Some Joint Distributions Concerning Random Walk in a Plane

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Abstract. This paper deals with the joint distributions of some characteristics of the two-dimensional simple symmetric random walk in which a particle at any stage moves one unit in any one of the four directions, namely, north, south, east, and west with equal probability.

1. Introduction.

In this paper we consider a random walk in a plane (that is, the 2-dimensional simple symmetric walk) in which a particle starting from the origin moves at any stage one unit in any one of the four directions, namely, north, south, east, and west with equal probability. In this random walk, since every path of length d in the plane has the probability $(1/4)^d$, we can determine the distribution of any characteristic of the random walk when the particle starting from the origin reaches a fixed point (a, b) after d steps, if we know the number of paths corresponding to the characteristic under consideration and the number of all paths of length d from the origin to (a, b). DeTemple and Robertson (1984) and DeTemple, Jones, and Robertson (1988) have derived distributions of some characteristics of this random walk. Later Csáki, Mohanty, and Saran (1990) have derived distributions related to the boundaries $y - x = k_1$ and $y + x = k_2$ such as touchings, arrivals, crossings, etc. In this paper we consider the above mentioned 2-dimensional simple symmetric walk and derive, using combinatorial methods, the joint distributions of some characteristics related to the boundary $y - x = k (k \geq 0)$, namely, arrivals, touchings, crossings, steps above the boundary, index of the ith arrival, and index of the ith touch.

2. Notations.

We introduce the following symbols.

$E_{a,b;d}$	:	a path of length d from $(0,0)$ to (a,b)
$N(a,b;d;k,r)$	:	the number of $E_{a,b;d}$ paths reaching the line $y-x=k$ exactly r times, that is, having exactly r arrivals.
$N^-(a,b;d;k,r)$	:	the number of $E_{a,b;d}$ paths never crossing the line $y-x=k$ and reaching it from below exactly r times.
$N^+(a,b;d;k,r)$	:	the number of $E_{a,b;d}$ paths reaching the line $y-x=k$ from above exactly r times (known as positive arrivals).
$N^\star(a,b;d;k,r)$	:	the number of $E_{a,b;d}$ paths crossing the line $y-x=k$ exactly r times.
$N(a,b;d;k,r;i,d_1)$	:	the number of paths of type $N(a,b;d;k,r)$ where the ith arrival occurs in d_1 steps.
$N^-(a,b;d;k,r;i,d_1)$	:	the number of paths of type $N^-(a,b;d;k,r)$ where the ith touch occurs in d_1 steps.
$N^+(a,b;d;k,r;i,d_1)$	:	the number of paths of type $N^+(a,b;d;k,r)$ where the ith positive arrival occurs in d_1 steps.
$N^\star(a,b;d;k,r;i,d_1)$	:	the number of paths of type $N^\star(a,b;d;k,r)$ where the ith crossing occurs in d_1 steps.
$M^+(a,b;d;k,r,r_1)$	:	the number of $E_{a,b;d}$ paths having exactly r arrivals and r_1 positive arrivals on the line $y-x=k$.
$M^\star(a,b;d;k,r,r_1)$	:	the number of $E_{a,b;d}$ paths reaching the line $y-x=k$ exactly r times and crossing the line $y-x=k$ exactly r_1 times.
$M^{+\star}(a,b;d;k,r,r_1,j)$	:	the number of paths of type $M^+(a,b;d;k,r,r_1)$ having exactly j crossings of the line $y-x=k$.
$M^{2g+}(a,b;d;k,r,r_1)$	:	the number of paths of type $M^+(a,b;d;k,r,r_1)$ having exactly $2g$ steps above the line $y-x=k$.
$M^{2g+\star}(a,b;d;k,r,r_1,j)$	:	the number of paths of type $M^{+\star}(a,b;d;k,r,r_1,j)$ having exactly $2g$ steps above the line $y-x=k$.

3. Some auxiliary results.

Some basic results needed in the sequel are quoted from Csáki, Mohanty, and Saran (1990) and others are easily derivable.

(i) For $a > b$

$$N^-(a,b;d;0,0) = \frac{a-b}{d}\binom{d}{\frac{d-a-b}{2}}\binom{d}{\frac{d-a+b}{2}}$$

where $d \geq a + b$ and $d - a - b \equiv 0 \pmod 2$.

(ii)

$$N^{-}(a,a;d;0,1) = \frac{1}{d-1}\binom{d-1}{\frac{d-2}{2}}\binom{d}{\frac{d-2a}{2}}$$

= the number of paths of length d from $(0,0)$ to (a,a) lying entirely below the line $y=x$ except at the end point.

(iii) For $a \geq b-k$, $k \geq 0$ and $r \geq 1$

$$N^{-}(a,b;d;k,r) = \frac{a-b+2k+r-1}{d-r+1}\binom{d-r+1}{\frac{d-a+b-2k-2r+2}{2}}\binom{d}{\frac{d-a-b}{2}}.$$

When $k=0$, the starting point is counted as a touch.

(iv) For $a \geq b-k$, $k \geq 0$ and $r \geq 1$

$$N(a,b;d;k,r) = 2^{r-1}N^{-}(a,b;d;k,r).$$

When $k=0$, the starting point is counted as an arrival.

(v) For $k > 0$ and $r \geq 1$

$$N^{\star}(x,k+x;d;k,r) = \frac{k+1+2r}{d+1}\binom{d+1}{\frac{d-k-2r}{2}}\binom{d}{\frac{d-k-2x}{2}}.$$

(vi) For $a > b$ and $r \geq 1$

$$N^{\star}(a,b;d;0,r) = \frac{a-b+1+2r}{d+1}\binom{d+1}{\frac{d-a+b-2r}{2}}\binom{d}{\frac{d-a-b}{2}}.$$

(vii) For $r \geq 1$

$$N^{\star}(a,a;d;0,r) = \frac{4(r+1)}{d}\binom{d}{\frac{d-2(r+1)}{2}}\binom{d}{\frac{d-2a}{2}}.$$

4. Joint distributions.

Theorem 1.

(a) *For* $a \geq b-k$, $k > 0$, $r \geq 1$, $d_1 - k \geq 2(i-1)$ *and* $d - d_1 - (a-b+k) \geq 2(r-i)$

$$N^{-}(a,b;d;k,r;i,d_1) = \frac{(k+i-1)(r+a-b+k-i)}{(d_1-i+1)(d-d_1-r+i)}\binom{d_1-i+1}{\frac{d_1-k+2-2i}{2}} \cdot \binom{d-d_1-r+i}{\frac{d-d_1-a+b-k-2r+2i}{2}}\binom{d}{\frac{d-a-b}{2}}. \quad (1)$$

(b) *For* $k=0, a \geq b, r \geq 1, d_1 \geq 2i$ *and* $d-d_1-a+b \geq 2(r-i)$

$$N^-(a,b;d;0,r;i,d_1) = \frac{i(r-i+a-b)}{(d_1-i)(d-d_1-r+i)}\binom{d_1-i}{\frac{d_1-2i}{2}} \cdot \binom{d-d_1-r+i}{\frac{d-d_1-a+b-2r+2i}{2}}\binom{d}{\frac{d-a-b}{2}}. \tag{2}$$

Proof: Let the path, as envisaged in (1), touch the line $y-x=k$ for the ith time at the point $(x, k+x)$ in d_1 steps, where $d_1-k\equiv 0 \pmod 2$. Then the required number of paths is given by

$$\begin{aligned}
&N^-(a,b;d;k,r;i,d_1)\\
&=\sum_x N^-(x,k+x;d_1;k,i)\cdot N^-(a-x,b-k-x;d-d_1;0,r-i)\\
&=\sum_{x=-(d_1+k)/2}^{(d_1-k)/2}\frac{k+i-1}{d_1-i+1}\binom{d_1}{\frac{d_1-k-2x}{2}}\binom{d_1-i+1}{\frac{d_1-k-2i+2}{2}}\\
&\cdot\frac{r-i+a-b+k}{d-d_1-r+i}\binom{d-d_1-r+i}{\frac{d-d_1-a+b-k-2r+2i}{2}}\binom{d-d_1}{\frac{d-d_1-a-b+k+2x}{2}},
\end{aligned}$$

by (iii) of Section 3. Since d_1-k is a multiple of 2, therefore, on putting $u=(d_1-k)/2$, the above expression reduces to

$$\begin{aligned}
&N^-(a,b;d;k,r;i,d_1)\\
&=\frac{(k+i-1)(r+a-b+k-i)}{(d_1-i+1)(d-d_1-r+i)}\binom{d_1-i+1}{\frac{d_1-k-2i+2}{2}}\\
&\cdot\binom{d-d_1-r+i}{\frac{d-d_1-a+b-k-2r+2i}{2}}\sum_{x=-(u+k)}^{u}\binom{k+2u}{u-x}\binom{d-k-2u}{\frac{d-a-b}{2}-u+x},
\end{aligned}$$

which on using the convolution identity

$$\sum_{x=0}^{n}\binom{a}{x}\binom{b}{n-x}=\binom{a+b}{n}$$

leads to (1). Proceeding in a similar manner, as for $k>0$, one can easily prove (2). ∎

Deductions:

(i) Summing (1) over d_1, we get for $a \geq b-k$, $k>0$ and $r \geq 1$

$$N^-(a,b;d;k,r) = \sum_{\substack{d_1=k+2i-2\\ d_1-k\equiv 0 \pmod 2}}^{d-a+b-2r+2i-k}\frac{(k+i-1)(r+a-b+k-i)}{(d_1-i+1)(d-d_1-r+i)} \cdot\binom{d_1-i+1}{\frac{d_1-k-2i+2}{2}}\binom{d-d_1-r+i}{\frac{d-d_1-a+b-k-2r+2i}{2}}\binom{d}{\frac{d-a-b}{2}}$$

which on putting $u = (d_1 - k)/2$ gives

$$N^-(a,b;d;k,r) = \sum_{u=i-1}^{(d-a+b-2r+2i-k)/2} \frac{(k+i-1)(r+a-b+k-i)}{(k+2u-i+1)(d-k-2u-r+i)} \cdot \binom{k+2u-i+1}{u+k} \binom{d-k-2u-r+i}{\frac{d-2k-2u-a+b-2r+2i}{2}} \binom{d}{\frac{d-a-b}{2}}.$$

Further, making the substitution $j = u - i + 1$ and using the convolution identity in Mohanty (1979, p. 25), we get

$$N^-(a,b;d;k,r) = \frac{a-b+2k+r-1}{d-r+1} \binom{d-r+1}{\frac{d-a+b-2k-2r-2}{2}} \binom{d}{\frac{d-a-b}{2}}$$

which is equivalent to (6) in Csáki, Mohanty, and Saran (1990) for $k > 0$.

(ii) Summing (2) over d_1, we get (6) in Csáki, Mohanty, and Saran (1990).

Theorem 2.

(a) *For* $a > b - k$, $k > 0$ *and* $r \geq 1$

$$N^\star(a,b;d;k,r;i,d_1) = \frac{(k+2i-1)}{(d_1+1)} \frac{(a-b+k+2r-2i+1)}{(d-d_1+1)} \cdot \binom{d_1+1}{\frac{d_1-k-2i+2}{2}} \binom{d-d_1+1}{\frac{d-d_1-a+b-k-2r+2i}{2}} \binom{d}{\frac{d-a-b}{2}}. \quad (3)$$

(b) *For* $k > 0$ *and* $r \geq 1$

$$N^\star(a,a+k;d;k,r;i,d_1) = \frac{2(r-i+1)(k+2i-1)}{(d_1+1)(d-d_1)} \cdot \binom{d_1+1}{\frac{d_1-k-2i+2}{2}} \binom{d-d_1}{\frac{d-d_1-2r+2i-2}{2}} \binom{d}{\frac{d-2a-k}{2}}. \quad (4)$$

(c) *For* $a > b$ *and* $r \geq 1$

$$N^\star(a,b;d;0,r;i,d_1) = \frac{2i(a-b+2r-2i+1)}{d_1(d-d_1+1)} \binom{d_1}{\frac{d_1-2i}{2}} \cdot \binom{d-d_1+1}{\frac{d-d_1-a+b-2r+2i}{2}} \binom{d}{\frac{d-a-b}{2}}. \quad (5)$$

(d) *For* $r \geq 1$

$$N^\star(a,a;d;0,r;i,d_1) = \frac{4i(r-i+1)}{d_1(d-d_1)} \binom{d_1}{\frac{d_1-2i}{2}} \cdot \binom{d-d_1}{\frac{d-d_1-2r+2i-2}{2}} \binom{d}{\frac{d-2a}{2}} \quad (6)$$

Proof: Let the path as envisaged in (3) cross the line $y - x = k$ for the ith time at the point $(x, k+x)$ in d_1 steps where $d_1 - k \equiv 0 \pmod 2$. Then the required number of paths is given by

$$\begin{aligned}
&N^*(a,b;d;k,r;i,d_1) \\
&= \sum_x N^*(x,k+x;d_1;k,i-1)\cdot N^*(a-x,b-k-x;d-d_1;0,r-i) \\
&= \sum_{x=-(d_1+k)/2}^{(d_1-k)/2} \frac{k+1+2(i-1)}{d_1+1}\binom{d_1+1}{\frac{d_1-k-2(i-1)}{2}}\binom{d_1}{\frac{d_1-k-2x}{2}} \\
&\quad \cdot \frac{a-b+k+1+2(r-i)}{d-d_1+1}\binom{d-d_1+1}{\frac{d-d_1-a+b-k-2(r-i)}{2}}\binom{d-d_1}{\frac{d-d_1-a-b+k+2x}{2}},
\end{aligned}$$

by (v) and (vi) of Section 3, leading to (3) after simplification. Proceeding in a similar manner, one can prove (4) to (6). ∎

Deductions: Summing (3) to (6) each over d_1, we get (11) to (14), respectively, in Csáki, Mohanty, and Saran (1990).

Theorem 3.

(a) *For* $a \geq b-k, k > 0, r \geq 1, d_1 - k \geq 2(i-1)$ *and* $d - d_1 - (a-b+k) \geq 2(r-i)$

$$\begin{aligned}
N(a,b;d;k,r;i,d_1) &= 2^{r-1}\frac{(k+i-1)(a-b+k+r-i)}{(d_1-i+1)(d-d_1-r+i)}\binom{d_1-i+1}{\frac{d_1-k-2i+2}{2}} \\
&\quad \cdot \binom{d-d_1-r+i}{\frac{d-d_1-a+b-k-2r+2i}{2}}\binom{d}{\frac{d-a-b}{2}}.
\end{aligned} \tag{7}$$

(b) *For* $a \geq b, r \geq 1, d_1 \geq 2i$ *and* $d - d_1 - (a-b) \geq 2(r-i)$

$$\begin{aligned}
N(a,b;d;0,r;i,d_1) &= \frac{2^{r-1}i(a-b+r-i)}{(d_1-i)(d-d_1-r+i)}\binom{d_1-i}{\frac{d_1-2i}{2}} \\
&\quad \cdot \binom{d-d_1-r+i}{\frac{d-d_1-a+b-2r+2i}{2}}\binom{d}{\frac{d-a-b}{2}}.
\end{aligned} \tag{8}$$

Proof: Let the path as envisaged in (7) reach the line $y - x = k$ for the ith time at the point $(x, k+x)$ in d_1 steps where $d_1 - k \equiv 0 \pmod 2$. Then

$$\begin{aligned}
&N(a,b;d;k,r;i,d_1) \\
&= \sum_x N(x,k+x;d_1;k,i)\cdot N(a-x,b-k-x;d-d_1;0,r-i)
\end{aligned}$$

which on using (iv) of Section 3 leads to (7). Similarly (8) can be proved. ∎

Deductions:

(i) Summing (7) over d_1 from $k+2(i-1)$ to $d-a+b-k-2(r-i)$, we get

$$N(a,b;d;k,r) = 2^{r-1}\frac{a-b+2k+r-1}{d-r+1}\binom{d-r+1}{\frac{d-a+b-2k-2r+2}{2}}\binom{d}{\frac{d-a-b}{2}}$$

which is equivalent to (10) in Csáki, Mohanty, and Saran (1990) for $k>0$.

(ii) Summing (8) over d_1, we get (10) in Csáki, Mohanty, and Saran (1990) for $k=0$.

Theorem 4.

(a) *For* $a>b-k, k>0, 0\le r_1\le r, r\ge 1$ *and* $d-2k-a+b\ge 2(r-i)$

$$M^\star(a,b;d;k,r,r_1) = \begin{cases} \binom{r}{r_1}\frac{a-b+2k+r-1}{d-r+1}\binom{d-r+1}{\frac{d-a+b-2k-2r+2}{2}}\binom{d}{\frac{d-a-b}{2}} & \text{when } r_1 \text{ is even,} \\ 0 & \text{when } r_1 \text{ is odd.}\end{cases} \tag{9}$$

(b) *For* $k>0, 0\le r_1\le r-1, r\ge 1$ *and* $d-k\ge 2(r-1)$

$$M^\star(a,a+k;d;k,r,r_1) = \binom{r-1}{r_1}\frac{k+r-1}{d-r+1}\binom{d}{\frac{d-k-2a}{2}}\binom{d-r+1}{\frac{d-k-2r+2}{2}} \tag{10}$$

(c) *For* $a>b, 0\le r_1\le r, r\ge 1, d-a+b\ge 2r$

$$M^\star(a,b;d;0,r,r_1) = \binom{r}{r_1}\frac{a-b+r}{d-r}\binom{d-r}{\frac{d-a+b-2r}{2}}\binom{d}{\frac{d-a-b}{2}} \tag{11}$$

(d) *For* $1\le r_1\le r-1, r\ge 1$ *and* $d\ge 2r$

$$M^\star(a,a;d;0,r,r_1) = 2\binom{r-1}{r_1}\frac{r}{d-r}\binom{d-r}{\frac{d-2r}{2}}\binom{d}{\frac{d-2a}{2}} \tag{12}$$

and

$$M^\star(a,a;d;0,r,0) = \frac{r}{d-r}\binom{d-r}{\frac{d-2r}{2}}\binom{d}{\frac{d-2a}{2}}. \tag{13}$$

Proof: To establish (9), we observe that on reflecting the path segments lying above the line $y - x = k$ about this line itself, we get a path of length d from $(0,0)$ to (a,b), $a > b - k$, lying entirely below the line $y - x = k$ and touching it exactly r times. Thus,

$$M^{*}(a,b;d;k,r,r_1) = \binom{r}{r_1} N^{-}(a,b;d;k,r),$$

since r_1 crossing points can be selected out of r arrivals in $\binom{r}{r_1}$ ways, leading to (9) on using (iii) of Section 3.

Proceeding in a similar manner, one can prove (10) to (12). In result (12), the factor 2 appears due to the fact that the path may start either with a north/west step or with a south/east step. ∎

Deductions:

(i) For the special case $r_1 = 0$, results (9), (10), (11), and (13) each reduce to result (6) of Csáki, Mohanty, and Saran (1990).

(ii) Summing (9) to (12) each over r_1, we get, results equivalent to (10) in Csáki, Mohanty, and Saran (1990).

Theorem 5. *For* $a \geq b-k$, $k \geq 0$, $0 \leq r_1 \leq r-1$, $r \geq 1$ *and* $d-a+b-2k \geq 2(r-1)$

$$M^{+}(a,b;d;k,r,r_1) = \binom{r-1}{r_1} \frac{r+a-b+2k-1}{d-r+1} \binom{d-r+1}{\frac{d-a+b-2k-2r+2}{2}} \binom{d}{\frac{d-a-b}{2}}. \tag{14}$$

When $k = 0$, *the starting point is counted as an arrival.*

Proof: On reflecting the segments of the path lying above the line $y - x = k$ about this line, we get a path of length d from $(0,0)$ to (a,b) lying entirely below the line $y - x = k$ and touching it exactly r times and the number of such paths is given by $N^{-}(a,b;d;k,r)$. Now since r_1 positive arrivals can be selected out of $(r-1)$ arrivals in $\binom{r-1}{r_1}$ ways, we get (14). ∎

Deductions:

(i) For the special case $r_1 = 0$, result (14) reduces to result (6) of Csáki, Mohanty, and Saran (1990).

(ii) Summing (14) over r_1 from 0 to $r-1$, we get (10) in Csáki, Mohanty, and Saran (1990).

(iii) Summing (14) over r from $r_1 + 1$ to $(d - a + b - 2k + 1)/2$, making the substitution $j = r - r_1 - 1$ and using the relation $\frac{n-2i}{n}\binom{n}{i} = \binom{n-1}{i} - \binom{n-1}{i-1}$, we get the following new result:

Corollary. *For $a \geq b-k$, $k \geq 0$ and $r_1 \geq 0$*

$$N^+(a,b;d;k,r_1) = \frac{a-b+2k+2r_1+1}{d+1}\binom{d+1}{\frac{d-a+b-2r_1-2k}{2}}\binom{d}{\frac{d-a-b}{2}}. \tag{15}$$

When $k=0$, the starting point is counted as a positive arrival.

Theorem 6. *For $a \geq b-k$, $k \geq 0$ and $r \geq 1$*

$$N^+(a,b;d;k,r;i,d_1) = \frac{(k+2i)(a-b+k+2r-2i+1)}{d_1(d-d_1+1)}\binom{d_1}{\frac{d_1-k-2i}{2}} \cdot \binom{d-d_1+1}{\frac{d-d_1-2r+2i-a+b-k}{2}}\binom{d}{\frac{d-a-b}{2}}. \tag{16}$$

Proof: Let the ith positive arrival occur at the point $(x, k+x)$ in d_1 steps where $d_1 - k \equiv 0 \pmod 2$. Then the required number of paths is given by

$$\begin{aligned} N^+(a,b;d;k,r;i,d_1) = \sum_x &\text{(the number of paths from } (0,0) \text{ to } (x,k+x) \text{ of length } d_1 \\ &\text{with } i \text{ positive arrivals at } y-x=k \text{ such that} \\ &\text{the } i\text{th positive arrival occurs at the point } (x,k+x)) \\ &\cdot N^+(a-x, b-k-x; d-d_1; 0, r-i) \end{aligned} \tag{17}$$

where

$$N^+(a-x,b-k-x;d-d_1;0,r-i) = \frac{a-b+k+2r-2i+1}{d-d_1+1} \cdot \binom{d-d_1+1}{\frac{d-d_1-2r+2i-a+b-k}{2}}\binom{d-d_1}{\frac{d-d_1-a-b+k+2x}{2}}, \tag{18}$$

by (15) and, the first factor, under the summation sign, on the right-hand side of (17) equals

$$\begin{aligned} &N^+(x,k+x;d_1;k,i) \\ &\quad - [N^+(x,k+x-1;d_1-1;k,i) + N^+(x+1,k+x;d_1-1;k,i)] \\ &= \frac{k+2i}{d_1}\binom{d_1}{\frac{d_1-k-2x}{2}}\binom{d_1}{\frac{d_1-k-2i}{2}}, \end{aligned} \tag{19}$$

by (15). Hence, on using (18) and (19), we get (16). ∎

Deduction: Summing (16) over d_1, we get

$$N^+(a,b;d;k,r) = \frac{a-b+2k+2r+1}{d+1}\binom{d+1}{\frac{d-2k-2r-a+b}{2}}\binom{d}{\frac{d-a-b}{2}}$$

which is equivalent to (15).

Theorem 7.

(a) *For $a > b - k$, $k > 0$ and $r \geq 1$*

$$M^{++}(a,b;d;k,r,r_1,2j) = \binom{r_1-1}{j-1}\binom{r-r_1}{j} N^{-}(a,b;d;k,r). \quad (20)$$

(b) *For $k > 0$ and $r \geq 1$*

$$M^{++}(a,a+k;d;k,r,r_1,2j) = \binom{r_1-1}{j-1}\binom{r-r_1-1}{j} N^{-}(a,a+k;d;k,r), \quad (21)$$

$$M^{++}(a,a+k;d;k,r,r_1,2j-1) = \binom{r_1-1}{j-1}\binom{r-r_1-1}{j-1} N^{-}(a,a+k;d;k,r). \quad (22)$$

(c) *For $a > b$ and $r \geq 1$*

$$M^{++}(a,b;d;0,r,r_1,2j) = \binom{r_1-1}{j-1}\binom{r-r_1}{j} N^{-}(a,b;d;0,r), \quad (23)$$

$$M^{++}(a,b;d;0,r,r_1,2j-1) = \binom{r_1}{j}\binom{r-r_1-1}{j-1} N^{-}(a,b;d;0,r). \quad (24)$$

(d) *For $r \geq 1$*

$$\begin{aligned} M^{++}&(a,a;d;0,r,r_1,2j) \\ &= \left[\binom{r_1-1}{j-1}\binom{r-r_1-1}{j} + \binom{r_1-1}{j}\binom{r-r_1-1}{j-1}\right] \\ &\quad \cdot N^{-}(a,a;d;0,r), \end{aligned} \quad (25)$$

$$M^{++}(a,a;d;0,r,r_1,2j-1) = 2\binom{r_1-1}{j-1}\binom{r-r_1-1}{j-1} N^{-}(a,a;d;0,r). \quad (26)$$

Proof: In order to prove (20), we observe that on reflecting the path segments lying above the line $y - x = k$ about this line, we get a path which accounts for the factor $N^{-}(a,b;d;k,r)$. For other factors in (20), there arise the following two contingencies:

(a_1) the last return point is a crossing point,

(a_2) the last return point is not a crossing point.

In case (a_1), the path consists of j positive segments (that is, the path segment between two consecutive crossings of the line $y - x = k$ and lying above this line) which can be constructed out of r_1 positive arrivals in $\binom{r_1-1}{j-1}$ ways and j negative segments (that is, the path segment between two consecutive crossings of the line $y-x=k$ and lying below this line) which can be constructed out of the remaining $r-r_1$ arrivals (called negative arrivals) in $\binom{r-r_1-1}{j-1}$ ways.

In case (a_2), the path consists of j positive segments which can be constructed out of r_1 positive arrivals in $\binom{r_1-1}{j-1}$ ways and $(j+1)$ negative segments which can be constructed out of $(r-r_1)$ negative arrivals in $\binom{r-r_1-1}{j}$ ways. On adding these two cases we get (20). Likewise other results can be proved. ∎

Deductions:

(i) Summing results (a), (b), (c), and (d), of Theorem 7, each over j generates result (14) given in Theorem 5.

(ii) Summing results (a), (b), (c), and (d), of Theorem 7, each over r_1 generates results (9) to (12) given in Theorem 4.

Theorem 8. *For $a \geq b-k$, $k \geq 0$ and $d-2g-(a+b-2k) \geq 2(r-r_1-1)$*

$$M^{2g+}(a,b;d;k,r,r_1) = \frac{r_1(a-b+2k+r-r_1-1)}{(2g-r_1)(d-2g-r+r_1+1)}\binom{r-1}{r_1}\binom{2g-r_1}{g} \cdot \binom{d-2g-r+r_1+1}{\frac{d-2g-a+b-2k-2(r-r_1-1)}{2}}\binom{d}{\frac{d-a-b}{2}}. \tag{27}$$

When $k=0$, the starting point is counted as an arrival.

Proof: To prove (27), we divide the requisite path into two parts, namely,

(a) by combining together the path segments of total length $2g$ lying above the line $y-x=k$ end to end, in order, and

(b) by combining the path segments of total length $d-2g$ lying below the line $y-x=k$ end to end, in order.

Supposing the coordinates of the end point of the path obtained in (a) above to be (x,x), the required number of paths is given by

$$M^{2g+}(a,b;d;k,r,r_1) = \sum_{x=(a+b-d+2g)/2}^{g}\binom{r-1}{r_1}N^{-}(x,x;2g;0,r_1) \cdot N^{-}(a-x,b-x;d-2g;k,r-r_1),$$

which on using (iii) of Section 3 leads to (27). ∎

Deduction: Summing (27) over g, we get (14) given in Theorem 5.

Theorem 9.

(a) *For* $a > b - k, k > 0$, *and* $d - 2g - a + b - 2k \geq 2(r - r_1 - 1)$

$$M^{2g+\star}(a, b; d; k, r, r_1, 2j) = \binom{r_1 - 1}{j - 1}\binom{r - r_1}{j} \cdot A \quad (28)$$

(b) *For* $k > 0$, *and* $d - 2g - a + b - 2k \geq 2(r - r_1 - 1)$

$$M^{2g+\star}(a, a + k; d; k, r, r_1, 2j) = \binom{r_1 - 1}{j - 1}\binom{r - r_1 - 1}{j} \cdot A \quad (29)$$

$$M^{2g+\star}(a, a + k; d; k, r, r_1, 2j - 1) = \binom{r_1 - 1}{j - 1}\binom{r - r_1 - 1}{j - 1} \cdot A \quad (30)$$

(c) *For* $a > b, r \geq 1$, *and* $d - 2g - a + b \geq 2(r - r_1)$

$$M^{2g+\star}(a, b; d; 0, r, r_1, 2j) = \binom{r_1 - 1}{j - 1}\binom{r - r_1}{j} \cdot B \quad (31)$$

$$M^{2g+\star}(a, b; d; 0, r, r_1, 2j - 1) = \binom{r_1}{j}\binom{r - r_1 - 1}{j - 1} \cdot B \quad (32)$$

(d) *For* $r \geq 1$, *and* $d - 2g \geq 2(r - r_1)$

$$M^{2g+\star}(a, a; d; 0, r, r_1, 2j) = \left[\binom{r_1 - 1}{j - 1}\binom{r - r_1 - 1}{j} + \binom{r_1 - 1}{j}\binom{r - r_1 - 1}{j - 1}\right] \cdot C \quad (33)$$

$$M^{2g+\star}(a, a; d; 0, r, r_1, 2j - 1) = 2\binom{r_1 - 1}{j - 1}\binom{r - r_1 - 1}{j - 1} \cdot C \quad (34)$$

where

$$A = \frac{r_1(a - b + 2k + r - r_1 - 1)}{(2g - r_1)(d - 2g - r + r_1 + 1)}\binom{2g - r_1}{g} \cdot \binom{d - 2g - r + r_1 + 1}{\frac{d-2g-a+b-2k-2r+2r_1+2}{2}}\binom{d}{\frac{d-a-b}{2}},$$

$$B = \frac{r_1(a - b + r - r_1)}{(2g - r_1)(d - 2g - r + r_1)}\binom{2g - r_1}{g} \cdot \binom{d - 2g - r + r_1}{\frac{d-2g-a+b-2r+2r_1}{2}}\binom{d}{\frac{d-a-b}{2}},$$

$$C = \frac{r_1(r - r_1)}{(2g - r_1)(d - 2g - r + r_1)}\binom{2g - r_1}{g} \cdot \binom{d - 2g - r + r_1}{\frac{d-2g-2r+2r_1}{2}}\binom{d}{\frac{d-2a}{2}}.$$

The result follows on using similar arguments as used in proving Theorem 7 and Theorem 8.

Deductions:

(i) Summing results (a), (b), (c), and (d) of Theorem 9 each over g and using the convolution identity (Mohanty (1979), p. 25) generates the corresponding results in Theorem 7.

(ii) Summing results (a), (b), (c), and (d) of Theorem 9 each over j generates the result given in Theorem 8.

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