Note on the edge reconstruction of planar graphs

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Abstract. In this paper, we prove that if G is a 3-connected planar graph and contains no vertex of degree 4, then G is edge reconstructible. This generalizes a result of J. Lauri [J1].

In this paper, we will follow the notation and terminology of [BM]. All graphs $G = (V(G), E(G))$ considered will be finite and simple. The number of vertices and edges of G are denoted by $n(G)$ and $m(G)$, respectively. For $v \in V(G)$, a neighbor of v is denoted by $N(v)$. We use $d_G(v)$ to denote the degree of vertex v in G. If $d(v) = q$, we say that v is a q-vertex. The set of q-vertices is denoted by S_q. The minimum degree of graph G is denoted by $\delta(G)$. The path $C[a, b]$ is denoted by $av_1 \ldots v_t b$ and the vertices $v_1, \ldots, v_t$ are called internal vertices of $C[a, b]$. A family of paths in G is said to be internally disjoint if no vertex of G is an internal vertex of more than one path of the family. If $a = b$, $C[a, b]$ is called a circuit.

The connectivity (or vertex-connectivity) $k(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or in the trivial graph.

A set S is said to be a separating set of G if the deletion of the vertices of S from G disconnects G. We shall use the following well-known theorem:

Theorem A (Menger). *If G is a k-connected graph, then for any pair of vertices u and v of G, there are k internally disjoint paths from u to v.*

Let G be a 2-connected planar graph that is embedded in the plane E and let Γ be a circuit of G. Then Γ partitions $E - \Gamma$ into two open regions, the interior of Γ, $Int(\Gamma)$, and the exterior of Γ, $Ext(\Gamma)$, the unbounded region. If Γ is q-circuit such that $Int(\Gamma) \cap G = \phi$ (or $Ext(\Gamma) \cap G = \phi$), then $Int(\Gamma)$ (or $Ext(\Gamma)$), is called a q-face.

A graph G is said to be edge reconstructible if it can be determined uniquely, up to isomorphism, from the collection (edge-deck) $D(G) = \{G_e: G_e = G - e, e \in E(G)\}$ of edge-deleted subgraphs of G. The edge form of the reconstruction conjecture states that every graph with at least four edges is edge reconstructible. We know the following recognizing theorem.

Theorem B [F]. *A connected graph of order at least 7 and minimum degree at least 3 is planar iff every edge deleted subgraph is planar.*

In this paper, by restricting our attention to 3-connected graphs, we extend the following result by allowing the presence of 3-vertices.

Theorem C [J1]. *Every planar graph of minimum degree at least 5 is edge reconstructible.*

Our main theorem is:

Theorem. *Let G be a 3-connected planar graph such that G has no vertex of degree 4. Then G is edge reconstructible.*

In order to prove this result, we need several Lemmas.

Lemma 1. *Let G be a simple graph such that G has no $(\delta(G) + q)$-vertex and $(\delta(G) + 1)$-vertex $(q \geq 1)$. Let G have a $(\delta(G) + k)$-vertex w such that w is adjacent to at least $k - q$ vertices of degree $\delta(G)$, where $k > q$. Then G is edge reconstructible.*

Proof: We use induction on k. First assume that $k = q + 1$, then there exists a vertex w such that $d_G(w) = \delta + k$. And w is adjacent to at least $k - q = 1$ vertex v of degree δ. Let $e = wv \in E(G)$. Then $d_{G_e}(w) = \delta + k - 1 = \delta + q$ and $d_{G_e}(v) = \delta - 1$. Since G has no such vertices, and since the degree sequence of G is edge reconstructible, we can uniquely reconstruct G by adding $e = wv$.

Now suppose that $k > q + 1$. We assume that our conclusion is true for $k - 1$. Since w is adjacent to at least $k - q$ vertices of degree δ, there exists a δ-vertex v such that $e = wv \in E(G)$. Obviously, $d_{G_e}(w) = \delta + k - 1$ and $d_{G_e}(v) = \delta - 1$. We claim that G can be uniquely reconstructed from G_e by adding $e = wv$. Otherwise, let H be an edge reconstruction from G_e such that $e \notin E(H)$. Since the degree sequence of G is edge reconstructible, and G has no $\delta + 1$-vertex, it follows that w is adjacent to at least $k - 1 - q$ vertices of degree δ. By induction, H is edge reconstructible. This contradiction shows that our claim is true. This proves Lemma 1. ∎

As a consequence of Lemma 1, we have that

Corollary 1. *Let G be a planar graph. If G has no 5-vertex and $\delta(G) = 4$, then G is edge reconstructible. If G has no 4-vertex or 5-vertex, and $\delta(G) = 3$, then G is edge reconstructible.*

Proof: Since $\delta(G - S_4) \leq 5$, there is a $(4 + k)$-vertex w adjacent to at least $k - 1$ vertices of degree 4. By Lemma 1, G is edge reconstructible. This proves the first conclusion. The proof of the second one is similar. ∎

Lemma 2. *Let G be a simple graph containing no $(\delta + 1)$-vertex. Suppose that $e = uv \in E(G)$, $d_G(u) = \delta + k$, and $d_G(v) = \delta + h$ where $k \geq 2$ and $h \geq 2$.*

If u is adjacent to at least $k-2$ vertices of degree δ and v is adjacent to at least $h-2$ vertices of degree δ, then G is edge reconstructible.

Proof: Note that $d_{G_e}(u) = \delta + k - 1$ and $d_{G_e}(v) = \delta + h - 1$. We claim that, in order to reconstruct G from G_e, the only choice is to add edge $e = uv$. In fact, since G contains no $(\delta + 1)$-vertex and the degree sequence of G is edge reconstructible, it follows that the reconstruction H from G_e must be isormorphic to G. Otherwise, H has the property that at least one of the vertices u and v has degree $\delta + k'$ (where $k' = k-1$ or $h-1$) such that it is adjacent to $k'-1$ vertices of degree δ. By Lemma 1, H is edge reconstructible. This is a contradiction. The proof is finished. ∎

By Lemma 2, we have that

Corollary 2. *Let G be a planar graph containing no 4-vertex. And let $\delta(G) \geq 3$ and $\delta(G - S_3) = 5$. Suppose that there are two vertices u and v such that both u and v are 5-vertex in $G - S_3$ and $uv \in E(G - S_3)$. Then G is edge reconstructible.*

Proof: Let $d_G(u) = 3 + k$ and $d_G(v) = 3 + h$. Since both u and v are 5-vertices in $G - S_3$, it follows that u is adjacent to at least $k-2$ vertices of degree 3 and v is adjacent to at least $h-2$ vertices of degree 3 for some k and some h. Then the edge uv satisfies Lemma 2. ∎

Denote by S_5' the set of 5-vertices of $G - S_3$.

Lemma 3. *Let G be a 3-connected planar graph containing no 4-vertex. And let $S_3 \cup S_5$ be an independent set in G and S_5' be also an independent set in $G - S_3$. If $\delta(G - S_3) = 5$, then G has a vertex v satisfying the following conditions: (1) v is a 5-vertex in $G - S_3$; (2) the faces incident to v in G are either 4-face or 3-face; (3) if v is incident to a 4-face F, then the unique non-adjacent vertex must be 3-vertex in G on F.*

Proof: We prove the lemma by assuming that G does not satisfy our requirement and construct another graph $G^\star$ from G which is planar such that $\delta(G^\star - S_3) \geq 6$. Therefore, we get a contradiction.

Suppose that G is embedded in the plane so that no vertex of degree 5 in $G - S_3$ satisfies our requirement. Then each 5-vertex of $G - S_3$ occurs on some k-face F in G with $k \geq 4$.

If v is a unique 5-vertex of $G - S_3$ on F, then there is some vertex z on F to which v is not adjacent. Since S_3 is an independent set in G, we can choose z such that $d_G(z) \neq 3$. Clearly, z is not adjacent to v since G is 3-connected. But then, we can obtain a graph $G^\star$ from G in which the degree of v in $G^\star - S_3$ is 6, by joining v to z.

If F has at least 2 non-adjacent 5-vertices of $G - S_3$ on its boundary, then we can join them by one or several edges to increase their degree in $G^\star$. This can be done because S_5' is an independent set in $G - S_3$.

It is easy to see that $\delta(G^* - S_3) \geq 6$, this contradiction completes our proof. ∎

Lemma 4. *Let G be a 3-connected planar graph containing no 4-vertex. Let both $S_3 \cup S_5$ and S_5' be two independent sets and $\delta(G - S_3) = 5$. Suppose that v is a vertex of S_5' satisfying the conclusion of Lemma 3. Then there exists an edge e incident to v such that G_e is 3-connected.*

Proof: Let the faces incident to v be $F_0, F_1, \ldots, F_t$. Let the edges incident to v be $e_0 = vv_0, e_1 = vv_1, \ldots, e_t = vv_t$, such that e_i is incident to the faces F_{i-1}, F_i (modulo $t+1$). If F_i is a 4-face, then there is a 3-vertex of G on F_i which is not adjacent to v. Denote this vertex by z_i.

Without loss of generality, assume that $d_G(v_0) > 3$. If G_{e_0} is 3-connected, then we have nothing to prove. Therefore, assume that $G - e_0$ has connectivity 2, so there exists a separating pair $\{x_1, x_2\}$ in $G - e_0$ which is not separating in G. Clearly, $\{v, v_0\} \cap \{x_1, x_2\}$ is empty, and also $\{x_1, x_2\}$ separates v and v_0 in $G - e_0$, since $\{x_1, x_2\}$ is not a separating set in G. Therefore, $\{x_1, x_2\} = \{v_1, v_t\}$.

Now, let H be that component of $(G - e_0) - \{v_1, v_t\}$ which contains the vertex v_0. Clearly, v_0 can not be the only vertex of H, since its degree in G is greater than 3. Therefore, let $w \in V(H), w \neq v_0$; w and v are separated in G by $\{v_0, v_1, v_t\}$.

Since G is 3-connected, there exist in G, by Menger's Theorem, three internally disjoint paths P_1, P_2, P_3 joining w to v, and since $\{v_0, v_1, v_t\}$ separate w and v, we may assume that $v_1 \in V(P_1)$, $v_t \in V(P_2)$, $v_0 \in V(P_3)$. Also, we may assume that e_1, e_t, e_0, are edges of P_1, P_2, P_3, respectively, and that if F_0 (or F_t) is a 4-face, then $z_0 v_1$ (respectively, $v_t z_t$) is also an edge of P_1.

We shall now show that $G - e_1$ is 3-connected. First we note that v_1 can not have degree 3 in G. Otherwise F_0 would have to be a 4-face (because if not, the 3-vertices v_1 and z_0 would be adjacent); but then the edges $v_0 v_1$, $v_1 v$, $v_1 v_2$, and the path P_1 would already give that v has degree at least 4.

Now suppose that $G - e_1$ is not 3-connected. Then, as above, there exists a vertex w' such that v and w' are separated by $\{v_0, v_1, v_2\}$. Also, we can let P_1', P_2', P_3' be three internally disjoint paths from w' to v such that $v_2 \in V(P_1')$, $v_0 \in V(P_2')$, and $v_1 \in V(P_3')$. Also as above, we may assume that e_2, e_0, e_1 are edges of P_1', P_2', P_3', repectively, and that if F_0 is a 4-face, then $z_0 v_0$ is also an edge of P_2'.

Now let $Q_1 = P_1 \cup P_2 - v$ and $Q_2 = P_1' \cup P_2' - v$. Since G is planar, $V(Q_1) \cap V(Q_2)$ is not empty. Let q be the first vertex on Q_1 (as traversed in the direction from v_t to v_1) that lies also on Q_2. The vertex p can not be z_0 (if F_0 is a 4-face), since z_0 is a 3-vertex. Then let Q_1' be that part of Q_1 between v_t and q (inclusive) and let Q_2' be that part of Q_2 between q and w' (inclusive). Therefore, $Q_1' \cup Q_2' \cup \{e_t\}$ is a path joining w' to v, passing through none of the vertices v_0, v_1, v_2, a contradiction.

This completes the proof of Lemma 4. ∎

Now we are in the position to prove our theorem.

Proof of Theorem: The case that $n(G) \leq 6$ is easy. Therefore, we can assume that $n(G) \geq 7$. Let G be a 3-connected planar graph containing no 4-vertex. The planarity of G can be recognized from its edge-deck. We can assume that $S_3 \cup S_5$ is an independent set. Otherwise, the reconstructibility of G is trivial. If $\delta(G - S_3) \leq 4$, then G has a vertex w such that $d_G(w) = 3 + k$ ($k \geq 3$) and w is adjacent to at least $(k-1)$ 3-vertices. By Lemma 1 (letting $q = 1$), G is edge reconstructible. Then we may assume that $\delta(G - S_3) = 5$. By Corollary 2, we can assume that S_5' is an independent set. Now by Lemma 3, we can find a vertex v such that: (1) v is a 5-vertex of $G - S_3$; (2) v is incident to either 3-face or 4-face; (3) if v is incident to 4-face, then the unique non-adjacent vertex must be a 3-vertex on that face in G. By Lemma 4, there is an edge $e = wv$ such that G_e is 3-connected. But a 3-connected planar graph has a unique planar representation (Theorem 2.4.2 [O]). Claim that we can uniquely reconstruct G from G_e by adding wv. In fact, let $d_G(v) = 3 + k$. Then, v is adjacent to $(k-2)$ vertices of degree 3. We observe that G has no $(3 + k - 1)$-vertex adjacent to $(k-2)$ vertices of degree 3, otherwise, $\delta(G - S_3) < 5$, a contradiction. So the reconstruction H from G_e is to add an edge between v and the unique vertex w of degree bigger than 3 on the face to which v is not adjacent.

This finishes the proof of Theorem. ∎

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