On locally n-(arc)-strong digraphs*

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ABSTRACT. In 1974, G. Chartrand and R.E. Pippert first defined locally connected and locally n-connected graphs and obtained some interesting results. In this paper we first extend these concepts to digraphs. We obtain generalizations of some results of Chartrand and Pippert and establish relationships between local connectedness and global connectedness in digraphs.

1 Introduction

The connectedness in graphs and digraphs, one of the most important properties that a graph or digraph can possess, has been extensively studied (see, for instance, the surveys [1] and [5]). In 1974, G. Chartrand and R.E. Pippert [3] first defined locally connected and locally n-connected graphs and obtained some interesting results, among which are the following:

Theorem A. *Every connected and locally n-connected graph ($n \geq 1$) is $(n+1)$-connected.*

Theorem B. *Let G be a graph of order p such that for every pair x, y of vertices*

$$\deg x + \deg y > \frac{4}{3}(p-1).$$

Then G is locally connected.

Theorem C. *Let G be a graph of order p such that for every pair x, y of vertices*

$$\deg x + \deg y > \frac{4}{3}(p + \frac{n-3}{2}), \text{ where } 1 \leq n \leq p-2.$$

*This research is partially supported by the RDG grant of the Penn State University

Then G is locally n-connected.

Following [3], a variety of research [6-11] has been devoted to locally connected graphs. In the present paper we first extend the concept of local connectedness to digraphs. We obtain generalizations of the above theorems and establish relationships between local connectedness and global connectedness in digraphs.

2 Definitions

We follow the standard terminology and notation. A digraph $D = (V(D), A(D))$ is a finite nonempty set $V(D)$ of vertices together with a (possibly empty) set $A(D)$ of ordered pairs of distinct vertices of D called arcs. An ordered pair $(u, v) \in A(D)$ is also called an arc from u to v. A digraph D is said to be weakly connected if its underlying undirected graph is connected. If there is a dipath from u to v for any pair u and v of vertices in D, then the digraph D is said to be strongly connected, or simply said to be strong. The subdigraph induced by a nonempty subset $W \subset V(D)$ is denoted $\langle W \rangle_D$. Let $u, v \in V(D)$. We say u is a neighbor of v if $(u, v) \in A(D)$ or $(v, u) \in A(D)$. The set of neighbors of v in D is denoted $N_D(V)$. The induced subdigraph $\langle N_D(V) \rangle_D$ is said to be the neighborhood of v. The outdegree of v is denoted as *od* v and the indegree of v is denoted as *id* v. Let S and T be two disjoint proper subsets of $V(D)$. We use $(S,T)_D$ to denote the set of arcs (s,t) in D with $s \in S$ and $t \in T$. When there is no confusion, we may simply use $\langle W \rangle$, $\langle N(v) \rangle$ and (S,T) to denote the corresponding $\langle W \rangle_D$, $\langle N_D(v) \rangle_D$ and $(S,T)_D$, respectively.

Recall the following definitions from [3]:

A graph G is locally connected if the neighborhood of every vertex of G is connected (where the neighborhood of a vertex v is the subgraph induced by all the vertices in G adjacent with v).

A graph G is locally n-connected if the neighborhood of every vertex of G is n-connected.

Now let us extend these definitions to digraphs.

Let $n \geq 1$. A digraph D is said to be n-strong [n-arc-strong, resp.] if the removal of fewer than n vertices [arcs, resp.] always results in a nontrivial strong digraph. Clearly, every n-strong digraph is n-arc-strong. Every n-strong [n-arc-strong, resp.] digraph is also m-strong [m-arc-strong, resp.] for $1 \leq m < n$. It should also be noted that D is 1-strong iff D is 1-arc-strong iff D is a nontrivial strong digraph. The trivial strong digraph consisting of a single vertex is the only digraph that is strong but not 1-strong (or not 1-arc-strong).

We define D to be locally strong [locally n-strong, locally n-arc-strong, resp.] if the neighborhood of every vertex of D is strong [n-strong, n-arc-strong, resp.].

For other terminologies not defined here we refer the reader to the book [2].

3 Main results

Theorem 1. *Any weakly connected and locally n-arc-strong digraph is $(n+1)$-arc-strong.*

To prove Theorem 1, we need the following lemma.

Lemma 1. *A nontrivial digraph D is n-arc-strong if and only if $|(S,\overline{S})_D| \geq n$ for every nonempty proper subset S of $V(D)$ (where $\overline{S} = V(D) - S$).*

The proof of Lemma 1 is easy and so omitted.

Now the proof of Theorem 1 goes as follows.

Proof of Theorem 1: Suppose D is weakly connected and locally n-arc-strong but not $(n+1)$-arc-strong. Then by Lemma 1, there is a nonempty proper subset S of $V(D)$ such that $|(S,\overline{S})| \leq n$. We distinguish the following two cases.

Case 1 $0 < |(S,\overline{S})| \leq n$. Let $(x,y) \in (S,\overline{S})$. Then $N(x) \cap \overline{S} \neq \emptyset$ and $N(y) \cap S \neq \emptyset$. We claim that $N(x) \cap S = \emptyset$. If not, then by Lemma 1, $|(N(x) \cap S, N(x) \cap \overline{S})| \geq n$ since $\langle N(x) \rangle$ is n-arc-strong. Note that $(S,\overline{S}) \supseteq (N(x)\cap S, N(x)\cap\overline{S}) \cup (x,y)$ and that $(x,y) \notin (N(x)\cap S, N(x)\cap\overline{S})$. Then we have $|(S,\overline{S})| \geq n+1$, which is a contradiction. Thus we have shown $N(x) \cap S = \emptyset$, i.e., $N(x) \subseteq \overline{S}$. Similarly, we can show that $N(y) \cap \overline{S} = \emptyset$, i.e., $N(y) \subseteq \overline{S}$. Thus we have $N(x) \cap N(y) = \emptyset$.

Since $y \in N(x)$ and $\langle N(x) \rangle$ is n-arc-strong, we have $N(x) - y \neq \emptyset$ and $(y, N(x) - y) \neq 0$. This implies that $N(y) \cap N(x) \neq \emptyset$, a contradiction.

Case 2 $|(S,\overline{S})| = 0$. Since D is weakly connected, there exists an arc $(y,x) \in (\overline{S},S)$. Then, as in Case 1, $N(x) \cap \overline{S} \neq \emptyset$ and $N(y) \cap S \neq \emptyset$. We also have $N(x) \cap S = \emptyset$, i.e., $N(x) \subseteq \overline{S}$. (Otherwise, since $\langle N(x) \rangle$ is n-arc-strong, $|(N(x) \cap S, N(x) \cap S)| \geq n$ by Lemma 1. This contradicts the condition $|(S,\overline{S})| = 0$.). Similarly, we have $N(y) \cap \overline{S} = \emptyset$, i.e., $N(y) \subseteq S$. Then, $N(x) \cap N(y) = \emptyset$ and the same argument as in Case 1 also leads to a contradiction.

This completes the proof of Theorem 1. □

Theorem 2. *Any weakly connected and locally n-strong digraph is $(n+1)$-strong.*

Proof: Suppose D is weakly connected and locally n-strong but not $(n+1)$-strong. Then there exists a minimal set T of k $(\geq n)$ vertices of D such that $D - T$ is not strong or $D - T$ is a single vertex. If $D - T$ is a single vertex, then D has only $k+1$ vertices, implying that the neighborhood of a vertex of D has at most $k \leq n$ vertices and that, consequently, no neighborhood is n-strong. Thus, $D-T$ is not strong. Let $D_1 = D - T$. Then certainly D_1 is not 1-arc-strong. By Lemma 1 there is a nonempty proper subset S of $V(D_1)$ such that $(S, V(D_1) - S)_{D_1} = \emptyset$. Note that D is strong by Theorem 1. Hence T is not empty. Let $v \in T$, $S_1 = N_D(V) \cap S$ and $\overline{S}_1 = N_D(V) \cap (V(D_1) - S)$. By the choice of T, T is a minimal set of D such that $D - T$ is not strong. So we must have $S_1 \neq \emptyset$ and $\overline{S}_1 \neq \emptyset$. Notice that

$$N_D(V) - N_D(V) \cap T = S_1 \cup \overline{S}_1, (S_1, \overline{S}_1)_{\langle N_D(v)\rangle} \subseteq (S, V(D_1) - S)_{D_1} = \emptyset,$$

and $|N_D(v) \cap T| \leq |T - V| \leq k - 1 \leq n - 1$. Then $\langle N_D(v)\rangle$ is not n-strong. This contradicts the assumption that D is locally n-strong. □

Theorem 3. *Let D be a digraph of order $p \geq 2$ such that for every pair x, y of distinct vertices*

$$od\ x + id\ y > \frac{4}{3}(p-1).$$

Then D is locally strong.

Proof: It is obvious for $p = 2$. Now we assume that $p \geq 3$.

Suppose D satisfies the hypothesis of the theorem but D is not locally strong. Then there is a vertex v such that $\langle N(v)\rangle$ is not strong. Clearly, $\langle N(v)\rangle$ is a nontrivial digraph which is not 1-arc-strong. By Lemma 1, there is a nonempty proper subset S of $N(v)$ such that there is no arc coming out from S to $N(v) - S$ in $\langle N(v)\rangle$. Let $u \in S$, $w \in N(v) - S$, and let $a = |S|$ and $b = |N(V) - S|$. Then it is easily seen that $od\ v + id\ v \leq 2(a+b)$, $od\ u \leq p - (b+1)$ and $id\ w \leq p - (a+1)$. Thus, $(od\ v + id\ v) + 2od\ u + 2id\ w \leq 4(p-1)$. That is, $(od\ v + id\ w) + (od\ u + id\ w) + (od\ u + id\ v) \leq 4(p-1)$. This contradicts the given inequality in the hypothesis. □

Similarly, we can prove the following

Theorem 4. *Let D be a digraph of order $p \geq 3$ such that for every pair x, y of distinct vertices*

$$od\ x + id\ y > \frac{4}{3}\left(p + \frac{n-3}{2}\right), \quad \text{where } 1 \leq n \leq p-2.$$

Then D is locally n-strong.

It is easy to see that Theorems A, B and C follow from Theorems 2, 3 and 4, respectively. Also, we may obtain the following corollary from Theorem 1, which is parallel to Theorem A.

Corollary 1. *Every connected and locally n-edge-connected graph ($n \geq 1$) is $(n+1)$-edge-connected.*

Acknowledgement

I would like to thank Professor Gary Chartrand for kindly sending me a copy of his joint paper [3] with Professor Raymond E. Pippert which inspired this work. I am also grateful to Professors C.Y. Chao, Fuji Zhang and a referee for their helpful comments and suggestions.

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