On Complete Bipartite Decomposition of Complete Multigraphs

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Abstract. Let $K(n \mid t)$ denote the complete multigraph containing n vertices and exactly t edges between every pair of distinct vertices, and let $f(n;t)$ be the minimum number of complete bipartite subgraphs into which the edges of $K(n \mid t)$ can be decomposed. Pritikin [3] proved that $f(n;t) \geq \max\{n-1, t\}$, and that $f(n;2) = n$ if $n = 2,3,5$, and $f(n;2) = n-1$, otherwise. In this paper, for $t \geq 3$ using Hadamard designs, skew-Hadamard matrices and symmetric conference matrices [6], we give some complete multigraph families $K(n \mid t)$ with $f(n;t) = n-1$.

1. Introduction.

Graham and Pollak [1] proved that $n-1$ is the minimum number of complete bipartite subgraphs into which the edges of K_n can be decomposed. In [5] Tverberg gave a simple proof of that result.

Let $[i,j]$ denote the integer interval including i and j. Let $K(n \mid t)$ (or $K(A \mid t)$) denote the complete multigraph with the vertex set $[1,n]$ (or A), containing exactly t edges between every pair of distinct vertices (but containing no loops). For two disjoint subsets S,T of $[1,n]$ (or A) let $K(S,T)$ denote the complete bipartite subgraph of $K(n \mid t)$ (or $K(A \mid t)$) having partite sets S,T. Let $f(n;t)$ be the minimum number of complete bipartite subgraphs into which the edges of $K(n \mid t)$ can be decomposed. Using Tverberg's [5] technique, Pritikin [3] proved that $f(n;t) \geq \max\{n-1, t\}$, and that if $n = 2,3,5$, $f(n;2) = n$; otherwise $f(n;2) = n-1$. In this paper, for $t \geq 3$ we give some complete multigraph families $K(n \mid t)$ with $f(n;t) = n-1$. Our results are Theorems 3.2, 3.8, 3.15, 3.16, and 3.17.

For terms and notations not defined on the block design see Hughes and Piper [2].

2. Preliminaries.

We need the following result by Pritikin [3].

Lemma 2.1. *$f(n;t) \geq \max\{n-1, t\}$ for $n > 1$.*

Lemma 2.2. *If $f(n_i;t) = n_i - 1$ for $i = 1,2$, then $f(n_1 + n_2 - 1;t) = n_1 + n_2 - 2$; in particular, $f(2n_i - 1;t) = 2n_i - 2$ for $i = 1,2$.*

Proof: Let $K(n_1 \mid t)$ be decomposed into $n_1 - 1$ complete bipartite subgraphs $K(S_1,T_1), \ldots, K(S_{n_1-1}, T_{n_1-1})$. Without loss of generality, we may assume that the single vertex set $\{n_1\}$ is exactly contained in $T_1, \ldots, T_k$. Let $T = [n_1 +$

$1, n_1+n_2-1]$. Then $K(n_1+n_2-1 \mid t)$ can be decomposed into $K(T\cup\{n_1\} \mid t)$ and the following $n_1 - 1$ complete bipartite graphs:

$$K(S_1,T_1\cup T),\dots,K(S_k,T_k\cup T),K(S_{k+1},T_{k+1}),\dots,K(S_{n_1-1},T_{n_1-1}).$$

Since $|T\cup\{n_1\}| = n_2$ and $f(n_2;t) = n_2-1$, $K(T\cup\{n_1\} \mid t)$ can be decomposed into $n_2 - 1$ complete bipartite graphs. Therefore, $K(n_1 + n_2 - 1 \mid t)$ has a decomposition of n_1+n_2-2 complete bipartite graphs, that is, $f(n_1+n_2-1;t) = n_1 + n_2 - 2$ by Lemma 2.1. ∎

Corollary 2.2.1. *If $f(n_k;t) = n_k - 1$ for $k = 1,2$, then $f(i(n_1-1)+j(n_2-1)+1;t) = i(n_1-1)+j(n_2-1)$ where $i,j = 0,1,\dots$.*

Corollary 2.2.2. *If $f(n_i;t) = n_i - 1$ for $i = 1,2$, and $n_1 - 1$ and $n_2 - 1$ are relatively prime numbers, then $f(n;t) = n-1$ whenever $n \geq (n_1-2)(n_2-2)+1$.*

Proof: A well known result by Sylvester says that every $n-1 \geq (n_1-2)(n_2-2)$ is a combination, with nonnegative integral coefficients of n_1-1 and n_2-1. Now apply Corollary 2.2.1. ∎

Lemma 2.3. *If $K(n \mid t)$ can be decomposed into n complete bipartite subgraphs $K(S_1,T_1),\dots,K(S_n,T_n)$, and for any vertex of $K(n \mid t)$ there exist exactly t partite sets T_i's containing it, then $f(n+1 \mid t) = n$.*

Proof: In fact, $K(S_1\cup\{n+1\},T_1),\dots,K(S_n\cup\{n+1\},T_n)$ form a decomposition of $K(n+1 \mid t)$. ∎

3. Construction.

In this section, using Hadamard designs, skew-Hadamard matrices, and symmetric conference matrices [6], we shall give some multigraph families $K(n \mid t)$ with $f(n;t) = n-1$.

First let H be any 2-$(4k-1, 2k-1, k-1)$ Hadamard design, and let $B(H)$ be the complement design of H. Then $B(H)$ is a 2-$(4k-1, 2k, k)$ symmetric design. It is widely conjectured that Hadamard designs exist for all $k \geq 2$.

Lemma 3.1. *If a 2-$(4k-1,2k-1,k-1)$ design H exists, then $f(4k-1;2k) \leq 4k-1$.*

Proof: Let S_i be a block of H, and T_i a block of $B(H)$ such that $S_i\cap T_i = \emptyset$, where $i = 1,\dots,4k-1$. Then the complement of $K(S_i,T_i)$ in K_{4k-1} is a copy of $K_{2k-1}\cup K_{2k}$ for $i = 1,\dots,4k-1$. Since the $4k-1$ copies of $K_{2k-1}\cup K_{2k}$ form a decomposition of $K(4k-1 \mid 2k-1)$, $K(S_1,T_1),\dots,K(S_{4k-1},T_{4k-1})$ form a complete bipartite decomposition of $K(4k-1 \mid 2k)$. ∎

Theorem 3.2. *If a 2-$(4k-1, 2k-1, k-1)$ design H exists, then $f(4k; 2k) = 4k-1$, and $f(i(4k-1)+1; 2k) = i(4k-1)$ where $i = 1, 2, \ldots$.*

Proof: Following the notation in the proof of Lemma 3.1, we note that the replication number of the design $B(H)$ is $2k$, so the first result holds by Lemma 2.3. And the latter is from Corollary 2.2.1. ∎

Next we consider the skew-Hadamard matrix. The following Definition 3.3, Definition 3.4, and Lemma 3.5, are from p. 292 of [6].

Definition 3.3: A skew-Hadamard matrix H of order $h \equiv 0 \pmod 4$, has every element $+1$ or -1, and is of the form $H = S + I$ where S is skew-symmetric, $SS^T = (h-1)I$ and I is the identity matrix.

Definition 3.4: The core of a skew-Hadamard matrix H of order h is that matrix W of order $h-1$ obtained from H by first multiplying the columns so that the first row has only $+1$ elements and then multiplying the rows so every element in the first column (bar the first) is -1; then H becomes

$$\begin{bmatrix} 0 & e \\ -e^T & W \end{bmatrix} + I,$$

where $e = [1, 1, \ldots, 1]$ is a $1 \times (h-1)$ matrix.

Lemma 3.5. *If W of order $h-1$ is the core of a skew-Hadamard matrix, then W satisfies*

$$WW^T = (h-1)I - J, \quad WJ = 0, \quad W^T = -W$$

where J has every element $+1$.

We also give the following definition.

Definition 3.6: If S_i and T_i are the blocks of symmetric 2-(v, k_1, λ_1) design A and 2-(v, k_2, λ_2) design B with the same vertex set, respectively, where $i = 1, 2, \ldots, v$; and if $S_i \cap T_i = \emptyset$ for $i = 1, 2, \ldots, v$, then we call them a pair of block-disjoint designs.

For example, a symmetric design and its complement form a pair of block-disjoint designs.

Let a matrix W of order $4k-1$ $(k \geq 2)$ be the core of a skew-Hadamard matrix. It is obvious by Lemma 3.5 that $\frac{1}{2}(J + W - I)$ and $\frac{1}{2}(J - W - I)$ are incidence matrices of a pair of block-disjoint 2-$(4k-1, 2k-1, k-1)$ Hadamard designs.

Lemma 3.7. *If a skew-Hadamard matrix of order $4k$ exists, then $f(4k-1; 2k-1) \leq 4k-1$.*

Proof: Let S_i and T_i be the blocks of above two block-disjoint 2-$(4k-1, 2k-1, k-1)$ designs, respectively, where $S_i \cap T_i = \emptyset$, and $i = 1, \ldots, 4k-1$. Then the complement of $K(S_i,T_i)$ in K_{4k-1} is a copy of $K_{2k-1} \cup K_{2k-1} \cup K_{1,4k-2}$ for $i = 1,\ldots,4k-1$ where $K_{1,4k-2}$ is a star. Note that $4k-1$ copies of $K_{2k-1} \cup K_{2k-1}$ form a decomposition of $K(4k-1 \mid 2k-2)$, and $4k-1$ copies of $K_{1,4k-2}$ form that of $K(4k-1 \mid 2)$. Therefore, $K(S_1,T_1),\ldots,K(S_{4k-1},T_{4k-1})$ form a complete bipartite decomposition of $K(4k-1 \mid 2k-1)$. ∎

Theorem 3.8. *If a skew-Hadamard matrix of order* $4k (k \geq 2)$ *exists, then* $f(4k; 2k-1) = 4k-1$, *and* $f(i(4k-1)+1; 2k-1) = i(4k-1)$ *where* $i = 1,2,\ldots$.

Proof: This is analogous to that of Theorem 3.2. ∎

Now we consider the symmetric conference matrix. The following Definition 3.9, and Definition 3.10, are from p. 293 of [6].

Definition 3.9: A symmetric conference matrix N of order $n \equiv 2 \pmod 4$ has every element $+1$ or -1, and is of the form $N = R + I$ where R is symmetric, and $RR^T = (n-1)I$.

Definition 3.10: The core of a symmetric conference matrix N of order n is that matrix W of order $n-1$ obtained from N by first multiplying the rows and columns so that the first row and column has only $+1$ elements; then N becomes

$$\begin{bmatrix} 0 & e \\ e^T & W \end{bmatrix} + I,$$

where $e = [1,1,\ldots,1]$ is a $1 \times (n-1)$ matrix.

Lemma 3.11. *If* W *of order* $n-1$ *is the core of a symmetric conference matrix, then* W *satisfies*

$$WW^T = (n-1)I - J, \quad WJ = 0, \quad W^T = W.$$

Proof: See p. 306 of [6], or [4]. ∎

Definition 3.12: If a design D has the property that its blocks can be arranged in disjoint pairs so there is a vertex missing from each pair and each vertex is omitted just once from a disjoint pair, then D is called a block pair disjoint design.

By Lemma 3.11, we easily obtain the following result.

Lemma 3.13. *If* W *of order* $4k+1$ *is the core of a symmetric conference matrix, then the* $(4k+1) \times (8k+2)$ *matrix*

$$\begin{bmatrix} \frac{1}{2}(J + W - I) & \frac{1}{2}(J - W - I) \end{bmatrix}$$

is the incidence matrix of a $2\text{-}(4k+1, 2k, 2k-1)$ *design, and the design is block pair disjoint.*

We note that the $2\text{-}(4k+1, 2k, 2k-1)$ design has $4k+1$ pairs of blocks.

Example: We give a symmetric conference matrix of order 6

$$\begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 & -1 & +1 \\ +1 & +1 & +1 & -1 & +1 & -1 \\ +1 & -1 & +1 & +1 & -1 & +1 \end{bmatrix}, \text{ its core } W \text{ of order 5 } \begin{bmatrix} 0 & -1 & +1 & +1 & -1 \\ -1 & 0 & -1 & +1 & +1 \\ +1 & -1 & 0 & -1 & +1 \\ +1 & +1 & -1 & 0 & -1 \\ -1 & +1 & +1 & -1 & 0 \end{bmatrix}$$

and the matrix $\left[\frac{1}{2}(J + W - I) \quad \frac{1}{2}(J - W - I)\right]$

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Lemma 3.14. *If a symmetric conference matrix of order* $4k + 2$ *exists, then* $f(4k + 1; 2k) \leq 4k + 1$.

Proof: Let S_i and T_i be the blocks of two block sets of the block pair disjoint 2-$(4k+1, 2k, 2k-1)$ design in Lemma 3.13, where $i = 1, \ldots, 4k+1$; S_i's are decided by $\frac{1}{2}(J + W - I)$, and T_i's by $\frac{1}{2}(J - W - I)$; and $S_i \cap T_i = \emptyset$. Then using the method from the proof of Lemma 3.7, we obtain that $K(S_1, T_1), \ldots, K(S_{4k+1}, T_{4k+1})$ form a complete bipartite decomposition of $K(4k + 1 \mid 2k)$. ∎

Theorem 3.15. *If a symmetric conference matrix of order* $4k+2$ *exists, then* $f(4k+2; 2k) = 4k + 1$, *and* $f(i(4k+1)+1; 2k) = i(4k+1)$ *where* $i = 1, 2, \ldots$.

Proof: We note that in each row of the matrix

$$\left[\frac{1}{2}(J + W - I) \quad \frac{1}{2}(J - W - I)\right]$$

of Lemma 3.13 $\frac{1}{2}(J + W - I)$ has $2k$ +1's, and so does $\frac{1}{2}(J - W - I)$. Then following the notation in the proof of Lemma 3.14, we obtain that any vertex of $K(4k + 1 \mid 2k)$ is contained in exactly $2k$ T_i's. By Lemma 2.3 and Corollary 2.2.1, the result holds. ∎

Since $4k-1$ and $4k+1$ are relatively prime numbers, by Theorem 3.2, Theorem 3.15, Corollary 2.2.1, and Corollary 2.2.2, we obtain the following result:

Theorem 3.16. *If a symmetric conference matrix of order* $4k+2$ *and a Hadamard matrix of order* $4k$ *exist, then* $f(n;2k) = n-1$ *whenever* $n = i(4k-1) + j(4k+1) + 1$ $(i,j = 0,1,\ldots)$, *and, in particular, when* $n \geq 4k(4k-2)+1$.

Remarks: For some known symmetric conference matrices, Hadamard matrices and skew-Hadamard matrices, see Appendices of [6]; note that all of them are infinite families. It is strange that for some k there are no symmetric conference matrices of order $4k+2$. (See p. 295 of [6].)

On the other hand, we point out that for every t for which one has an n_0 with $f(n_0;t) = n_0 - 1$ one gets an information on $f(n;t)$ for all n. Writing n as $q(n_0-1)+r$, with $2 \leq r \leq n_0$ one gets, by Corollary 2.2.1,

$$f(n;t) \leq f((q+1)(n_0-1)+1;t) = (q+1)(n_0-1) \leq n+n_0-3.$$

(Clearly f is non-decreasing in n for fixed t.)

It can be checked that $f(n;3) > n-1$ for $n < 8$. Besides, note that $K(13 \mid 3)$ can be decomposed into the following 12 complete bipartite subgraphs:

$$\begin{array}{ll}
K(\{1,2,3\},\{4,5,6,11,12,13\}), & K(\{4,5,6\},\{7,8,9,11,12,13\}), \\
K(\{7,8,9\},\{1,2,3,11,12,13\}), & K(\{1,4,7,11\},\{2,5,8,10,13\}), \\
K(\{2,5,8,11\},\{3,6,9,10,13\}), & K(\{3,6,9,11\},\{1,4,7,10,13\}), \\
K(\{1,5,9,12\},\{2,6,7,10,11\}), & K(\{2,6,7,12\},\{3,4,8,10,11\}), \\
K(\{3,4,8,12\},\{1,5,9,10,11\}), & K(\{1,6,8,13\},\{2,4,9,10,12\}), \\
K(\{2,4,9,13\},\{3,5,7,10,12\}), & K(\{3,5,7,13\},\{1,6,8,10,12\}).
\end{array}$$

And by Theorem 3.8, $f(8;3) = 7$. Therefore, by Corollary 2.2.1 and Corollary 2.2.2, we have

Theorem 3.17. *If* $n = 7i + 12j + 1$ $(i,j = 0,1,\ldots)$, *and, in particular, if* $n \geq 67$, *then* $f(n;3) = n-1$.

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References

1. R.L. Graham and H.O. Pollak, *On embedding graphs in squashed cubes*, Springer Lecture Notes **303** (1973), 99–110.
2. D.R. Hughes and F.C. Piper, "Design Theory", Cambridge Univ. Press, London/New York, 1985.
3. D. Pritikin, *Applying a proof of Tverberg to complete bipartite decompositions of digraphs and multigraphs*, J. Graph Theory **10** (1986), 197–201.
4. R.J. Turyn, *On C-matrices of arbitrary powers*, Canad. J. Math. **23** (1971), 531–535.
5. H. Tverberg, *On the decomposition of K_n into complete bipartite graphs*, J. Graph Theory **6** (1982), 493–494.
6. W.D. Wallis, A.P. Street, and J.S. Wallis, *Combinatorics: room squares, sum-free sets, and Hadamard matrices*, in "Lecture Notes in Mathematics", Springer-Verlag, Berlin-Heidelberg-New York, 1972.