Primary Graphs

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Abstract. Primal graphs and primary graphs are defined and compared. All primary stars, paths, circuits, wheels, theta graphs, caterpillars and echinoids are found, as are all primary graphs of the form $K_{2,n}$ with $n \leqslant 927$.

1. Introduction

If G is a graph and k is a positive integer, let kG denote the union of k vertex-disjoint copies of G; we shall call it a *multiple* of G. Dewdney [3] introduced the class of primal graphs, which is defined inductively as follows: a simple graph G with no isolated vertices is *primal* if it cannot be expressed as an edge-disjoint union of distinct primal graphs different from G. The twelve primal graphs with at most seven vertices are listed in [2]. They are K_2, $2K_2$, $K_{1,2}$, $2K_{1,2}$, $K_{1,4}$, $K_{2,2}$, $K_{2,4}$, $C_5 \cup K_2$, $P \cup K_2$, $A \cup K_2$, Y and S_3 (see Figure 1), where $\cup$ denotes disjoint union.

Chinn and Lin [1] introduced another class of graphs with a similar inductive definition: a simple graph G with no isolated vertices is *primary* if it cannot be expressed as an edge-disjoint union of multiples of distinct primary graphs different from G. So a primary graph can be used more than once in the decomposition of G, provided that all its occurrences are

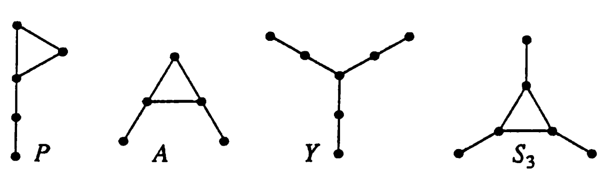


Figure 1. The graphs P, A, Y and S_3.

totally disjoint (vertex-disjoint). According to [1], the ten primary graphs with at most seven vertices are K_2, $K_{1,2}$, $K_{1,4}$, $K_{2,2} = C_4$, $K_{2,4}$, C_5, C_7, P, A and Y. We now compare and contrast these concepts.

(1.1) Primary decompositions, like primal decompositions, are not generally unique. For example, the graph consisting of C_4 and C_5 with an edge in common can be decomposed into the primary subgraphs C_5, $K_{1,2}$ and K_2, or into C_4, $K_{1,2}$ and two copies of K_2.

(1.2) It is easy to see that all primary graphs are connected. However, many primal graphs are not connected.

(1.3) It is easy to see that K_2 is the only primal graph with an odd number of edges. However, there are arbitrarily large primary graphs with an odd number of edges (see (1.5) below).

(1.4) The only primary paths are K_2 and $K_{1,2}$; every other path can be decomposed using these. This is the same result as for primal paths.

(1.5) The circuit C_n is primary if and only if n is not a multiple of 3 [1]. This is because the only primary proper subgraphs of C_n are the paths K_2 and $K_{1,2}$, and it is quite different from the result [2] that C_4 is the only primal circuit.

(1.6) It is easy to see inductively [1] that the star $K_{1,n}$ is primary if and only if n is a power of 2. This is the same result as for primal graphs [2], and it holds because n can be expressed as a sum of distinct powers of 2 smaller than n if and only if n is not itself a power of 2.

(1.7) The wheel W_4 with four vertices has a primary decomposition into C_4 and two copies of K_2. And if $n \geqslant 5$ and $2^k \leqslant n-1 < 2^{k+1}$ then W_n can be decomposed into $K_{1,2^k}$ and a graph whose primary decomposition cannot use $K_{1,2^k}$. Thus there are no primary wheels. The same argument shows also that there are no primal wheels.

(1.8) The theta graph $\theta_{a,b,c}$ consists of three internally disjoint paths of lengths a, b and c joining the same pair of vertices. If not all of a, b and c are multiples of 3, then $\theta_{a,b,c}$ is the union of a primary circuit C_n and a path whose primary decomposition cannot use C_n. If all of a, b and c are multiples of 3, then $\theta_{a,b,c}$ is the union of the primary graph Y and a graph whose primary decomposition, if it uses Y at all, cannot use it in a position where it touches the first Y. Thus there are no primary theta graphs. There are also no primal theta graphs, but that seems to be harder to prove.

(1.9) It is stated in [2] that $K_{2,n}$ is primal if and only if n is a power of 2. *Pace* [1], the analogous result does not hold for primary graphs. In Section 3 we shall determine the primary graphs of the form $K_{2,n}$ for $n \leqslant 927$.

(1.10) In the next section we shall characterize the primary caterpillars and echinoids; there are only finitely many of them apart from the infinite families mentioned in (1.6) and (1.5). The corresponding problem for primal graphs seems to be harder and is still open.

We are indebted to the referee for drawing our attention to references [1] and [4]. Theorem 1 below (the characterization of primary caterpillars) was given with a different proof in [1], and Theorem 3 below corrects an error in [1].

2. Caterpillars and echinoids

A *caterpillar* is a graph such that the removal of all end-vertices leaves a path; equivalently, it is a tree that does not contain Y (Figure 1) as a subgraph. We propose to use the term *echinoid* for a graph such that the removal of all end-vertices (if any) leaves a circuit. The echinoids include the suns from [2].

A vertex in a caterpillar or echinoid will be called a *junction* if it has degree at least 3. Two junctions in a caterpillar are *consecutive* if the unique path connecting them contains no other junction. An *end-path* in a caterpillar C is a segment P of a longest path in C such that P connects an end-vertex of C to the nearest junction. A 0(3)-*path* is a path whose length is congruent to 0 (modulo 3). We shall sometimes find it convenient to represent parts of caterpillars and echinoids by diagrams such as ⊢—○—●—△—●—, in which the symbols ⊢, ○, △ and ● denote vertices with, respectively, degree 1, degree 2, degree at least 3 (a junction) and unspecified degree.

We shall now determine the primary caterpillars and echinoids.

Theorem 1. *A caterpillar is primary if and only if it is a primary star* $K_{1,2^k}$ *or one of the four graphs* H_1, H_2, H_3 *and* H_4 *shown in Figure* 2.

Proof. We start with some observations that will dispose of all caterpillars that do not have H_1 as a subgraph, and we then use a general argument for the caterpillars that contain H_1.

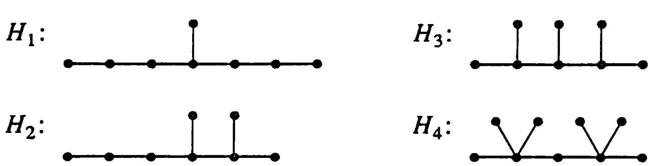
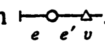


Figure 2. Four primary graphs.

(2.1) A primary caterpillar cannot have an end-path of length greater than 3. For, if it does, remove the end three edges of this path, and observe that a primary decomposition of what is left can easily be extended to these three edges using K_2 and $K_{1,2}$.

(2.2) A primary caterpillar G cannot have an end-path of length 2, as in the diagram . For, in a primary decomposition of $G-e$, it cannot be essential for e' to be covered by K_2 (since there is at least one other end-edge at v, which can be interchanged with e'), and so e can be covered by K_2.

(2.3) A similar argument shows that two consecutive junctions in a primary caterpillar G cannot be connected by a 0(3)-path. For, if they were, then there would be a primary decomposition of G in which the second edge of this path was covered by K_2.

(2.4) *Stars and double stars.* Primary stars were classified in (1.6). By a *double star* we mean a caterpillar G of diameter 3 with two junctions, with degrees $a \geqslant b$, say. The decomposition of $K_{1,a}$ into primary stars uses at least one primary star that is not used in the decomposition of $K_{1,b-1}$; thus we can join these two decomposed stars so as to form a primary decomposition of G. Hence there are no primary double stars.

(2.5) *Caterpillars with one junction.* By (2.1) and (2.2), a primary caterpillar G of this type is a star or a star with one or two paths of length 2 adjoined at end-vertices. It is easy to see that if G is not a star then G has a primary decomposition unless two paths of length 2 are adjoined to $K_{1,3}$, which gives the primary graph H_1.

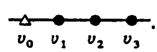
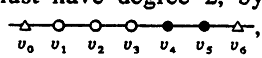
(2.6) *A double star with a path attached.* By (2.1) and (2.2), a primary caterpillar G of this type consists of a double star with a path of length 2 adjoined at an end-vertex. If both junctions have degree 3 then we have the primary graph H_2; if not, then G has a primary decomposition using H_2.

(2.7) *Two stars with an end-vertex in common.* Such a graph G is easily seen to be non-primary unless both stars are of the form $K_{1,2^k}$ for the same integer k; $k < 2$ gives no problem, $k = 2$ gives the primary graph H_4, and if $k > 2$ then G has a primary decomposition using H_4.

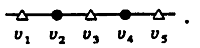
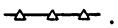
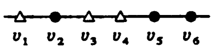
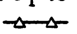
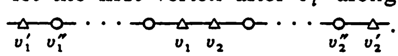
(2.8) *Caterpillars of diameter 4.* The case where there are fewer than three junctions has been dealt with in (1.4), (2.5), (2.6) and (2.7). If there are three junctions with degree 3 then we have the primary graph H_3; otherwise there is a primary decomposition using H_3.

(2.9) *Two stars connected by a path.* In view of (1.4), (2.4), (2.5) and (2.7), we may suppose that there are two junctions connected by a path of length $l \geqslant 3$. It is not difficult to see that a primary decomposition of the central $l-2$ edges of this path can be extended to a primary decomposition of the whole graph. (This is easy unless one of the junctions has degree 3. But then at least one of the three ways of decomposing its incident edges into a K_2 and a $K_{1,2}$ must work.)

(2.10) *Caterpillars not containing H_1 as a subgraph.* In view of (2.2), a primary caterpillar G of this type is a path with a star, a double star or nothing at each end. In view of (1.4) and (2.4)–(2.9), it remains only to consider the case when there is a double star at one end and a star or double star at the other. Remove H_2 from G. The primary decomposition of what is left cannot use H_2 in a position where it touches the first H_2, by (2.3) and the absence of H_1 as a subgraph. Thus there are no new primary caterpillars of this type.

It remains to consider the case where G is a caterpillar having H_1 as a proper subgraph. We shall show that G is not primary. Suppose it is. Draw G with a longest path horizontal, and let H denote the leftmost copy of H_1 in G with six horizontal edges. Then a primary decomposition of $G-H$ must use a copy H' of H_1 on the right of H and touching H. Note that there is no H_2 to the left of H, by (2.3) and the fact that H is the leftmost copy of H_1 in G. Note also that a caterpillar of diameter at most 3, which we shall refer to as a *PDDS* (for *possibly degenerate double star*), certainly does not use H_1 or H_2 in its primary decomposition. Thus we can deal with the juxtaposition of H and H' as follows. Let v_0 be the central vertex of H, so that part of G has diagram . Then v_1 must have degree 2, since otherwise we could replace H by a copy of H_2 and a PDDS containing the edge v_2v_3. Also, v_2 must have degree 2, since otherwise we could turn up the rightmost edge of H at v_2 so that H is separated from H' by a PDDS. Now v_3 must have degree 2, by (2.3). Thus the diagram of G continues in the form , where v_6 is the central vertex of H'. Now v_4 must have degree 2 for the same reason as v_2. And if v_5 did not have degree 2, then we could replace H' by a copy of H_2 and a K_2 covering the edge v_3v_4, unless this copy of H_2 touched another to its right, in which case we could use instead a copy of H_1 centred on v_5 with a PDDS separating it from the copy of H_2 on its right, leaving on its left a graph that would not contain H_1. Thus v_5 must have degree 2. Now v_0 and v_6 violate (2.3). This contradiction completes the proof of Theorem 1. □

Theorem 2. *An echinoid is primary if and only if it is a primary circuit or the graph A in Figure* 1.

Proof. Let G be a primary echinoid that is not a circuit but contains the circuit C_n. Evidently C_n is not primary, so n is a multiple of 3. The cases $n = 3$ and $n = 6$ are left to the reader; we suppose $n \geq 9$. We now list the diagrams of various configurations that cannot occur within G.
(A)  . For, if this occurs within G, let H be a copy of H_1 centred on v_3 with six edges in C_n, and consider a primary decomposition of $G-H$. If this uses one or two copies of H_1 touching H on the left or right or both, then we can turn up the leftmost or rightmost edge of H at v_1 or v_3 or both, creating one or two PDDSs, to obtain a primary decomposition of G with no two copies of H_1 touching.
(B)  . For, if this occurs, then G has a primary decomposition using H_3. (Note that, by (A), G cannot contain two touching copies of H_3.)
(C)  . If this occurs, let H be a copy of H_1 centred on v_3 with six edges in C_n, and consider a primary decomposition of $G-H$. If this uses a copy H' of H_1 touching H on the left of H, then we can turn up the leftmost edge of H at v_1, thereby creating a PDDS which separates H from H'. If there is a copy of H_1 touching H on the right of H, replace H by a copy H'' of H_2 and a PDDS that contains the edge v_5v_6. This gives a primary decomposition of G unless there is a copy of H_2 touching H'' on the left of H'', in which case we can turn up the leftmost edge of H'' at v_1 to create a PDDS and avoid the contact.
(D)  . If this occurs, let the junctions closest to v_1 and v_2 on the left and right be v_1' and v_2' respectively, let the distance between v_i and v_i' along the junction-free path be l_i, and let the first vertex after v_i' along this path towards v_i be v_i'' $(i = 1, 2)$: .
Then $l_i \geq 3$ by (B) and (C), and $l_i \equiv 1$ or $2 \pmod 3$ by the argument of (2.3). If $l_1 \equiv l_2 \equiv 1 \pmod 3$, then use a copy of H_1 centred on v_2, a copy of K_2 and (if $l_1 > 4$ or $l_2 > 4$) an equal number of additional copies of K_2 and $K_{1,2}$, to decompose the segment from v_1'' to v_2'' in such a way that v_1'' is incident with a copy of K_2 and v_2'' is incident with a copy of K_2 or H_1. In a primary decomposition of the rest of G, it is not necessary for the edge $v_1'v_1''$ to be covered by K_2, nor for the edge $v_2'v_2''$ to be covered by K_2 or H_1 (as appropriate), by arguments that have been used several times already, and so there exists a primary decomposition of G. If $l_1 \equiv 1$ and $l_2 \equiv 2 \pmod 3$, or *vice versa*, then we use the same argument with an additional copy of K_2. And if $l_1 \equiv l_2 \equiv 2 \pmod 3$ then we use H_2, $K_{1,2}$ and two copies of K_2 (plus an equal number of additional copies of K_2 and $K_{1,2}$ if necessary) in an exactly similar way.

We can now complete the argument as follows. Let H be a copy of H_1 in G with six edges in C_n, and consider a primary decomposition of $G-H$. This gives a primary decomposition of G unless it uses a copy H' of H_1 that touches H. Suppose that H and H' are centred on v and v'. By (A), (D) and the argument of (2.3), v and v' are separated by a path of six edges that passes through exactly one other junction v'', which is at distance 2 from v or v'. Thus we can avoid the contact between H and H' by turning up the end edge of H or H' at v'', creating a PDDS. We can do the same independently if there is a copy of H_1 touching H on the other side of H, and so we obtain a primary decomposition of G. □

3. The graphs (a, b, c).

Let (a, b, c) denote the union of $K_{1,a+b}$ and $K_{1,b+c}$ with an independent set of b vertices in common, labelled as in Figure 3. Then $(0, b, 0) = K_{2,b}$, and every connected subgraph of $K_{2,n}$ is either a star or a graph (a,b,c) for some a, b and c $(b \geqslant 1)$. This section is devoted to a proof of the following theorem.

Theorem 3. *The primary graphs* (a, b, c) *with* $1 \leqslant b \leqslant 927$ *are the following:*

(a) $K_{2,1}$, $K_{2,2}$, $K_{2,4}$, $K_{2,8}$ *and* $(3, 1, 3) = H_4$ *(Figure 2),*

(b) *the fourteen graphs* $(13, 37, 13)$, $(12, 38, 12)$, $\ldots$, $(1, 49, 1)$, $(0, 50, 0)$ $= K_{2,50}$, *each with* 100 *edges, and*

(c) $(0, 100, 0) = K_{2,100}$.

We conjecture that in fact there are no more primary graphs (a, b, c) with $b < 1637$, and that the 206 graphs $(205, 1637, 205)$, $(204, 1638, 204)$, $\ldots$, $(0, 1842, 0) = K_{2,1842}$ are all primary.

We show first that each of the graphs in Theorem 3 is either primary or contains a primary subgraph that is not listed there. For the graphs in (a), this is easy and is left to the reader. For a graph G in (b), note that the largest independent set of vertices in G has cardinality at most 63, and so

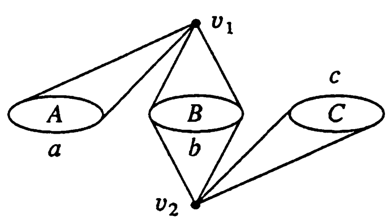


Figure 3. The graph (a, b, c).

in a primary decomposition of G not both of v_1 and v_2 can be contained in a copy of $K_{1,32}$. Suppose v_1 is not. Then every edge incident with v_1 is covered by one of the graphs in (a) or one of the stars $K_{1,1} = K_2$, $K_{1,2} = K_{2,1}$, $K_{1,4}$, $K_{1,8}$ and $K_{1,16}$. But these graphs can cover at most 49 edges at v_1, whereas v_1 has degree 50 in G. Thus either G is primary or its primary decomposition uses a primary graph that we do not yet know about. Since, for each graph G in (b), $K_{2,100} - G \cong G$, a primary decomposition of $K_{2,100}$ cannot use any of the primary graphs in (b), and so a similar argument works for $K_{2,100}$. A similar argument also shows that each graph G in $\{(205, 1637, 205), \ldots, (0, 1842, 0) = K_{2,1842}\}$ is either primary or contains a new primary subgraph. For, the largest independent set in G has cardinality at most 2047, so v_1 (say) is not covered by $K_{1,1024}$; thus the primary graphs available to cover v_1 are the graphs in Theorem 3 and the primary stars $K_{1,1}, \ldots, K_{1,512}$, which can cover at most 1841 edges at v_1, whereas v_1 has degree 1842 in G.

It remains to prove that there are no primary graphs (a, b, c) with $b \leqslant 927$ other than those listed in Theorem 3. The following lemma will help with this.

Lemma 3.1. *The graph (a, b, c) has a primary decomposition into primary stars unless $a+b = 2^k+x$, $b+c = 2^k+y$ and $x+y < b$, for some non-negative integers k, x and y.*

Proof. Write $a+b = 2^k+x$ and $b+c = 2^l+y$, where k and l are as large as possible subject to k, l, x and y being non-negative integers. If $k \neq l$, w.l.o.g. $k > l$, use $K_{1,2^k}$ to cover 2^k edges from v_1 including all edges from v_1 to B; this is possible because $2^l > \frac{1}{2}(b+c) \geqslant \frac{1}{2}b$ and $2^k \geqslant 2^{l+1} \geqslant b$. The remaining edges can now easily be covered by primary stars. If $k = l$ and $x+y \geqslant b$, partition B into sets B_1 and B_2 satisfying $b-x \leqslant |B_1| \leqslant y$ and $b-y \leqslant |B_2| \leqslant x$, which is clearly possible, and use two copies of $K_{1,2^k}$, one to cover 2^k edges from v_1 including all edges from v_1 to B_1 and none from v_1 to B_2 (which is possible since $|B_1| \leqslant y < 2^l = 2^k = a+b-x \leqslant a+|B_1|$) and one to cover 2^k edges from v_2 including all edges from v_2 to B_2 and none from v_2 to B_1. As before, the remaining edges can now easily be covered. □

We now complete the proof of Theorem 3 in seven cases, which are exhaustive though not exclusive. Let $G = (a, b, c)$.

Case 1: $b = 1$. This case was dealt with in (2.7).

Case 2: $2 \leqslant b \leqslant 15$. The reader is assumed to have verified that $K_{2,2}$ is primary and $K_{2,4}$ and $K_{2,8}$ have no primary decomposition into primary graphs that we already know about (that is, the primary stars and the graphs in Theorem 3(a)). If $2 \leqslant b \leqslant 3$ and $G \neq K_{2,2}$, then $G - K_{2,2}$ does not contain $K_{2,2}$, and so a primary decomposition of it can be extended to G by using $K_{2,2}$; thus G is not primary. It follows that $K_{2,4}$ has no new primary subgraphs, and so it is primary. If $4 \leqslant b \leqslant 7$ and $G \neq K_{2,4}$, then $G-K_{2,4}$ does not contain $K_{2,4}$; thus G is not primary. It follows that $K_{2,8}$ is primary. If $8 \leqslant b \leqslant 15$ and $G \neq K_{2,8}$, then $G-K_{2,8}$ does not contain $K_{2,8}$; thus G is not primary.

Case 3: $7 \leqslant b \leqslant 21$, b odd. Remove H_4 and some of $K_{2,2}$, $K_{2,4}$ and $K_{2,8}$ from G to leave $(a+3,0,c+3)$, which does not contain any of the graphs removed. Thus G is not primary.

Case 4: $10 \leqslant b \leqslant 22$, b even. Remove H_4 and some of $K_{2,2}$, $K_{2,4}$ and $K_{2,8}$ from G to leave $(a+3,1,c+3)$, which can be decomposed into primary stars by Lemma 3.1 unless $a+4 = c+4 = 2^k$ for some k. In this last case, remove H_4, $K_{2,1}$ and some of $K_{2,2}$, $K_{2,4}$ and $K_{2,8}$ from G to leave $(a+3,2,c+3)$, which can be decomposed using $K_{1,a+4}$ twice and $K_{1,1}$ twice.

Case 5: $21 \leqslant b \leqslant 35$. Remove H_4, $K_{2,2}$, $K_{2,4}$ and $K_{2,8}$ from G to leave $(a+3,b-21,c+3)$, which can be decomposed by Lemma 3.1 unless

$$a+b-18 = 2^k+x, \qquad b+c-18 = 2^k+y \quad \text{and} \quad x+y < b-21. \tag{1}$$

In this last case, let d be the unique even integer satisfying

$$b-21-x-y \leqslant d \leqslant b-20-x-y \tag{2}$$

(note $0 < d \leqslant 35-20 = 15$, so $2 \leqslant d \leqslant 14$), and remove H_4 and some of $K_{2,2}$, $K_{2,4}$ and $K_{2,8}$ from G to leave $G' = (a+3,b-21+d,c+3)$. Now

$$a+b-18+d = 2^k+x', \quad b+c-18+d = 2^k+y' \quad \text{and} \quad x'+y' \geqslant b-21+d$$

by (2), since $x'+y' = x+y+2d$. Thus we can decompose G' into primary stars by Lemma 3.1 unless *both* $a+b-18+d \geqslant 2^{k+1}$ *and* $b+c-18+d \geqslant 2^{k+1}$. But this is impossible, since it implies $x' \geqslant 2^k$, $y' \geqslant 2^k$ and

$$2^{k+1}+x+y \leqslant x'+y'+x+y = 2(x+y+d) \leqslant 2b-40$$

by (2), whereas (1) gives

$$2^{k+1}+x+y = (a+b-18)+(b+c-18) \geqslant 2b-36.$$

Case 6: $36 \leqslant b \leqslant 50$. The argument of Case 5 works as long as $b-20-x-y \leqslant 15$, $x+y \geqslant b-35$. So we may suppose that

$$a+b-18 = 2^k+x, \qquad b+c-18 = 2^k+y \quad \text{and} \quad x+y \leqslant b-36. \tag{3}$$

If $b = 36$ then $x = y = 0$ and $18 \leqslant a+b-18 = b+c-18 = 2^k \geqslant 32$; thus

$$a+b = b+c = 2^k+18 < 2^{k+1} \quad \text{and} \quad 18+18 = b,$$

whence G has a primary decomposition by Lemma 3.1.

If $37 \leqslant b \leqslant 50$ and $\min(a, c) < 50-b$, say $a < 50-b$, then (3) gives

$$2^k \leqslant a+b-18 < 32,$$

so $2^k \leqslant 16$ and $x = a+b-18-2^k \geqslant b-34$. Thus G has a primary decomposition since (3) does not hold. It follows that none of the graphs in Theorem 3(b) has a primary subgraph that we did not already know about, and so all these graphs are primary. Moreover, there are no other primary graphs with $37 \leqslant b \leqslant 50$, since if $\min(a, c) \geqslant 50-b$ then G has the primary graph $(50-b, b, 50-b)$ as a subgraph.

Case 7: $51 \leqslant b \leqslant 927$. If $51 \leqslant b \leqslant 99$, then $G-K_{2,50}$ does not contain $K_{2,50}$, and so G is not primary. Thus $K_{2,100}$ does not have any primary subgraphs that we did not already know about, and so it is primary. If $100 \leqslant b \leqslant 199$ and $G \neq K_{2,100}$, then $G-K_{2,100}$ does not contain $K_{2,100}$, and so G is not primary. Note that $K_{2,200}$ is the edge-disjoint union of the four graphs $(3, 47, 3)$, $(2, 48, 2)$, $(1, 49, 1)$ and $(0, 50, 0) = K_{2,50}$, and $K_{2,206}$ is the edge-disjoint union of these four graphs and two stars, and so if $200 \leqslant b \leqslant 206+46$ then we can remove these four graphs from G so as to leave a graph that does not contain any of them. If $250 \leqslant b \leqslant 256+46$ then we can do the same by using $K_{2,100}$ in place of $K_{2,50}$. It is now easy to see that if

$$b \leqslant 50+51+\ldots+63+100+36 = 927$$

then we can remove from G some or all of the primary graphs in Theorem 3(b) and (c) so as to leave a graph (a', b', c') with $b' \leqslant 36$, which cannot contain any of the graphs removed. Thus the next primary graph in the family must have $b \geqslant 928$. This completes the proof of Theorem 3. □

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