Symmetric $(49, 16, 5)$ Designs

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ABSTRACT. In this paper the existence of the 12140 non-isomorphic symmetric $(49, 16, 5)$ designs with an involutory homology (this is a special kind of involution acting on a design) is propagated. The automorphism groups are cyclic of orders 2, 4, 8, or 10. 218 designs are self-dual. The 40 designs with an automorphism group of order 10 were already given in [2]. A computer (IBM 3090) was used for about 36 hours CPU time. According to [2,4] now there are known 12146 symmetric (49,16,5) designs.

1. Preliminaries

Definition 1: An involutory homology τ acting on a symmetric (v, k, λ) design is an involution (i.e. an automorphism of order 2) with $k+1$ fixed points and (therefore) $k+1$ fixed blocks such that there is one fixed point—called the center—being incident with all fixed blocks excepted one—called the axis—being incident with all fixed points excepted the center. □

Remark: The previous definition implies $v + k \equiv 1 \pmod 2$. □

Definition 2: Let τ be an involutory homology acting on the symmetric (v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \varepsilon)$ with set $\mathcal{P}$ of points and set $\mathcal{B}$ of blocks (which are k-subsets of $\mathcal{P}$), let $\mathcal{P}_i = \{P_i\}$, $\mathcal{B}_i = \{B_i\}$, $0 \le i \le k$, be the $\langle\tau\rangle$-orbits of fixed points resp. blocks, and let $\mathcal{P}_i = \{P_i, Q_i\}$, $\mathcal{B}_i = \{B_i, C_i\}$, $k+1 \le i \le \frac{v+k-1}{2}$, be the $\langle\tau\rangle$-orbits of non-fixed points resp. blocks.

By $A = A(\mathcal{D}, \tau)$ we define the *orbit matrix* $A = (a_{ij})$ with entries

$$a_{ij} = |\{P \in \mathcal{P}_j : P \in B \text{ for a } B \in \mathcal{B}_i\}|, 0 \le i, j \le \frac{v+k-1}{2},$$

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and by $\overline{A} = \overline{A}(\mathcal{D}, \tau)$ the *dual orbit matrix* $\overline{A} = (\overline{a}_{ij})$ with entries

$$\overline{a}_{ij} = |\{B \in \mathcal{B}_j : P \in B \text{ for a } P \in \mathcal{P}_i\}|, 0 \leq i,j \leq \frac{v+k-1}{2}.$$

□

Proposition 1. *The above defined matrices $A = A(\mathcal{D}, \tau)$ and $\overline{A} = \overline{A}(\mathcal{D}, \tau)$ fulfill the* orbit matrix equation

$$XSX^T = (k-\lambda)S + \lambda J \qquad (*)$$

with $X \in \{A, \overline{A}\}$, $S = \begin{vmatrix} I_f & 0 \\ 0 & \frac{1}{2}I_w \end{vmatrix}$, $f = k+1$, $w = \frac{v-f}{2}$, (m,m)-identity matrix I_m, $m \in \{f, w\}$, and $(f+w, f+w)$-all-1 matrix J.

Proof: In the case $X = A$ (*) counts the intersections of any two blocks (in a modified way), and in the case $X = \overline{A}$ (*) describes the dual situation. □

Definition 3: The orbit matrices A and A' are called *isomorphic* ($\cong$), if both of them fulfill the same orbit matrix equation (*) and if one is received from the other one by row and column permutations (that is a renumbering of the point and block orbits). In this sense we call $\overline{A} \cong \overline{A'}$ iff $A \cong A'$. □

2. Results

By aid of a computer as a first result we get the

Theorem 1. *Let $(v, k, \lambda) = (49, 16, 5)$. Up to isomorphism there are five matrices A_1 up to A_5 (and therefore $\overline{A_1}$ up to $\overline{A_5}$) which fulfill the orbit matrix equation $(*)$ and hence could come from some symmetric $(49, 16, 5)$ design with an involutory homology. All solutions can be written in the form*

$$A_i = \begin{array}{|c|c|c|} \hline & j^T & \\ \hline j & M & 2N_i \\ \hline & (N_i)^T & J-M \\ \hline \end{array}, 1 \leq i \leq 5,$$

with a $(16,1)$-all-1 vector j, a symmetric $(16,16)$-incidence matrix M (i.e. $M = M^T$) of the semibiplane $SB(16,5)$ (that is a semisymmetric $(16,5,[2])$ design, cf. [3],196ff.), the $(16,16)$-all-1 matrix J, and a $(16,16)$-incidence matrix N_i of some tactical configuration $TC(16,5)$ (that is a 1-$(16,5,5)$ design), $1 \leq i \leq 5$.

In particular, the resulting matrices are as follows, where for easier reading the entries 0 are omitted and an entry 1 is replaced by the number of the column

$M =$

											12	13	14	15	16
								9	10	11				15	16
						7	8			11			14		16
					6		8		10			13			16
					6	7		9			12				16
			4	5						11			14	15	
		3		5					10			13		15	
		3	4					9			12			15	
	2			5			8					13	14		
	2		4			7					12		14		
	2	3			6						12	13			
1				5			8		10	11					
1			4			7		9		11					
1		3			6			9	10						
1	2				6	7	8								
1	2	3	4	5											

$N_1 =$

1										11	12	13			16
1							8	9					14	15	
		3	4						10				14		16
	2				6				10	11				15	
				5		7		9	10		12				
	2			5		7								15	16
			4			7	8		10			13			
		3			6	7				11			14		
				5	6			9				13	14		
			4		6		8				12			15	
	2	3					8	9							16
		3		5			8			11	12				
	2		4					9		11		13			
1			4	5	6										16
1		3				7						13		15	
1	2								10		12		14		

$N_2 =$

1										11	12	13			16
1						7			10				14	15	
	2			5				9						15	16
			4				8	9	10			13			
		3			6			9		11			14		
		3	4				8						14		16
	2				6		8			11				15	
				5		7	8	9			12				
				5	6				10		12		14		
			4		6	7						13		15	
	2	3				7			10						16
		3		5		7				11		13			
	2		4						10	11	12				
1			4	5	6										16
1		3					8				12			15	
1	2							9				13	14		

$N_3 =$

1										11	12	13			16
1							8	9					14	15	
	2			5		7								15	16
	2				6				10	11			14		
			4			7	8		10		12				
		3	4						10				14		16
				5		7		9	10			13			
		3			6	7				11				15	
			4		6		8					13		15	
				5	6			9			12		14		
	2	3				8	9								16
		3		5			8			11	12				
	2		4					9		11		13			
1			4	5	6										16
1		3				7						13	14		
1	2								10		12			15	

$N_4 =$

1				5							12	13			16
	2	3								11			14		16
1					6		8						14	15	
	2		4				8	9				13			
	2			5		7			10					15	
						7		9	10			13	14		
		3				7	8				12			15	
			4		6					11		13		15	
	2				6				10	11	12				
		3	4				8		10						16
		3		5	6			9					14		
			4	5	6	7									16
				5			8	9		11	12				
1			4						10		12		14		
1		3				7				11		13			
1	2							9						15	16

$N_5 =$

1				5							12			15	16
		3			6					11			14		16
			4		6		8					13		15	
	2					7			10				14	15	
	2							9		11	12	13			
1								9	10			13	14		
			4				8		10	11	12				
		3				7	8	9						15	
		3		5		7				11		13			
	2	3	4						10						16
			4	5	6			9					14		
	2			5			8	9							16
				5	6	7			10		12				
1			4			7						13			16
1		3					8				12		14		
1	2				6					11				15	

The automorphism groups of the five matrices are

$$\begin{aligned} &\mathrm{Aut}(A_1) \cong \mathrm{Aut}(A_2) \cong Z_2 && \text{(cyclic group of order 2)},\\ &\mathrm{Aut}(A_3) \cong D_4 && \text{(dihedral group of order 8)},\\ &\mathrm{Aut}(A_4) \cong \mathrm{Aut}(A_5) \cong Z_5 && \text{(cyclic group of order 5)}. \end{aligned}$$

□

By the aid of a computer we "reconstruct" from the above five matrices A_1 up to A_5 all symmetric $(49, 16, 5)$ designs with an involutory homology, where the determination of the automorphism groups and the distinction of non-isomorphic designs is done by counting the intersection sizes of all the $\binom{49}{3}$ triples of pairwise different blocks and by showing that the only involutory homology is the presumed one.

Theorem 2. *Up to isomorphism there are* 12140 *symmetric* $(49, 16, 5)$ *designs with an involutory homology. In particular, the results are as follows, where we denote by* $\mathrm{Aut}(\mathcal{D})$ *the automorphism group of the design* $\mathcal{D}$*, by* Z_j *the cyclic group of order* j*, and by* $d = 1$ *resp.* $d = 2$ *the partition of all designs into self-dual and non-self-dual ones:*

A_1, A_2 :

$\mathrm{Aut}(\mathcal{D})$	$d=1$	$d=2$	all
Z_2	28	3932	3960 ($\to$ 7920),

A_3 :

$\mathrm{Aut}(\mathcal{D})$	$d=1$	$d=2$	all
Z_8	-	4	4
Z_4	10	44	54
Z_2	152	810	962
all	162	858	1020,

A_4, A_5 :

$\mathrm{Aut}(\mathcal{D})$	$d=2$
Z_{10}	20
Z_2	1580
all	1600 ($\to$ 3200).

□

3. Examples

We specify one example for each non-vanishing entry of the above tables excepted the case of $\mathrm{Aut}(\mathcal{D}) \cong Z_{10}$ (cf. [2]). According to theorem 1 it suffices to give the "indexed" matrix $J - M$ (the above matrix M has now to be rewritten as a $\{0, 1\}$-matrix), where "indexing" means replacing an entry 0 by the $(2, 2)$-null matrix, denoted by a "blank", and replacing an entry 1 by a $(2, 2)$-permutation matrix $P = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ or I $(= P^0)$, denoted by 1 resp. 0.

4. Outlook

The previous results could give rise to the following future projects:

1. Does a symmetric $(49, 16, 5)$ design with a trivial automorphism group exist?

2. The existence of a symmetric $(85, 28, 9)$ design is still in doubt. Is there such a design with an involutory homology?

$(A_1, Z_2, d=1)$

0	1	1	1	0	1	0	0	0	1	0					
0	1	0	0	1	0	0	0				0	1	1		
0	0	1	0	1	0			0	1		1	0		1	
1	1	1	0	1		1		0		0	0		0	0	
0	0	0	0	0			0		0	0		0	0	0	
1	1	1			0	0	0	1	0		1	0			1
1	0		1		0	0	0	0		1	0		0		0
1	0			0	0	1	0		1	0		1	1		1
1		0	0		1	0		0	1	1	1			0	1
1		0		1	1		0	1	1	0		0		1	0
0			1	1		1	0	1	1	1			0	0	1
	1	0	1		0	1		0			1	0	1	0	0
	0	1		1	1		0		0		1	1	1	0	0
	1		0	0		1	0			1	1	1	0	1	0
		0	1	1				0	0	0	1	1	0	1	1
					1	1	0	0	0	1	0	0	1	1	1

$(A_1, Z_2, d=2)$

1	1	0	1	0	0	1	0	1	1	1					
0	0	1	1	1	0	1	0				1	0	1		
0	1	1	1	0	1			0	1		0	0		0	
0	1	0	1	1		0		0		0	1		0	1	
0	0	0	0	0			0		0	0		0	0	0	
1	1	1			1	0	0	1	0		1	0			0
1	0		1		0	0	0	0		1	0		0		0
0	1			0	0	0	0		0	1		1	1		1
1		0	1		1	1		0	0	1	1			0	1
0		0		1	1		0	1	1	1		1		0	0
1			0	1		0	0	0	1	0			1	0	1
	1	1	0		0	1		0			1	1	0	0	0
	0	1		0	1		0		1		1	1	0	1	1
	1		0	1		1	0			1	0	0	0	1	1
		0	0	0				0	1	1	1	0	1	1	0
					1	1	0	0	0	0	0	1	1	1	0

$(A_2, Z_2, d=1)$

1	0	1	1	0	0	0	0	1	1	0					
0	1	1	1	1	1	0	0				1	0	1		
1	0	1	0	0	1			0	0		1	0		0	
0	1	0	0	0		0		1		0	1		0	0	
0	0	1	1	1			1		0	0		1	0	0	
0	0	0			0	0	0	0	0		0	0			0
1	1		1		0	0	1	0		1	1		0		0
0	0			0	1	0	1		1	1		1	1		0
1		0	1		1	0		1	0	1	0			0	1
0		1		0	0		1	1	0	1		0		1	1
1			0	1		0	1	1	0	0			1	1	0
	1	1	0		0	0		0			0	1	1	0	1
	1	1		0	1		0		0		0	1	0	1	0
	0		0	1		0	0			1	1	1	0	1	1
		0	1	0				0	0	0	1	1	1	1	1
					1	0	1	0	1	0	0	0	0	1	1

$(A_2, Z_2, d{=}2)$

1	1	1	0	1	0	0	0	0	1	1					
0	1	1	1	0	0	0	1				1	0	0		
1	0	1	0	0	1			0	0		1	0		0	
1	1	0	0	0		0		1		0	0		0	0	
0	1	1	0	0			0		0	0		1	1	1	
0	0	0			0	0	0	0	0		0	0			0
1	0		1		0	0	0	1		0	1		1		1
0	0			0	1	0	0		1	1		1	0		1
1		1	1		1	0		1	0	1	0			1	0
1		0		0	0		1	0	0	1		1		1	1
0			0	1		0	1	1	0	1			1	0	1
	1	0	1		1	0		0			1	1	1	0	0
	1	0		1	1		0		0		1	0	0	1	1
	0		0	1		0	1			0	1	1	0	1	0
		1	1	1				0	0	0	0	1	0	0	1
					1	0	1	0	1	0	0	0	1	1	1

$(A_3, Z_8, d{=}2)$

1	0	0	0	0	1	0	0	1	1	0					
1	1	1	0	1	0	0	0				0	0	1		
0	0	0	0	0	0			0	0		0	0		0	
0	1	0	0	1		0		0		0	1		0	1	
0	1	0	1	0			0		1	1		0	1	1	
1	1	0			1	1	0	0	0		0	1			0
1	0		1		0	0	0	0		1	1		0		0
0	0			1	1	0	0		0	1		1	1		1
0		1	1		1	0		0	1	0	0			0	0
0		1		0	0		0	1	0	0		1		1	0
0			0	1		1	0	1	1	1			0	0	0
	0	1	0		1	1		0			1	0	1	1	0
	1	1		0	1		0		0		1	0	0	0	1
	0		1	1		1	0			0	0	0	0	1	1
		1	0	0				0	1	1	0	1	0	1	1
					0	1	0	0	1	0	1	1	1	0	1

$(A_3, Z_4, d{=}1)$

0	0	1	1	0	1	1	0	0	1	1					
1	0	0	1	0	0	0	0				1	1	1		
0	0	0	0	0	0			0	0		0	0		0	
1	0	1	1	1		0		0		0	0		0	0	
0	1	0	1	1			0		1	0		0	1	0	
1	1	1			0	1	0	0	0		1	0			0
1	1		0		1	0	0	0		1	0		1		1
0	1			0	1	0	0		0	0		1	0		0
0		1	0		0	0		0	1	0	1			1	1
0		1		1	0		0	1	0	1		1		0	1
1			0	0		1	0	1	1	0			0	0	1
	1	0	1		0	1		0			0	1	0	1	1
	0	0		1	1		0		0		1	0	0	1	1
	0		0	1		1	0			0	0	1	1	1	0
		0	0	1				0	1	1	1	1	0	0	0
					0	0	0	1	1	1	0	0	0	1	0

$(A_3, Z_4, d{=}2)$

1	0	1	0	0	0	1	0	1	1	0					
0	1	1	0	1	0	0	0				0	1	1		
0	0	0	0	0	0			0	0		0	0		0	
1	0	1	1	1		0		0		0	0		0	0	
0	1	1	1	0			0		0	0		0	0	1	
1	1	0			0	0	0	0	1		1	0			0
1	1		0		1	1	0	0		1	0		0		1
0	0			0	1	0	0		1	1		1	0		0
0		1	1		0	1		0	1	1	1			0	1
0		0		1	1		0	1	1	0		0		0	1
1			1	0		0	0	1	0	1			1	0	1
	0	1	0		1	0		0			1	0	1	1	1
	0	0		1	0		0		0		1	1	0	1	1
	0		1	1		1	0			1	0	0	1	1	0
		0	1	0				0	1	0	0	1	1	1	1
					1	1	0	0	0	0	1	1	1	0	0

$(A_3, Z_2, d{=}1)$

0	1	1	1	1	0	1	0	0	0	1					
0	1	0	1	0	1	0	0				0	1	0		
0	0	0	0	0	0			0	0		0	0		0	
1	0	1	1	1		0		0		0	0		0	0	
1	0	0	0	1			0		0	1		1	0	1	
1	1	0			0	0	0	0	1		1	0			0
1	1		0		1	1	0	0		0	0		1		1
0	0			1	1	0	0		1	1		0	1		1
0		1	0		1	0		0	0	0	1			1	0
1		1		0	1		0	1	0	1		0		0	0
0			0	1		1	0	1	1	0			0	0	0
	0	0	1		1	1		0			1	1	1	0	0
	0	1		0	0		0		1		0	1	1	1	0
	0		1	0		1	0			0	1	0	0	1	1
		1	0	0				0	1	1	1	1	0	0	1
					0	0	0	1	0	0	1	1	1	0	1

$(A_3, Z_2, d{=}2)$

0	1	1	1	1	0	1	0	0	0	1					
0	1	0	1	0	1	0	0				0	1	0		
0	0	0	0	0	0			0	0		0	0		0	
1	0	1	1	1		0		0		0	0		0	0	
0	0	1	1	0			0		1	0		0	1	1	
1	1	0			0	0	0	0	1		1	0			0
1	1		0		1	1	0	0		0	0		1		1
1	0			0	0	1	0		1	1		1	0		1
0		1	0		1	0		0	1	1	1			0	1
1		1		0	1		0	1	0	1		0		0	0
0			0	1		1	0	1	1	0			0	0	0
	0	0	1		1	1		0			1	1	1	0	0
	0	0		1	1		0		0		1	0	0	1	1
	0		0	1		0	0			1	0	1	1	1	0
		1	0	0				0	0	0	1	1	0	1	0
					0	0	0	1	0	0	1	1	1	0	1

$(A_4, Z_2, d{=}2)$

0	1	0	1	1	0	1	1	0	1	1					
1	1	1	0	1	0	1	0				0	0	1		
0	0	1	1	1	0			1	0		0	1		0	
1	1	0	1	0		0		1		1	0		1	0	
0	1	1	0	0			0		1	1		1	0	0	
1	0	0			0	1	0	1	1		1	1			0
1	0		0		0	0	1	0		1	0		0		0
1	1			0	0	1	1		0	0		1	0		1
0		0	0		0	0		1	1	0	0			1	1
0		0		0	0		0	0	0	0		0		0	0
1			1	1		0	0	0	1	0			0	0	1
.	1	1	1		0	0		1			1	0	0	1	0
.	0	1		0	0		1		1		1	0	1	0	1
.	0		1	0		1	0			1	0	0	0	1	1
.		1	1	0				0	1	0	0	1	1	1	0
.					0	0	0	0	0	1	1	1	1	1	1

$(A_5, Z_2, d{=}2)$

1	1	0	0	1	1	1	1	0	1	0					
1	0	1	0	0	1	0	1				0	1	0		
0	1	1	1	1	1			0	0		0	1		1	
1	1	1	1	0		0		0		0	1		1	0	
1	0	0	1	1			1		0	1		1	1	0	
0	0	0			1	0	0	0	1		1	1			1
1	0		1		0	1	0	0		0	0		0		1
0	0			0	1	1	1		0	0		0	1		1
0		0	0		0	0		0	0	0	0			0	0
1		0		0	1		0	0	0	1		0		1	0
0			1	0		1	1	0	1	1			0	0	0
.	0	1	0		0	1		0			1	1	1	1	0
.	1	0		0	0		1		0		1	1	0	1	1
.	1		0	0		1	0			1	0	1	1	0	1
.		1	0	1				0	0	1	1	0	0	0	1
.					0	0	1	0	1	1	0	0	1	1	1

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