The Linear 2-arboricity of Complete Bipartite Graphs *

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ABSTRACT. A forest in which every component is path is called a path forest. A family of path forests whose edge sets form a partition of the edge set of a graph G is called a path decomposition of a graph G. The minimum number of path forests in a path decomposition of a graph G is the *linear arboricity* of G and denoted by $\ell(G)$. If we restrict the number of edges in each path to be at most k then we obtain a special decomposition. The minimum number of path forests in this type of decomposition is called the *linear k-arboricity* and denoted by $\ell a_k(G)$. In this paper we concentrate on the special type of path decomposition and we obtain the answers for $\ell a_2(G)$ when G is $K_{n,n}$. We note here that if we restrict the size to be one, the number $\ell a_1(G)$ is just the chromatic index of G.

1 Introduction

A path decomposition is a special case of an edge decomposition and is the type of decomposition we will study in this paper. There are many interesting and important results and problems in this area. A good survey of them is provided by Chung and Graham. [5] Among other things, the *chromatic index* (the minimum number of matchings required to decompose a graph), the *arboricity* (the minimum number of forests needed to decompose a graph), the *linear arboricity* (the minimum number of path forests required to decompose a graph) or the *tree number* (the minimum number of

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trees needed to decompose a graph) have all been studied. [1,2,4,7,8,9,10] In some cases, exact formulas for these numbers have been found. An k-path coloring of G is an edge-coloring of G so that each component of each color class is a path of length at most k. Let $\ell a_k G) = \min\{c|G$ has an k-path coloring with c colors$\}$. The number $\ell a_k(G)$ is called the *linear k-arboricity* of G. It is clear that $\ell a_1(G) = \chi'(G)$. In this paper, we completely determine $\ell a_2(K_{n,n})$.

2 Main results

A proper edge-coloring of a graph is an assignment of colors to its edges so that no two incident edges have the same color. If a graph G can colored by no more than k colors, then this graph is called k-colorable and the number $\chi'(G) = \min\{k\colon G$ is k-colorable$\}$ is the chromatic index of G. As mentioned in Section 1, $\chi'(G)$ is the minimum number of matchings required to edge-decompose a graph G. Similarly, $\ell a_2(G)$ can be considered as the minimum number of colors required to color the graph G so that each component of each color class has at most 2 edges; call such an edge-coloring a p_3-coloring. If we focus on the p_3-colorings of $K_{n,n}$, then we can use an $n \times n$ array to represent the coloring. It is well-known that a $K_{n,n}$ with proper coloring can be represented by a Latin square of order n. But, if we consider a p_3-coloring, it is slightly different from a Latin square. Figure 2.1 is an example of $K_{6,6}$ with $\ell a_2(K_{6,6}) = 5$. As can easily be seen, in this array, $L = [\ell_{i,j}]$, a number occurs in each row and each column at most twice and furthermore if $\ell_{i,j} = \ell_{i',j'}$, $i \neq i'$ and $j \neq j'$, then $\ell_{i,j'} \neq \ell_{i,j}$ and $\ell_{i',j} \neq \ell_{i,j}$.

$\ell a_2(K_{6,6})$:

1	1	2	2	3	3
3	3	1	1	2	2
2	4	3	5	1	4
2	5	3	4	1	5
4	2	5	3	4	1
5	2	4	3	5	1

Figure 2.1

We note here that the array in Figure 2.1 provides an upper bound on $\ell a_2(K_{6,6})$ and that this upper bound equals the lower bound in Lemma 2.1.

The following results are necessary to obtain $\ell a_2(K_{n,n})$.

Lemma 2.1. [3]. $\ell a_2(K_{n,n}) \geq \left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil$.

Lemma 2.2. [3].

(1) $\ell a_2(K_n) + 1 \geq \ell a_2(K_{n,n}) \geq \ell a_2(K_{n-1,n-1})$.

(2) $\ell a_2(K_{in,in}) \leq i \cdot \ell a_2(K_{n,n})$.

Theorem 2.3 [2,3]. $\ell a_2(K_n) = \left\lceil \frac{\frac{n(n-1)}{2}}{\lfloor \frac{2n}{3} \rfloor} \right\rceil$.

The Lemma 2.1 gives the lower bounds of $\ell a_2(K_{n,n})$. The main result in this paper is to prove that the equality holds, as is formally stated in Theorem 2.12. The following propositions establish this fact.

Proposition 2.4. $\ell a_2(K_{12v,12v}) = 9v$.

Proof: By Lemma 2.1 and Figure 2.2, $\ell a_2(K_{12,12}) = 9$. Let $L = [\ell_{ij}]$ be a Latin square of order v. By using direct product of L and M (Figure 2.2), we obtain a $12v \times 12v$ array which corresponds a p_3-coloring of $K_{12v,12v}$. Hence $\ell a_2(K_{12v,12v}) \leq 9v$. Again by Lemma 2.1, $\ell a_2(K_{12v,12v}) = 9v$. □

1	1	2	2	3	3	8	7	5	4	6	9
8	4	4	5	5	6	6	7	9	3	2	1
4	6	7	7	8	8	9	9	5	3	2	1
4	6	9	1	1	2	2	3	3	8	7	5
3	2	1	8	4	4	5	5	6	6	7	9
3	2	1	4	6	7	7	8	8	9	9	5
8	7	5	4	6	9	1	1	2	2	3	3
6	7	9	3	2	1	8	4	4	5	5	6
9	9	5	3	2	1	4	6	7	7	8	8
2	3	3	8	7	5	4	6	9	1	1	2
5	5	6	6	7	9	3	2	1	8	4	4
7	8	8	9	9	5	3	2	1	4	6	7

Figure 2.2. M.

Corollary 2.5. $\ell a_2(K_{12v+11,12v+11}) = 9v + 9$.

Proof:

$$\begin{aligned} 9v + 9 &\leq \ell a_2(K_{12v+11,12v+11}) \\ &\leq \ell a_2(K_{12(v+1),12(v+1)}) = 9(v+1). \end{aligned}$$

□

Proposition 2.6. $\ell a_2(K_{12v+1,12v+1}) = 9v + 1$.

Proof: From Lemma 2.1, Figure 2.3 and Figure A.1 (see the Appendix), $\ell a_2(K_{13,13}) = 10$ and $\ell a_2(K_{25,25}) = 19$, respectively. For $v \geq 3$, we will construct a $(12v+1)\times(12v+1)$ array which corresponds to a p_3-coloring of $K_{12v+1,12v+1}$.

1	1	2	2	3	3	8	7	5	4	6	a	9
8	4	4	5	5	6	6	7	a	3	2	9	1
4	6	1	1	8	8	a	a	5	3	2	9	7
4	6	a	9	9	2	2	3	3	8	7	1	5
3	2	9	8	1	1	5	5	6	6	7	a	4
3	2	9	4	6	7	7	8	8	a	a	1	5
8	7	5	4	6	a	9	9	1	1	3	3	2
6	7	a	3	2	9	8	4	4	5	5	6	1
a	a	5	3	2	9	4	1	7	7	8	8	6
2	3	3	8	7	5	4	1	a	9	9	2	6
5	5	6	6	7	a	1	2	9	8	4	4	3
7	8	8	a	a	5	1	2	9	4	6	7	3
9	9	7	7	4	4	3	6	2	2	1	5	a

=

A_1	b_1
c_1	a

Figure 2.3. $\ell a_2(K_{13,13})=10$.

A_1						b_1
	A_2					b_2
		$\ddots$		$P_{\ell ij}$		$\vdots$
			A_i			b_i
				$\ddots$		$\vdots$
					A_v	b_v
c_1	c_2	$\cdots$	c_i	$\cdots$	c_v	a

Figure 2.4.

It is well-known that there exists an idempotent Latin square of order $n(n\geq 3)$ [6]. Let $L=[\ell_{ij}]$ be an idempotent Latin square of order v. Now using A_1, b_1 and c_1 defined in Figure 2.3, construct a $(12v+1)\times(12v+1)$ array as in Figure 2.4 where $A_t(i,j)=A_1(i,j)$ if $A_1(i,j)=a$, $A_t(i,j)=A_1(i,j)+9(t-1)$ if $A_1(i,j)\neq a$, $b_t=b_1+9(t-1)$ and $c_t=c_1+9(t-1)$ and $P_t=M+9(t-1)$, $1\leq t\leq v$. It is a routine matter to check that $\ell a_2(K_{12v+1,12v+1})=9v+1$. □

Proposition 2.7. $\ell a_2(K_{12v+2,12v+2})=9v+2$.

Proof: From Lemma 2.1 and Figure 2.5, $\ell a_2(K_{14,14})=11$. Since $20\leq \ell a_2(K_{26,26})\leq 2\cdot\ell a_2(K_{13,13})\leq 2\cdot 10=20$, $\ell a_2(K_{26,26})=20$. For $v\geq 3$, we use the same technique in the proof of Proposition 2.6 and then $\ell a_2(K_{12v+2,12v+2})=9v+2$. □

8	8	2	2	3	3	*a*	7	5	4	1	*b*	9	6
a	9	9	5	5	6	6	7	*b*	3	2	1	8	4
4	6	8	8	*a*	*a*	*b*	*b*	5	9	2	1	3	7
4	6	*b*	9	9	2	2	3	3	*a*	7	8	1	5
3	2	1	*a*	8	8	5	5	6	6	9	*b*	7	4
3	2	1	4	6	9	9	*a*	*a*	*b*	*b*	8	7	5
a	7	5	4	6	*b*	1	1	8	8	3	3	2	9
6	7	*b*	3	2	1	*a*	4	4	9	5	6	5	8
b	*b*	5	3	2	1	4	8	7	7	*a*	*a*	6	9
2	3	3	*a*	7	5	4	8	*b*	1	9	2	6	1
5	5	6	6	7	*b*	8	9	1	*a*	4	4	3	2
7	*a*	*a*	*b*	*b*	5	8	9	1	4	6	7	2	3
1	1	7	7	4	4	3	6	9	2	8	5	*a*	*b*
9	4	4	1	1	7	7	6	2	3	8	5	*b*	*a*

Figure 2.5. $\ell a_2(K_{14,14}) = 11$.

Proposition 2.8. *$\ell a_2(K_{12v+3,12v+3}) = 9v+3$, $\ell a_2(K_{12v+4,12v+4}) = 9v+4$, and $\ell a_2(K_{12v+5,12v+5}) = \ell a_2(K_{12v+6,12v+6}) = 9v+5$.*

Proof: From Lemma 2.1, Lemma 2.2, and Theorem 2.3, we have

$$\begin{aligned} 9v+3 &\le \ell a_2(K_{12v+3,12v+3}) \le \ell a_2(K_{12v+3}) + 1 = 9v+3; \\ 9v+4 &\le \ell a_2(K_{12v+4,12v+4}) \le \ell a_2(K_{12v+4}) + 1 = 9v+4; \\ 9v+5 &\le \ell a_2(K_{12v+5,12v+5}) \le \ell a_2(K_{12v+6,12v+6}) \\ &\le \ell a_2(K_{12v+6}) + 1 = 9v+5. \end{aligned}$$

□

Proposition 2.9. *$\ell a_2(K_{12v+7,12v+7}) = 9v+6$.*

1	1	2	2	3	3	6
3	4	1	1	2	2	3
5	3	6	4	1	5	2
6	3	4	5	1	4	2
2	6	3	4	6	1	5
2	5	3	6	5	1	4
4	2	5	3	4	6	1

Figure 2.6. $\ell a_2(K_{7,7}) = 6$.

Proof: From Lemma 2.1 and Fi use 2.6, Figure 2.7, and Figure A.2, $\ell a_2(K_{7,7}) = 6$, $\ell a_2(K_{19,19}) = 15$, and $\ell a_2(K_{31,31}) = 24$, respectively. For

$v \geq 3$, we use the same technique in the proof of Proposition 2.6 and then $\ell a_2(K_{12v+7,12v+7}) = 9v + 6$. □

a	9	*a*	*b*	9	*c*	*d*	2	*f*	*e*	3	*e*	7	6	5	8	1	4	4
1	*a*	*c*	2	*b*	*b*	5	*d*	*d*	5	*e*	*f*	7	6	4	8	3	3	9
c	*a*	8	*b*	*c*	9	*d*	*f*	6	*f*	*e*	6	2	2	4	3	1	7	5
b	1	*c*	*a*	1	*a*	*e*	2	*e*	*d*	3	*f*	4	9	8	7	6	5	6
1	*b*	*b*	2	*a*	*c*	6	*e*	*f*	6	*d*	*d*	4	3	9	7	8	5	3
b	*c*	8	*c*	*a*	9	*f*	*e*	5	*d*	*f*	5	3	7	2	2	4	6	1
d	4	*f*	*e*	5	*e*	*a*	6	*a*	*b*	6	*c*	3	7	1	1	2	9	8
3	*d*	*d*	3	*e*	*f*	7	*a*	*c*	8	*b*	*b*	9	4	5	4	2	6	1
d	*f*	1	*f*	*e*	1	*c*	*a*	3	*b*	*c*	4	8	8	6	9	7	2	5
e	4	*e*	*d*	5	*f*	*b*	7	*c*	*a*	7	*a*	1	1	6	3	9	2	8
4	*e*	*f*	4	*d*	*d*	7	*b*	*b*	8	*a*	*c*	6	9	3	5	5	1	2
f	*e*	2	*d*	*f*	2	*b*	*c*	3	*c*	*a*	4	5	5	9	6	8	1	7
6	8	3	1	4	5	9	5	2	4	2	3	*b*	*b*	*a*	*a*	*c*	*c*	*f*
6	5	4	1	3	7	8	4	7	2	9	8	*c*	*d*	*b*	*b*	*a*	*a*	*c*
8	6	5	7	7	8	4	3	9	2	4	1	*e*	*c*	*f*	*d*	*b*	*e*	*a*
9	6	5	9	2	4	2	3	8	1	1	7	*f*	*c*	*d*	*e*	*b*	*d*	*a*
2	2	6	6	8	4	3	1	1	3	5	7	*a*	*f*	*c*	*d*	*f*	*b*	*e*
7	3	7	8	6	3	1	9	4	9	5	2	*a*	*e*	*c*	*f*	*e*	*b*	*d*
5	7	9	5	3	6	1	8	4	7	8	2	*d*	*a*	*e*	*c*	*d*	*f*	*b*

Figure 2.7. $\ell a_2(K_{19,19}) = 15$.

Proposition 2.10. $\ell a_2(K_{12v+8,12v+8}) = \ell a_2(K_{12v+9,12v+9}) = 9v + 7$.

Proof:

$$\begin{aligned} 9v + 7 &\leq \ell a_2(K_{12v+8,12v+8}) \leq \ell a_2(K_{12v+9,12v+9}) \\ &\leq \ell a_2(K_{12v+9}) + 1 = 9v + 7. \end{aligned}$$

□

Proposition 2.11. $\ell a_2(K_{12v+10,12v+10}) = 9v + 8$.

Proof: From Lemma 2.1 and Figure 2.8, Figure A.3, and Figure A.4, $\ell a_2(K_{10,10}) = 8$, $\ell a_2(K_{22,22}) = 17$, and $\ell a_2(K_{34,34}) = 26$, respectively. For $v \geq 3$, we use the same technique in the proof of Proposition 2.6 and then $p_2(K_{12v+10,12v+10}) = 9v + 8$. □

From the above propositions, we have

7	1	2	2	3	3	4	6	5	7
6	7	1	1	2	2	8	3	4	4
4	4	5	5	1	1	8	2	3	3
3	6	4	7	5	6	1	7	2	8
3	8	4	6	5	7	1	8	2	6
6	3	8	4	8	5	7	1	7	2
8	3	6	4	7	5	6	1	8	2
5	2	3	8	4	7	3	6	1	5
2	5	7	3	6	4	2	4	1	8
1	2	7	3	4	8	5	5	6	1

Figure 2.8. $\ell a_2(K_{10,10}) = 8$.

Theorem 2.12. $\ell a_2(K_{n,n}) = \left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil$.

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Appendix

21	21	22	22	23	23	28	27	25	24	26	29	10	10	12	12	13	13	18	17	15	14	16	19	11
28	24	24	25	25	26	26	27	29	23	22	21	18	14	14	15	15	16	16	17	19	13	12	11	10
24	26	27	27	28	28	29	29	25	23	22	21	14	16	10	10	18	18	19	19	15	13	12	11	17
24	26	29	21	21	22	22	23	23	28	27	25	14	16	19	11	11	12	12	13	13	18	17	10	15
23	22	21	28	24	24	25	25	26	26	27	29	13	12	11	18	10	10	15	15	16	16	17	19	14
23	22	21	24	26	27	27	28	28	29	29	25	13	12	11	14	16	17	17	18	18	19	19	10	15
28	27	25	24	26	29	21	21	22	22	23	23	18	17	15	14	16	19	11	11	10	10	13	13	12
26	27	29	23	22	21	28	24	24	25	25	26	16	17	19	13	12	11	18	14	14	15	15	16	10
29	29	25	23	22	21	24	26	27	27	28	28	19	19	15	13	12	11	14	10	17	17	18	18	16
22	23	23	28	27	25	24	26	29	21	21	22	12	13	13	18	17	15	14	10	19	11	11	12	16
25	25	26	26	27	29	23	22	21	28	24	24	15	15	16	16	17	19	10	12	11	18	14	14	13
27	28	28	29	29	25	23	22	21	24	26	27	17	18	18	19	19	15	10	12	11	14	16	17	13
10	10	12	12	13	13	18	17	15	14	16	19	11	11	22	22	23	23	28	27	25	24	26	29	21
18	14	14	15	15	16	16	17	19	13	12	11	28	24	24	25	25	26	26	27	29	23	10	21	22
14	16	10	10	18	18	19	19	15	13	12	11	24	26	17	17	28	28	29	29	25	23	22	21	27
14	16	19	11	11	12	12	13	13	18	17	10	24	26	29	21	21	22	22	23	23	28	25	15	27
13	12	11	18	10	10	15	15	16	16	17	19	23	22	21	28	14	14	25	25	26	26	27	29	24
13	12	11	14	16	17	17	18	18	19	19	10	23	22	21	24	26	27	27	28	28	29	25	15	29
18	17	15	14	16	19	11	11	10	10	13	13	28	27	25	24	26	29	21	21	12	12	23	23	22
16	17	19	13	12	11	18	14	14	15	15	16	26	27	29	23	22	21	28	24	24	25	10	26	25
19	19	15	13	12	11	14	10	17	17	18	18	29	29	25	23	22	21	24	16	27	27	28	28	26
12	13	13	18	17	15	14	10	19	11	11	12	22	23	23	28	27	25	24	16	29	21	21	22	26
15	15	16	16	17	19	10	12	11	18	14	14	25	25	26	26	27	29	13	22	21	28	24	24	23
17	18	18	19	19	15	10	12	11	14	16	17	27	28	28	29	29	25	13	22	21	24	26	27	23
11	11	17	17	14	14	13	16	12	12	10	15	21	21	27	27	24	24	23	26	22	22	29	25	19

Figure A.1. $\ell a_2(K_{25,25}) = 19$.

7	1	2	38	3	3	4	6	5	7	35	32	33	38	34	35	33	36	31	37	26	2	28	24	28	25	23	21	27	22	8
6	7	1	1	2	2	8	3	4	4	22	33	36	34	37	32	36	31	38	35	25	23	23	28	24	27	22	26	21	5	3
3	6	4	7	5	6	1	7	2	8	33	36	34	37	35	38	31	37	32	36	27	28	25	25	21	21	24	22	23	23	26
4	4	5	5	1	1	8	2	3	3	31	32	37	33	34	31	35	35	36	7	23	21	24	27	25	26	26	27	22	28	6
1	2	7	3	4	8	5	5	6	1	37	31	32	32	33	37	34	36	35	33	23	21	24	26	25	27	28	28	22	26	38
6	3	8	4	8	5	7	1	7	2	36	33	38	34	38	32	37	31	37	35	24	24	22	28	23	23	21	26	25	27	21
8	3	6	4	7	5	6	1	8	2	32	35	37	33	36	38	32	34	31	34	28	26	26	24	27	25	23	21	28	22	7
5	2	3	8	4	7	3	6	1	5	33	38	34	36	35	36	31	38	32	37	26	28	21	21	22	22	27	23	24	24	25
2	5	7	3	6	4	2	4	1	8	36	37	31	31	32	34	38	33	34	32	22	22	27	23	26	24	25	24	21	28	35
3	8	4	6	5	7	1	8	2	37	34	34	35	35	31	33	38	32	33	31	21	25	36	23	24	28	27	25	26	21	32
36	38	34	36	35	37	31	38	32	33	25	23	22	28	24	27	23	26	21	25	2	7	1	2	3	3	4	6	5	7	8
32	33	38	34	38	35	37	31	37	6	26	21	27	21	22	22	28	23	24	27	7	8	5	3	36	4	2	4	1	2	25
34	37	31	31	32	32	38	33	34	36	21	27	22	23	24	28	25	25	26	21	4	8	35	7	5	6	1	7	2	3	33
38	36	34	37	35	36	31	37	32	33	27	7	25	23	6	24	22	24	21	28	5	3	4	5	1	1	8	2	3	4	2
37	31	32	32	33	33	34	36	35	34	23	24	28	26	25	27	21	28	22	26	4	6	8	6	5	7	1	8	2	3	38
33	34	35	35	31	31	38	32	33	37	26	28	23	24	28	25	27	21	27	22	8	2	3	4	8	5	7	1	7	6	36
32	33	36	34	37	35	36	31	38	38	28	26	23	24	27	25	26	21	28	22	6	2	3	4	7	5	6	1	8	8	27
35	32	21	27	34	37	33	36	31	35	38	22	25	22	23	23	24	26	6	24	7	1	2	3	4	8	5	5	6	1	28
38	35	37	33	36	34	32	34	31	32	23	24	26	27	25	26	21	27	22	28	1	4	7	1	2	2	8	3	4	6	5
31	32	37	33	34	27	35	35	36	31	24	26	24	25	21	21	28	22	23	23	3	5	2	8	4	7	3	6	1	38	26
25	23	22	25	27	24	232	26	28	21	3	8	4	6	5	7	1	8	2	6	37	31	35	38	33	37	34	36	32	33	34
26	28	23	22	25	28	27	21	24	27	6	7	1	1	2	2	8	3	4	4	32	35	31	33	36	38	32	34	37	34	5
23	24	26	28	26	25	21	27	27	22	7	4	5	5	1	1	8	2	3	3	31	32	36	33	34	31	35	35	37	38	6
27	25	24	23	21	21	28	22	25	23	3	6	4	7	5	6	1	7	2	8	33	36	32	37	35	38	31	37	34	36	24
23	24	28	26	27	25	21	28	26	22	4	1	2	8	3	3	4	6	5	7	33	38	32	36	35	36	31	38	34	37	1
26	21	27	24	22	22	28	23	21	24	6	3	8	4	8	5	7	1	7	2	36	33	37	34	38	32	37	31	38	35	23
28	26	23	22	25	27	26	21	24	28	8	3	6	4	7	5	6	1	8	2	38	33	38	34	37	32	36	31	36	35	37
24	22	21	27	23	23	24	26	22	25	5	2	3	8	4	7	3	6	1	5	35	32	31	38	34	35	33	36	33	37	28
22	27	25	28	24	26	22	24	23	21	2	5	7	3	6	4	2	4	1	8	36	37	34	31	32	34	38	33	31	32	35
37	27	22	21	28	24	25	25	23	6	1	2	7	3	4	8	5	5	26	1	34	34	33	35	31	33	38	32	35	31	36
21	1	33	2	36	38	4	32	35	26	7	37	21	2	26	28	34	23	25	38	27	31	6	22	6	8	24	3	5	25	22

Figure A.2. $\ell a_2(K_{31,31}) = 24$.

15	15	22	22	23	23	20	27	25	24	26	21	11	16	16	11	13	12	19	18	17	14
20	24	24	25	25	26	26	27	21	24	22	28	14	15	11	15	16	16	11	13	12	19
24	26	27	27	20	20	21	21	25	23	22	18	14	11	13	13	19	19	17	17	12	16
24	26	21	16	16	22	22	23	23	20	27	25	15	13	12	19	18	17	14	15	11	11
23	22	13	20	24	24	25	25	26	26	27	21	19	11	17	12	14	14	15	12	18	16
23	22	13	24	26	27	27	20	20	21	21	25	19	18	17	14	15	11	12	16	16	15
20	27	25	24	26	21	19	19	22	22	23	23	17	17	12	16	18	15	14	11	13	13
26	27	21	23	22	12	20	24	24	25	25	26	16	18	15	14	11	13	13	19	19	17
21	21	25	23	22	12	24	26	27	27	20	20	16	13	15	19	11	17	18	14	14	18
22	23	23	20	27	25	24	26	21	14	14	22	18	12	18	16	13	15	19	11	17	12
25	25	26	26	27	21	23	22	11	20	24	24	13	19	19	17	17	12	16	18	15	14
27	20	20	21	21	25	23	22	11	24	26	27	12	14	14	18	12	18	16	13	15	19
11	11	18	12	12	17	13	13	15	16	19	14	27	20	22	22	23	23	24	26	25	27
18	14	16	14	15	17	15	16	19	12	11	13	26	27	20	20	22	22	21	23	24	24
14	16	19	17	17	19	18	18	15	12	11	13	24	24	25	25	20	20	21	22	23	23
18	16	12	19	11	13	11	12	13	17	15	14	23	26	24	27	25	26	20	27	22	21
13	12	15	11	18	15	14	14	16	17	19	16	23	21	24	26	25	27	20	21	22	26
13	12	17	11	14	18	16	17	18	19	15	19	26	23	21	24	21	25	27	20	27	22
14	17	11	18	19	11	16	15	12	13	13	12	21	23	26	24	27	25	26	20	21	22
16	17	18	19	13	14	12	11	14	15	16	15	25	22	23	21	24	27	23	26	20	25
19	19	14	15	13	16	12	11	17	18	18	17	22	25	27	23	26	24	22	24	20	21
12	13	14	31	18	16	17	15	19	11	12	11	20	22	27	23	24	21	25	25	26	20

Figure A.3. $\ell a_2(K_{22,22}) = 17$.

1	1	2	2	3	3	8	7	5	4	6	26	31	31	32	32	33	33	38	37	35	34	36	29	9	21	22	22	23	23	28	27	25	24
22	4	4	6	5	8	6	7	9	3	2	1	38	34	34	35	35	36	36	37	21	33	32	31	28	24	24	25	25	26	26	27	29	23
4	6	7	7	8	6	9	21	5	3	2	1	34	36	37	37	38	38	21	22	35	33	32	31	24	26	27	27	28	28	29	29	25	23
4	23	9	1	1	2	2	3	3	8	7	5	34	36	6	31	31	32	32	33	33	38	37	35	28	27	25	24	26	29	21	21	22	22
3	2	1	8	4	4	5	5	6	27	7	6	33	32	31	38	34	34	35	35	36	36	37	29	23	22	21	28	24	24	25	25	26	26
3	2	1	4	6	7	7	8	8	9	9	5	33	32	31	34	36	37	37	38	38	29	29	35	23	22	21	24	26	27	27	28	28	25
8	7	5	4	6	9	26	1	2	2	3	3	38	37	35	34	36	1	31	31	32	32	33	33	26	27	29	23	22	21	29	24	24	25
6	7	9	3	2	1	8	4	4	5	5	26	36	37	27	33	32	31	38	34	34	35	35	36	27	28	28	29	29	25	23	22	21	24
9	28	6	3	2	1	4	28	7	7	8	8	21	9	35	33	32	31	34	36	37	37	38	38	29	29	25	23	22	5	24	26	27	27
2	3	3	8	7	5	4	6	9	1	1	2	32	33	33	38	37	35	34	36	24	31	31	32	25	25	26	26	27	29	23	22	21	28
21	5	6	54	7	9	3	22	1	8	4	4	35	35	36	36	37	2	33	32	31	38	34	34	22	23	23	26	27	25	24	26	29	21
7	8	8	9	9	5	3	2	1	4	6	7	37	38	38	25	27	35	33	32	31	34	36	37	24	26	290	21	21	22	22	23	23	28
31	31	32	32	33	33	38	37	35	34	36	29	21	6	22	22	23	23	28	27	25	24	26	9	1	1	2	2	3	3	8	7	5	4
38	34	34	35	35	36	36	37	25	33	32	31	28	24	24	1	25	26	26	27	29	23	22	21	2	4	4	5	5	6	6	7	9	3
34	36	37	37	38	38	1	2	35	33	32	31	24	26	29	21	21	22	22	23	23	28	27	25	4	6	7	7	8	8	9	9	5	3
34	36	5	31	31	32	32	33	33	38	37	35	24	26	7	27	28	28	29	29	25	23	22	21	4	6	9	1	1	2	2	3	3	8
33	32	31	38	34	34	35	35	36	36	37	29	23	22	21	28	24	24	25	25	26	26	27	9	3	2	1	8	4	4	5	5	6	6
33	32	31	34	36	37	37	38	38	29	29	35	23	22	21	24	26	27	27	28	28	5	9	25	3	2	1	4	6	7	7	8	8	9
38	37	35	34	36	8	31	31	32	32	33	33	28	27	25	24	26	29	3	21	22	22	23	23	21	7	5	4	6	9	1	1	2	2
36	37	24	33	32	31	38	34	34	35	35	36	26	27	29	23	22	21	28	24	5	25	25	26	6	7	9	3	2	1	8	4	4	5
22	9	35	33	32	31	34	36	37	37	38	38	29	29	25	23	8	21	24	26	27	27	28	28	8	9	5	3	2	1	4	6	7	7
32	33	33	38	37	35	34	36	27	31	31	32	22	23	23	28	1	25	24	26	29	21	21	22	2	3	3	8	7	5	4	6	9	1
35	35	36	36	37	22	33	32	31	38	34	34	25	25	26	26	27	29	23	4	21	28	24	24	5	5	6	6	7	9	23	2	1	8
37	38	38	28	26	35	33	32	31	34	36	37	27	28	6	29	29	25	23	22	7	24	5	27	7	8	8	9	9	21	3	2	1	4
24	21	25	29	26	22	27	9	24	23	23	9	8	3	7	2	2	4	8	5	4	6	3	1	37	31	32	32	33	33	34	36	35	37
26	27	25	29	27	23	28	24	26	22	22	21	9	2	8	3	4	6	7	5	6	7	2	1	36	37	31	31	32	32	38	33	34	34
27	25	26	25	29	23	28	24	29	21	21	22	5	1	8	3	4	7	5	6	9	9	1	2	34	34	35	35	31	31	38	32	33	33
27	24	26	23	23	28	29	25	21	28	24	22	3	8	9	8	5	7	4	6	1	3	4	2	33	36	34	37	35	36	31	37	32	38
28	26	27	22	22	27	29	25	23	24	26	21	2	4	9	7	5	8	6	7	1	2	6	3	33	38	34	36	35	37	31	38	32	36
28	29	29	21	21	25	25	26	22	24	27	23	1	4	5	5	6	8	9	9	2	1	7	3	36	33	38	34	38	35	37	31	37	32
29	21	23	24	28	24	23	26	22	25	27	28	4	5	3	4	6	9	1	3	2	8	7	8	38	33	36	34	37	35	36	31	38	32
29	23	22	236	24	26	22	27	21	25	28	27	6	5	2	6	7	9	1	2	3	4	8	7	35	32	33	38	34	37	33	36	31	35
25	22	21	27	24	29	21	29	23	26	28	25	7	6	1	9	9	5	2	1	3	4	8	5	32	35	37	33	36	34	32	34	31	38
23	22	28	27	25	21	24	23	28	26	25	24	9	7	4	1	3	3	2	8	8	6	5	4	31	32	37	33	34	38	35	35	36	31

Figure A.4. $\ell a_2(K_{34,34}) = 26$.