A Ramsey Type Problem

H.L. Abbott
Department of Mathematics
University of Alberta
Edmonton, Alberta, T6G 2G1
Canada

and

M. Katchalski
Department of Mathematics
The Technion
Haifa
Israel

ABSTRACT. Ramsey's Theorem implies that for any graph H there is a least integer $r = r(H)$ such that if G is any graph of order at least r then either G or its complement contains H as a subgraph. For $n < r$ and $0 \leq e \leq \frac{1}{2}n(n-1)$, let $f(e) = 1$ if every graph G of order n and size e is such that either G or $\overline{G}$ contains H, and let $f(e) = 0$ otherwise. This associates with the pair (H, n) a binary sequence $S(H, n)$. By an interval of $S(H, n)$ we mean a maximal string of equal terms. We show that there exist infinitely many pairs (H, n) for which $S(H, n)$ has seven intervals.

Let H be a finite graph with no isolated vertices. Let $o(H)$ denote the order of H and let $r(H)$ be the Ramsey number of H; that is, $r(H)$ is the least integer r such that if G is any graph of order at least r then either G contains H or $\overline{G}$, the complement of G, contains H. We suppose that $3 \leq o(H) < r(H)$.

For $n \geq o(H)$ we define a function $f_{H,n}$ on $\{0, 1, 2, \ldots, \binom{n}{2}\}$ as follows:

$$f_{H,n}(e) = \begin{cases} 1 & \text{if every graph } G \text{ of order } n \text{ and size } e \text{ is such that} \\ & \text{either } G \text{ or } \overline{G} \text{ contains } H \\ 0 & \text{otherwise.} \end{cases}$$

We shall often write f instead of $f_{H,n}$ if there is no danger of confusion.

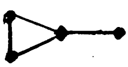
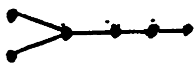
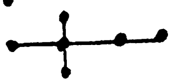
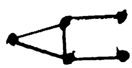
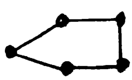
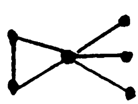
H	r(H)	n	S(H,n)
	7	6	(6,1,2,1,6)
	7	6	(5,2,2,2,5)
	7	6	(5,2,2,2,5)
	9	8	(12,1,3,1,12)
	9	8	(12,1,3,1,12)
	9	8	(12,1,3,1,12)
	9	8	(12,2,1,2,12)
	11	10	(20,1,4,1,20)
		9	(16,1,1,1,1,1,16)

Table 1

Let $S = S(H,n) = (f(0), f(1), f(2), \dots, f(\binom{n}{2}))$ be the sequence of values of f. By an interval of S we mean a maximal set of consecutive integers on which f is constant. It is clear that for $n \geq r(H)$, S has one interval. Since $f(e) = f(\binom{n}{2} - e)$, S has at least three intervals for $o(H) \leq n < r(H)$ and the number of intervals of S is odd. The question naturally arises as to whether S may have more than three intervals. We began our investigation of this question by examining some small graphs. For the purposes of this discussion, a small graph is one with at most six edges. The Ramsey numbers of the 113 small graphs with no isolated vertices are known and are given in the survey article of Burr [1]. We found that for many (we did not examine all of them) of these small graphs H, $S(H,n)$ does indeed have three intervals. However, we found several pairs (H,n) for which $S(H,n)$ has five intervals and one pair for which $S(H,n)$ has seven intervals. Some of these are shown in Table 1. If $S(H,n)$ has m intervals, we write it as an m-tuple; the entries represent the lengths of the intervals. For example, $(1,1,1,1,1,1,0,1,1,0,1,1,1,1,1,1)$ is written as $(6,1,2,1,6)$. Among the small graphs H for which $S(H,n)$ has three intervals for $o(H) \leq n \leq r(H)$ are K_3, K_4, C_4, C_6 and $K_{1,l}$ for $3 \leq l \leq 6$. Table 2 shows $S(K_4,n)$ for $4 \leq n < 18 = r(K_4)$.

n	$S(K_4,n)$
4	(1,5,1)
5	(2,7,2)
6	(3,10,3)
7	(5,12,5)
8	(7,15,7)
9	(9,19,9)
10	(14,18,14)
11	(19,18,19)
12	(26,15,26)
13	(33,15,33)
14	(41,10,41)
15	(50,6,50)
16	(60,1,60)
17	(68,1,68)

Table 2

These results suggest the following question. Is it true that for each odd positive integer $k \geq 3$ there exist infinitely many pairs (H,n) for which $S(H,n)$ has k intervals? The main purpose of this paper is to show that this is so for $k = 3$, 5 and 7. We do not know the answer for $k \geq 9$ and do not know of any pair (H,n) for which $S(H,n)$ has more than seven intervals.

There are relatively few infinite families of graphs for which the Ramsey numbers are known, One such family is the family of stars. Here it is known, and easy to prove, that $r(K_{1,2l}) = 4l-1$ and $r(K_{1,2l+1}) = 4l+2$, We shall prove that in this case S has three intervals.

Theorem 1. *Let* $4 \leq l+1 \leq n < r(K_{1,l})$ *and let* $t = \lceil \frac{n(n-l)}{2} \rceil$. *Then* $S(K_{1,l},n) = (t, \binom{n}{2} - 2t + 1, t)$.

For $l \geq 3$, let H_l be the graph obtained from $K_{1,l}$ by joining two vertices of degree 1. H_3, H_4 and H_5 are the first, fifth and last graphs shown in Table 1. It is known, and also easy to prove, that $r(H_l) = 2l+1$. Our main result is the following theorem.

Theorem 2.

(a) $S(H_{2l-1}, 4l-2)$ *has five intervals for each* $l \geq 2$.

(b) *$S(H_{2l-1}, 4l-3)$ has seven intervals for each $l \geq 3$.*

(c) *$S(H_{2l-1}, n)$ has three intervals for $l \geq 2$, $2l \leq n \leq 4l-4$.*

(d) *$S(H_{2l}, n)$ has five intervals for $l \geq 2$, $n = 4l-1$ or $4l$.*

(e) *$S(H_{2l}, n)$ has three intervals for $l \geq 2$, $2l+1 \leq n \leq 4l-2$.*

We use the following notation. If G is a graph then $\delta(G)$ and $\Delta(G)$ denote the minimal and maximal degrees of G. If there is no danger of confusion we write δ and Δ instead of $\delta(G)$ and $\Delta(G)$. If x is a vertex of G then $N(x)$ is the set of neighbors of x and $T(x)$ is the set of vertices of G (other than x) not adjacent to x. If U is a set of vertices of G then $G[U]$ denotes the subgraph of G induced by U.

Proof of Theorem 1: Observe first that if G is a graph of order n with $e < t$ edges, then $\delta(G) \leq n-l-1$ so that $\Delta(\overline{G}) \geq l$ and hence $\overline{G}$ contains $K_{1,l}$. Thus $f(e) = 1$ for $0 \leq e < t$. We next show that for $t \leq e \leq \frac{1}{2}\binom{n}{2}$, $f(e) = 0$. The theorem will then follow. To do this we need to show that there is a graph G with n vertices and e edges such that neither G nor $\overline{G}$ contains $K_{1,l}$. We give the details only in the case where n is odd and l is even, say $n = 2s+1$ and $l = 2u$. The other cases may be handled in a similar manner. Note that since $r(K_{1,2u}) = 4u-1$, we have $2u+1 \leq 2s+1 \leq 4u-2$ so that $u \leq s \leq 2u-2$. We first construct two auxiliary graphs Γ_1 and Γ_2. The vertices of these graphs will be $2s+1$ equispaced points on the circumference of a circle. The edge set of Γ_1 is obtained by joining each point to its $2u-2$ nearest neighbors and adding a maximal matching. Γ_1 has $n-1$ vertices of degree $l-1$ and one vertex of degree $l-2$. The conditions on u and s imply that Γ_1 has more than $\frac{1}{2}\binom{n}{2}$ edges and that neither Γ_1 nor $\overline{\Gamma}_1$ contains $K_{1,l}$. The edge set of Γ_2 is obtained by joining each point to its $2(s-u+1)$ nearest neighbors and then deleting a maximal matching. Γ_2 is a subgraph of Γ_1 and it has $n-1$ vertices of degree $n-l$ and one vertex of degree $n-l+1$. It therefore has t edges. It is easy to see that neither Γ_2 nor $\overline{\Gamma}_2$ contains $K_{1,l}$. Let E be the set of edges of Γ_1 that are not edges of Γ_2. Then for $t \leq e \leq \frac{1}{2}\binom{n}{2}$, we may, by adding to Γ_2 a suitable subset of E, obtain a subgraph G of Γ_1 with n vertices and e edges such that neither G nor $\overline{G}$ contains $K_{1,l}$.

Proof of Theorem 2:

(a) Let G be a graph with $4l-2$ vertices and $e \leq \lfloor \frac{1}{2}\binom{4l-2}{2} \rfloor = 4l^2-5l+1$ edges. We show that except when $e = 4l^2-6l+2$, either G or $\overline{G}$ contains H_{2l-1}. Let v be a vertex of G of degree Δ. If $\Delta \geq 2l$ then either $G[N(v)]$ contains an edge, in which case G contains H_{2l-1}, or $N(v)$ is an independent set, in which case $\overline{G}$ contains H_{2l-1}. We may therefore suppose that $\Delta \leq 2l-1$. If $\delta \leq 2l-3$, then $\Delta(\overline{G}) \geq$

$2l$ and the argument just used shows that either G or $\overline{G}$ contains H_{2l-1}, We may therefore suppose that $\delta \geq 2l-2$. Suppose that $\Delta = 2l-1$. If $G[N(v)]$ has an edge then G contains H_{2l-1}. Thus we may assume that $N(v)$ is an independent set. If some vertex of $N(v)$ is not joined to some vertex of $T(v)$ then $\overline{G}$ contains H_{2l-1}. Hence we may suppose that each vertex of $N(v)$ is joined to each vertex of $T(v)$. The condition $\Delta = 2l-1$ then ensures that $T(v)$ is an independent set. It then follows that $e = \Delta + |N(v)|\,|T(v)| = 2l-1+(2l-1)(2l-2) = 4l^2-4l+1$. However, this is false. Thus $\delta = \Delta = 2l-2$. G is a $(2l-2)$ regular graph and $e = \frac{1}{2}(4l-2)(2l-2) = 4l^2-6l+2$. Since the graph consisting of two vertex disjoint copies of K_{2l-1} is such neither it nor its complement contains H_{2l-1} we have that $f(4l^2-6l+2) = 0$ and $f(e) = 1$ for all other e satisfying $0 \leq e \leq 4l^2-5l+1$. This establishes (a). In fact, $S(H_{2l-1}, 4l-2) = (4l^2-6l+2, 1, 2l-2, 1, 4l^2-6l+2)$.

(b) Let G be a graph with $4l-3$ vertices and e edges, $l \geq 3$, $0 \leq e \leq \left[\frac{1}{2}\binom{4l-3}{2}\right] = 4l^2-7l+3$. We shall show that either G or $\overline{G}$ contains H_{2l-1} except when $e = 4l^2-8l+4$, $4l^2-7l+3$. Let v be a vertex of degree Δ and u a vertex of degree δ. If $\Delta \geq 2l$ then either $G[N(v)]$ contains an edge, in which case G contains H_{2l-1} or $N(v)$ is an independent set, in which case $\overline{G}$ contains H_{2l-1}. We may therefore suppose that $\Delta \leq 2l-1$. If $\delta \leq 2l-4$ then $\Delta(\overline{G}) \geq 2l$ and one sees, by the argument just used, that G or $\overline{G}$ contains H_{2l-1} Hence we may assume that $\delta \geq 2l-3$. Suppose that $\delta = 2l-3$. Then $|T(u)| = 2l-1$. If $G[T(u)]$ is not K_{2l-1} then G contains H_{2l-1}. We may therefore suppose that $G[T(U)] = K_{2l-1}$. If some vertex of $T(u)$ is joined to some vertex of $N(u)$, G contains H_{2l-1} We may therefore suppose that no vertex of $T(u)$ is joined to any vertex of $N(u)$. The condition $\delta = 2l-3$ then ensures that $G[N(u)\cup\{u\}] = K_{2l-2}$. Then $e = \binom{2l-2}{2} + \binom{2l-1}{2} = 4l^2-8l+4$, one of the exceptions noted above. Since the union of K_{2l-2} and K_{2l-1} is such that neither it nor its complement contains H_{2l-1}, $f(4l^2-8l+4) = 0$. The only other case left is $\delta \geq 2l-2$. Then $e \geq \lceil\frac{1}{2}(4l-3)(2l-2)\rceil = 4l^2-7l+3$. It follows that $e = 4l^2-7l+3$ and that G is $(2l-2)$-regular. Since the graph whose vertex set consists of $4l-3$ equispaced points on the circumference of a circle and whose edge set is obtained by joining each point to its $2l-2$ nearest neighbors is such that neither it nor its complement contains H_{2l-1}, $f(4l^2-7l+3) = 0$. This complete the proof of (b). Note that we have shown that for $l \geq 3$, $S(H_{2l-1}, 4l-3) = (4l^2-8l+4, 1, l-2, 1, l-2, 1, 4l^2-8l+4)$. In case $l = 2$, the two exceptional values of e are consecutive and one gets $S(H_3, 5) = (4, 3, 4)$.

(c) Let $4 \leq a \leq 2l$ and let $n = 4l-a$. We show that $S(H_{2l-1}, n)$ has three intervals. Let $t = \lceil \frac{1}{2}n(n-2l+1) \rceil = 4l^2 - (3a-2)l + \frac{1}{2}a(a-1)$, Observe that, by Theorem 1, if $t \leq e \leq \frac{1}{2}\binom{n}{2}$, there exists a graph of order n and size e such that neither it nor its complement contains H_{2l-1} and hence $f(e) = 0$, We show now that for $0 \leq e < t$, $f(e) = 1$. Let G be a graph of order n with e edges. We must establish that G or $\overline{G}$ contains H_{2l-1}. Let u be a vertex of degree δ and v a vertex of degree Δ. We may suppose, as in (a) and (b), that $\Delta \leq 2l-1$ and, since $\delta \leq 2l-a-1$ implies $\Delta(\overline{G}) \geq 2l$, that $\delta \geq 2l-a$. If $\delta \geq 2l-a+1$ then $e \geq \lceil \frac{1}{2}(4l-a)(2l-a+1) \rceil = 4l^2 - (3a-2)l + \frac{1}{2}a(a-l) = t$, and this is false. Hence we may suppose that $\delta = 2l-a$. Note then that $|T(u)| = 2l-1$, If $G[T(u)]$ is not K_{2l-1} then $\overline{G}$ contains H_{2l-1}. We may therefore suppose that $G[T(u)] = K_{2l-1}$. If there is an edge from $T(u)$ to $N(u)$, G contains H_{2l-1}. We may therefore suppose that there are no edges from $T(u)$ to $N(u)$. The condition $\delta = 2l-a$ then ensures that $G[T(u) \cup \{u\}] = K_{2l-a+1}$. Thus $e = \binom{2l-a+1}{2} + \binom{2l-1}{2} = 4l^2 - (2a+2)l + \frac{1}{2}(a^2 - a + 2)$. The condition $e < t$ then becomes $(a-4)l < -1$. This is false since $a \geq 4$. Thus (c) is established.

We do not give the proofs of (d) and (e) since the arguments parallel closely those for (a) and (c).

We conclude with some remarks concerning $S(K_4, n)$ and explain how the information in Table 1 was obtained. Most of the work is done by others. For $4 \leq n \leq 18 = r(K_4)$, let $e(n)$ denote the least integer e for which $f(e) = 0$, The values of $e(n)$ for $n = 4, 5, \ldots, 17$ are 1,2,3,5,7,9,14,19,26,33,41,50,60, 68. For $4 \leq n \leq 13$ and $n = 17$, these values are given in the paper of Walker [3] and the references given there. Walker also showed that $e(14) \geq 40$, $e(15) \geq 49$ and $e(16) \geq 58$. Garcia (unpublished, cited in [2]) proved that $e(14) = 41$, $e(15) = 50$ and $e(16) = 60$. One may deduce from these results that $S(K_4, n)$ has three intervals. For example, in case $n = 14$, the graph in Figure 1 is one which establishes $e(14) \leq 41$. Neither it nor its complement contains K_4. One may add any subset of $\{ab, px, qy, rz\}$ without forcing a K_4. Thus $f(e) = 0$ for $41 \leq e \leq 45 = \frac{1}{2}\binom{14}{2}$. By Garcia's result, $f(e) = 1$ for $0 \leq e \leq 40$ and this suffices. Similar considerations apply for the other values of n.

References

[1] S. Burr, Diagonal Ramsey numbers for small graphs, *Journal of Graph Theory*, **7**(1983), 57–69.

[2] F.K.R. Chung and C.M. Grinstead, A survey of bounds for classical Ramsey numbers, *Journal of Graph Theory*, **7**(1983), 25–37.

[3] K. Walker, An upper bound for the Ramsey number $M(5, 4)$, *Journal of Combinatorial Theory*, **11**(1971), 1–10.

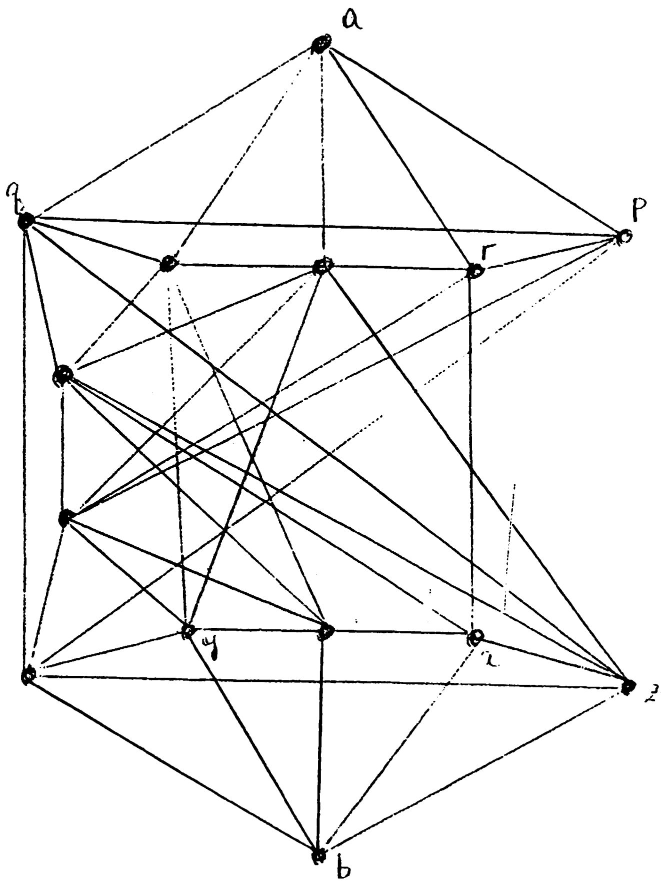


Figure 1