theorem 3.1 gives us the minimal t-sc graph.

Theorem 3.5. *Let $t = 2^r$. Then every non-trivial t-sc graph has at least $2t$ points.*

Proof: Let n be the number of points of a t-sc graph. Then $2^{r+1} = 2t$ divides $n(n-1)$, hence, $n = 0$ or $\pmod{2t}$. Hence, $n \geq 2t$. ∎

The next theorem gives a sufficient condition for the existence of t-sc graphs on less than t points.

Theorem 3.6. *Let $t = 2^r, s$, with $r \geq 1$ and $s \geq 3$. If s divides either $2^{r+1}-1$ or $2_{r+1}+1$ then there exists a t-sc graph on 2^{r+1} points or on $2^{r+1}+1$ points, respectively.*

Proof: Suppose $2^{r+1}-1$ is divisible by s. Let $n = 2^{r+1}$. Then by Theorem 1.1, K_{n+1} is the union of 2^r spanning cycles each of length $2^{r+1}+1$. Let these cycles be $C_1, C_2, \dots, C_{2^r}$.

Let $K_{n+1} = \{u_1, u_2, \dots, u_{n+1}\}$. Each C_i contains exactly 2 edges incident with u_{n+1}. Let D_i be the path of length $2^{r+1}-1$, obtained by deleting these two edges from C_i. Split D_i into s edge-disjoint paths of length $(2^{r+1}-1)/s$.

Let these paths be $P_{i1}, P_{i2}, \dots, P_{is}$. Let G_{ik} be the graph with $V(G_{ik}) = \{u_1, u_2, \dots, u_n\}$ and $E(G_{ik}) = E(P_{ik}), k = 1, 2, \dots, s, i = 1, 2, \dots, 2^r$. Then clearly, $G_{11}, G_{12}, \dots, G_{1s}, G_{21}, G_{22}, \dots, G_{2s}, \dots, G_{2^r1}, G_{2^r2}, G_{2^rs}$, are all isomorphic graphs. So $\mathcal{G} = (G_{11}, G_{12}, \dots, G_{2^rs})$, is a t-sc graph on $n(= 2^{r+1})$ points.

Suppose $2^{r+1}+1$ is divisible by s. Let $n = 2^{r+1}+1$. Then by Theorem 1.1, K_n is the union of 2^r spanning cycles each of length $2^{r+1}+1$. Let these cycles be $C_1, C_2, \dots, C_{2^r}$. Split each C_i into s edge-disjoint paths of length $(2^{r+1}+1)/s$. Let these paths be $P_{i1}, P_{i2}, \dots, P_{is}$. Then define G_{ik} as the graph with $V(G_{ik}) = V(K_n)$ and $E(G_{ik}) = E(P_{ik})$. Clearly, $G_{11}, G_{12}, \dots, G_{1s}, G_{21}, G_{22}, \dots, G_{2s}, \dots, G_{2^r1}, G_{2^r2}, \dots, G_{2^rs}$ are all isomorphic. So $\mathcal{G} = (G_{11}, G_{12}, \dots, G_{2^rs}$ is a t-sc graph on $n (= 2^{r+1}+1)$ points. ∎

Corollary 3.7. *Let $t = 2^r \cdot 3$. Then there exists a t-sc graph on 2^{r+1} points or one on $2^{r+1}+1$ points.*

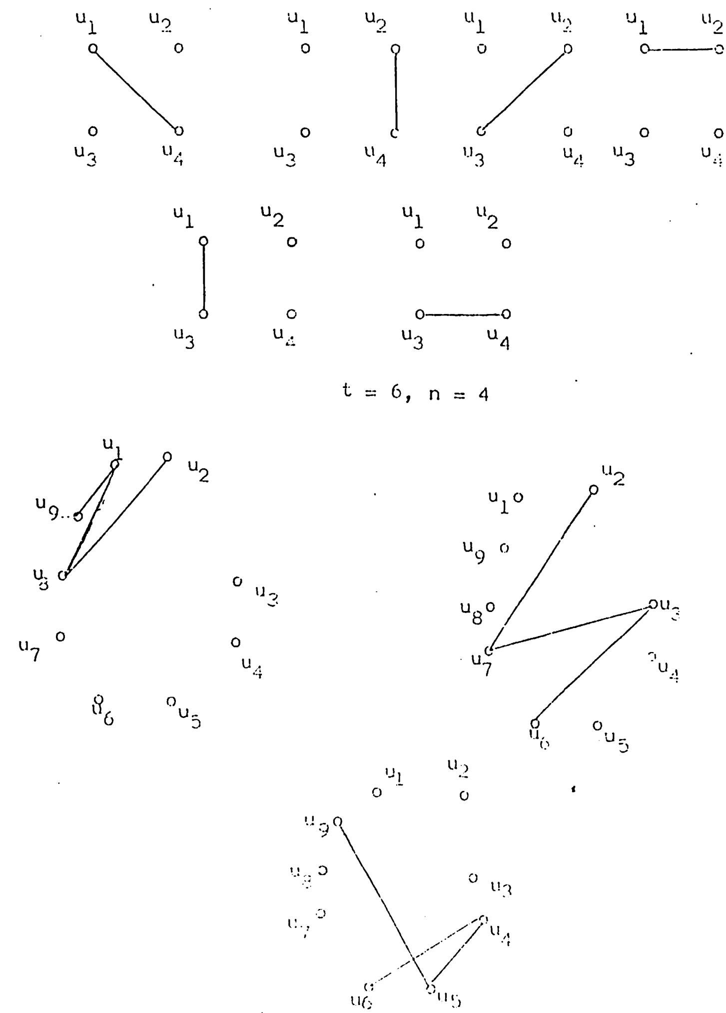


Figure 6. $t = 12$, $n = 9$

Proof: Follows from Theorem 3.6 since $s = 3$ divides either the $2^{r+1} - 1$ or $2^{r+1} + 1$. ∎

Figure 6 illustrates the construction described in the proof of Theorem 3.6 for $t = 6$ and $t = 12$.

The next theorem tells us that the graphs constructed in Theorem 3.6 are minimal.

Theorem 3.8. *Let $t = 2^r s$ with $r \geq 1$ and $s \geq 3$. Then every t-sc graph has at least $2^{r+1} + 1$ points. Further, if s divides $2^{r+1} + 1$ then every t-sc graph has at least $2^{r+1} + 1$ points.*

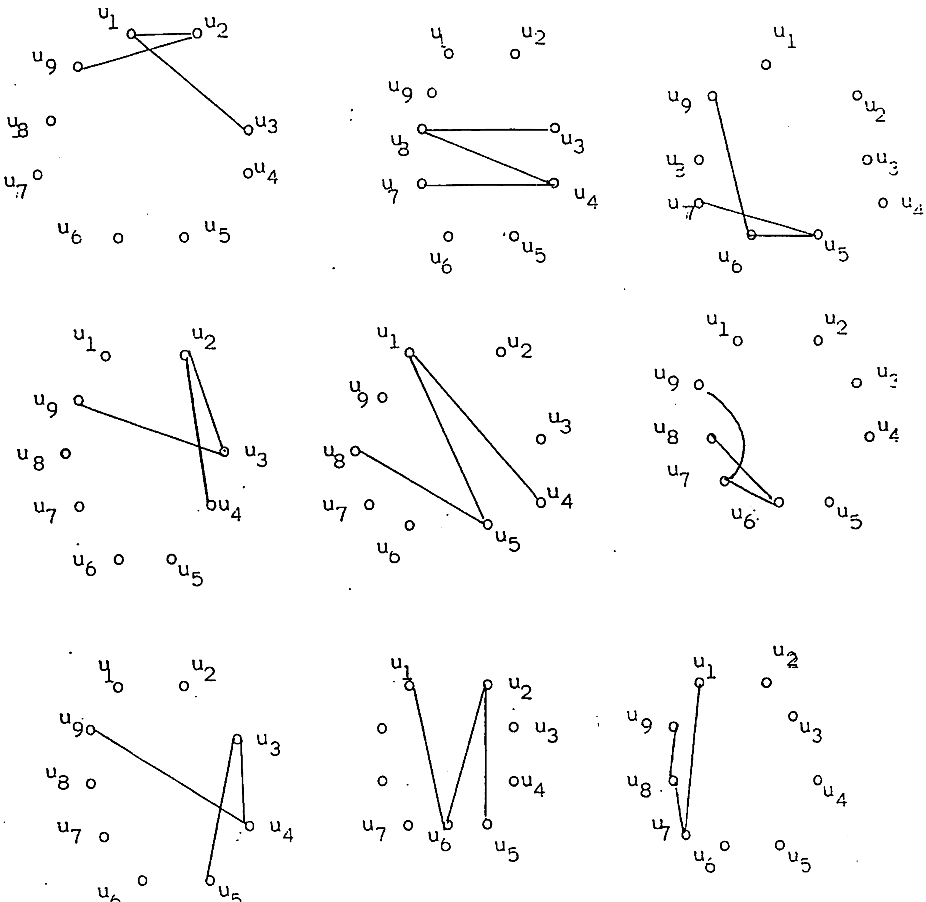


Figure 6 (continued)

Proof: Let n be the number of points of a t-sc graph. Now since $n(n-1)/2^{r+1}s$ is an integer, it follows that either n or $n-1$ is divisible by 2^{r+1}. So $n \geq 2^{r+1}$. Now, if $n = 2^{r+1}$ then $n(n-1)/2^{r+1}s = (n-1)/s$ and so s divides $n-1 = 2^{r+1} - 1$. Clearly, then s does not divide $2^{r+1} + 1$. Hence, if s divides $2^{r+1} + 1$, then $n > 2^{r+1}$. This proves the theorem. ∎

If $\mathcal{G}$ is a t-sc graph on n points then $n(n-1)/2t$ has to be an integer. In the next few theorems we investigate some sufficient conditions for the existence of t-sc graphs on n points.

Theorem 3.9. *Let $n = 2s + 1$. If t divides $n(n-1)/2$ and s divides t then there exists a t-sc graph on n points.*

Proof: By Theorem 1.1, K_n is the union of s spanning cycles, say $C_1, C_2, \ldots, C_s$. Let $t = ks$. Since t divides $n(n-1)/2$ it follows that k divides n. Now divide each C_i into k edge-disjoint paths $P_{i1}, P_{i2}, \ldots, P_{ik}$ each of length n/k. Define G_{ij}

to be the graph with $V(G_{ij}) = V(K_n)$ and $E(G_{ij}) = E(P_{ij}), j = 1,2,\dots,k$, $i = 1,2,\dots,s$. Then the graphs $G_{ij}, j = 1,2,\dots,k, i = 1,2,\dots,s$ are all isomorphic. Hence, $\mathcal{G} = (G_{11},\dots,G_{sk})$ is t-sc on n points. ∎

Theorem 3.10. *Let $n = 2s$. If t divides $n(n-1)/2$ and s divides t then there exists a t-sc graph on n points.*

Proof: Consider K_{2s+1}. Let $V(K_{2s+1}) = \{u_1, u_2, \dots, u_{2s}, u_{2s+1}\}$. By Theorem 1.1, K_{2s+1} is the union of s spanning cycles $C_1, C_2, \dots, C_s$. Let

$$P_i = C_i - u_{2s+1}, \quad i = 1,2,\dots,s.$$

Then P_i is a path of length $2s-1$. Now let $t = ks$. Then since t divides $s(2s-1)$ it follows that k divides $2s-1$. Thus, we can split each P_i into k edge-disjoint paths of length $(2s-1)/k$. Let these be $P_{i1}, P_{i2}, \dots, P_{ik}$. Now define G_{ij} as the graph with

$$V(G_{ij}) = \{u_1, u_2, \dots, u_n\} \text{ and } E(G_{ij}) = E(P_{ij})$$
$$j = 1,2,\dots,k; \qquad i = 1,2,\dots,s.$$

Then the graphs $G_{ij}, j = 1,2,\dots,k; i = 1,2,\dots,s$ are all isomorphic. Hence, $\mathcal{G} = (G_{11},\dots,G_{sk})$ is t-sc on n points. ∎

Theorem 3.11. *Let $n = 2s$. If $2t$ divides n then there is a t-sc graph on n points.*

Proof: Let $n = 2tk$. By Theorem 3.1, there is a t-sc graph $\mathcal{G} = (G_1, G_2, \dots, G_t)$ on $2t$ points where as in the proof of the theorem G_i is the path

$$u_i u_{i-1} u_{i+1} u_{i-2} u_{i+2} u_{i-3} \cdots u_{i+t-1} u_{i-t}$$

constructed on the points $u_1, u_2, \dots, u_{2t}$.

Notice that u_i and u_{i+t} (same as u_{i-t}) are both end points of G_i. Let $V(K_n) = \{v_{ij}, j = 1,2,\dots,k; i = 1,2,\dots,2t\}$ $(n = 2kt)$. Now for $m = 1,2,\dots,t$, we define H_m to be the spanning subgraph of K_n with $v_{ij}v_{i'j'} \in E(H_m)$ if and only if either

(i) $i = i' = m$ or $i = i' = m + t$ or
(ii) $u_i u_{i'} \in E(G_m)$.

Then clearly, $\mathcal{H} = (H_1, H_2, \dots, H_t)$ is t-sc on n points. This proves the theorem. ∎

Figure 7(a) and 7(b) illustrates the constructions embodied in Theorem 3.1 for $t = 4$ and Theorem 3.11 for $n = 16$ and $t = 4$.

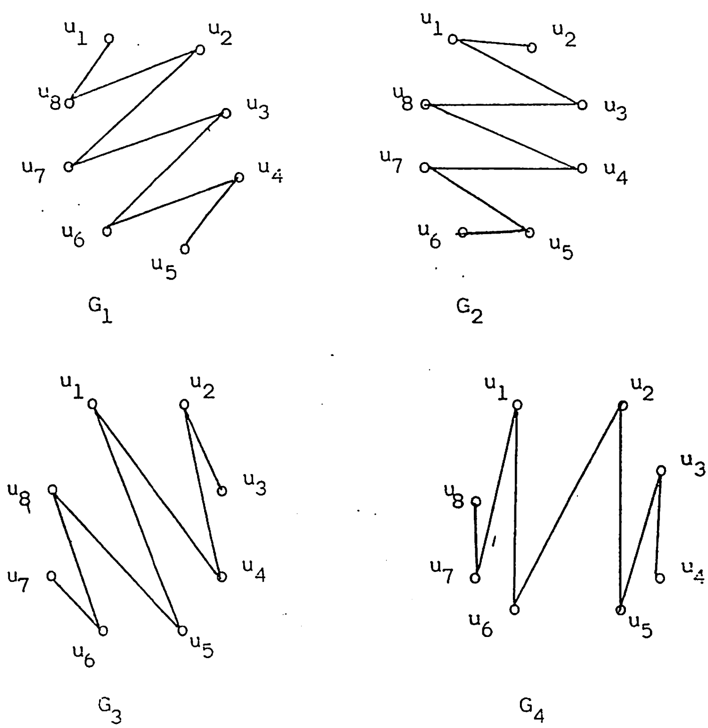


Figure 7(a)

4. Construction of an infinite class of t-sc graphs.

In this section, we give an inductive procedure for constructing infinite classes of t-sc graphs each with an scp.

Theorem 4.1. *Let* $\mathcal{G} = (G_1, G_2, \dots, G_t)$ *be t-sc with an scp* σ. *Let* $\mathcal{G} = (G'_1, G'_2, \dots, G'_t)$ *where* G'_i *is the graph with*

$$V(G'_i) = V(G_i) \cup \{w_1, w_2, \dots, w_t\}$$
$$E(G'_i) = E(G_i) \cup \{w_i, w_{i+1 \pmod t}\} \cup \left(\bigcup \{w_i u\}\right) \; u \in V(G_i)$$
$$i = 1, 2, \dots, t.$$

Then $\mathcal{G}'$ *is t-sc with* $\sigma' = \sigma(w_1 w_2 \dots w_t)$ *as an scp.*

Proof: We shall prove that for each i, σ' is an isomorphism from G'_i G'_{i+1}. Let $u, v \in V(G'_i)$.

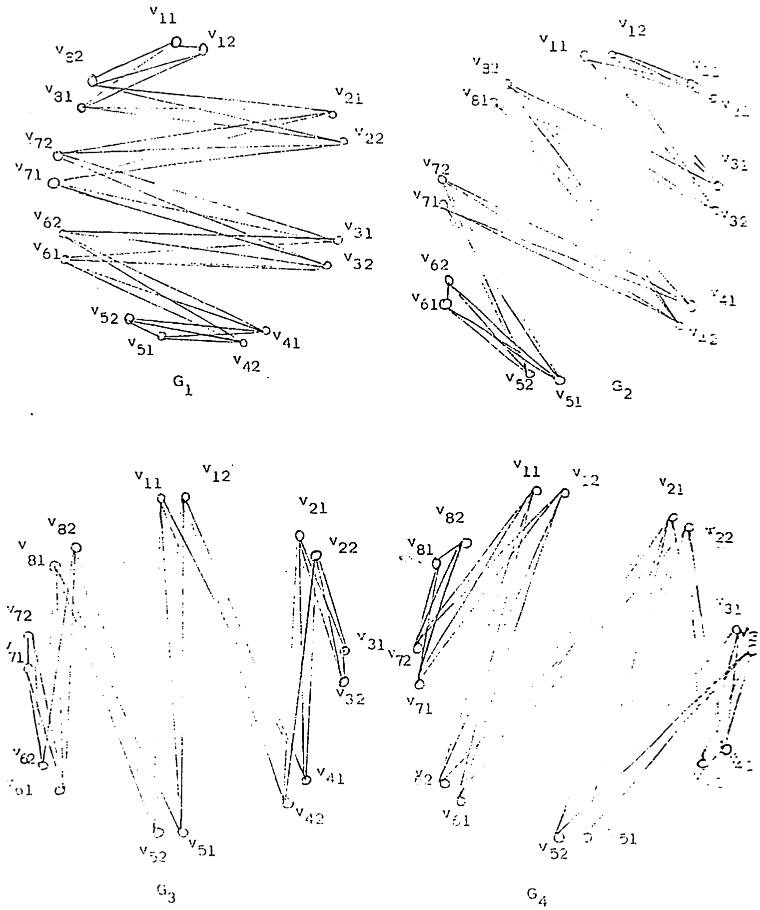


Figure 7(b)

Case 1. $u, v \in V(G_i)$. Then $\sigma'(u) = \sigma(u)$ and $\sigma'(v) = \sigma(v)$ and the result follows since σ is an isomorphism from G_i to G_{i+1} and since for all j, G_j is the subgraph induced by $V(G_j)$ in G'_j.

Case 2. $u \in V(G_i)$ and $v = w_k$. Then

$$uv \in E(G'_i) \Leftrightarrow v = w_i \Leftrightarrow \sigma(u) \in V(G_{i+1}) \text{ and } \sigma(v) = w_{i+1}$$
$$\Leftrightarrow \sigma'(u)\sigma'(v) \in E(G'_{i+1}).$$

Case 3. $u = w_j, v = w_k$. Then

$$uv \in E(G_i') \Leftrightarrow u = w_i, v = w_{i+1 \text{ (mod } t)}$$
$$\Leftrightarrow \sigma'(u) = w_{i+1 \text{ (mod } t)} \text{ and } \sigma'(v) = w_{i+2 \text{ (mod } t)}$$
$$\Leftrightarrow \sigma'(u)\sigma'(v) \in E(G_{i+1}').$$

This covers all cases and, thus, the theorem is proved. ∎

We demonstrate Theorem 4.1 in the construction shown in Figure 7.

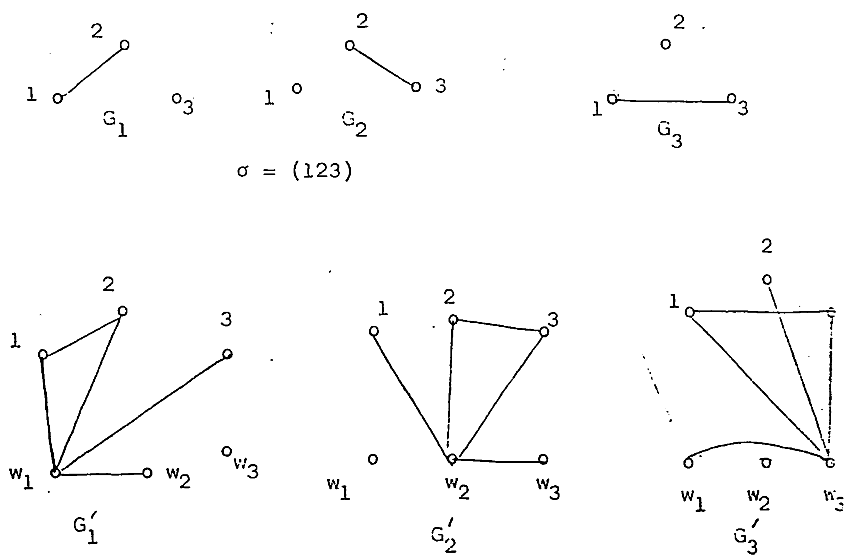


Figure 7. $\sigma' = (123)(w_1 w_2 w_3)$.

The graphs generated by Theorem 4.1 are all disconnected. Our next theorem inductively generates an infinite class of connected t-sc graphs.

Theorem 4.2. *Let $\mathcal{G} = (G_1, G_2, \ldots, G_t)$ be a connected t-sc graph with an scp σ. Then there exists a connected t-sc graph $\mathcal{G}' = (G_1', G_2', \ldots, G_t')$ where for each $i = 1, 2, \ldots, t$, $|V(G_i')| = |V(G_i)| + t$. Moreover, $\sigma' = \sigma(w_1 w_2 \ldots w_t)$ is an scp for $\mathcal{G}'$.*

Proof: By Theorem 2.7, σ may have at most one fixed point and every other cycle π has $|\pi| \equiv 0 \pmod t$. Let $\pi = (u_1, u_2, \ldots, u_{kt})$ be a cycle of σ with $|\pi| > 1$, and let (u_0) denote a possible fixed point of σ. Then we dfine the graph G_i as follows:

$$V(G_i') = V(G_i) \cup \{w_1, w_2, \dots, w_t\}$$

$$E(G_i') = E(G_i) \cup \{w_i, w_{i+1 \pmod t}\} \cup \left(\bigcup_{\substack{\pi \in \sigma \\ |\pi|>1}} \left(\bigcup_{r=0}^{k-1} \{u_{i+rt} w\} \right) \right) \cup \{u_0 w_i\}$$

where the edge $u_0 w_i$ is added to $E(G_i')$ only when σ has a fixed point, namely, 'u_0' and omitted otherwise.

We now prove that σ is an isomorphism from G_i' to G_{i+1}'. Let $u, v \in V(G_i')$.

If $u, v \in V(G_i)$ or if $u = w_j$ and $v = w_k$ this proof is similar to Case 1 and Case 3, respectively, in the proof of Theorem 4.1.

We consider the remaining cases below: Suppose $u \in V(G_i) - \{u_0\}$ and $v = w_j$. Then

$$\begin{aligned} & uV \in E(G_i') \\ & \Leftrightarrow u \in v(\pi) \text{ for some } \pi \in \sigma \text{ with } |\pi| > 1 \text{ and } v = w_i \\ & \Leftrightarrow \sigma'(u) \in V(\pi) \text{ for some } \pi \in \sigma \text{ with } |\pi| > 1 \text{ and } \sigma'(v) = w_{i+1} \\ & \Leftrightarrow \sigma'(u)\sigma'(v) \in E(G_{i+1}') \end{aligned}$$

Further if $u = u_0$ and $v = w_j$ then $\sigma'(u) = u_0$ and

$$\begin{aligned} uv \in E(G_i') \Leftrightarrow v = w_i & \Leftrightarrow \sigma'(v) = w_{i+1} \\ & \Leftrightarrow \sigma'(u)\sigma'(v) \in E(G_{i+1}') \end{aligned}$$

This covers all cases and proves our claim.

Finally, if G_i is connected and non-trivial then it is trivial to see that by construction G_i' is also connected. This proves the theorem completely. ∎

Conclusion.

The class of t-sc graphs exhibit many interesting properties. In separate papers, ([3], [4]), we construct a canonical stable complementing permutation for all t-sc graphs, and generalize a construction of Gibbs [8] for self-complementary graphs.

In conclusion, we would like to introduce the notion of a generalized factorization. A *t-factorization* of a graph G is a t-tuple $\mathcal{G} = (G_1, G_2, \dots, G_t)$ where:

i) each G_i is a spanning subgraph of G;
ii) $E(G_i) \cap E(G_j) = \emptyset \; \forall i \neq j$;
iii) $E(G) = \cup_{i=1}^{t} E(G_i)$;
iv) $G_1, G_2, \dots, G_t$, are all isomorphic.

If G is a complete graph then $\mathcal{G}$ becomes a t-sc graph. Similarly, another interesting class is obtained by taking G to be a complete r-partite graph. For such a G, the t-tuple is called a t-rpsc (t-r partite self-complementary) graph. In a separate paper [4.a] we study the properties of this class of graphs.

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