A Comparison of n-ary Design Definitions

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ABSTRACT. A generalization of (binary) balanced incomplete block designs is to allow a treatment to occur in a block more than once, that is, instead of having, blocks of the design as sets, allow multisets as blocks. Such a generalization is referred to as an n-ary design. There are at least three such generalizations studied in the literature. The present note studies the relationship between these three definitions. We also give some results for the special case when n is 3.

1 Introduction

A *balanced incomplete block design* (BIBD) with parameters (v, b, r, k, λ) is an arrangement of v treatments in b blocks (i.e. sets) each of size k, such that every treatment occurs 0 or 1 times in a block, every treatment occurs r times in the design and every pair of distinct treatments occurs λ times in the design. A BIBD (v, b, r, k, λ) can be represented by a $v \times b$ matrix, $N = [n_{ij}]$, called the *incidence matrix* of the design, where n_{ij} is 1 when treatment i (for some ordering of the treatments) is an element of block j (for some ordering of the blocks), and 0 when treatment i is not an element of block j. Each row sum of N is r, each column sum of N is k and the inner product of any two distinct rows of N is λ.

In 1952 K.D. Tocher [5] suggested the study of designs where elements could appear more than once in a block. His suggestion lead to what are today referred to as n-ary designs. (Here and throughout the paper n is assumed to be a positive integer greater than or equal to three.) Although in Tocher's original definition the requirement of constant replication was not required, today the most accepted definition (for example see [2]) of a *balanced n-ary design* with parameters (V, B, R, K, Λ) is an arrangement of V treatments in B blocks (i.e. multisets) each of size K, such that every treatment occurs $0, 1, 2, \ldots$,or $n-1$ times in a block, every treatment occurs R times in the design and every pair of distinct treatments occurs Λ times in the design. We refer to this definition for n-ary designs as Definition 1. For a Definition 1 n-ary design, the entries of the corresponding incidence matrix are 0's, 1's, ..., and $n-1$'s.

Example 1:

$$N = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

is the incidence matrix of a Definition 1 ternary design with parameters (3,3,3,3,2). Note in the case where $n = 3$ the design is referred to as a ternary design rather than a 3-ary design.

The literature contains at least two other interpretations of Tocher's suggestion to relax the restriction in a design on the number of times a treatment can appear in a block. In [4] an n-ary block design is defined to be a design whose incidence matrix has entries $m_0, m_1, \ldots, m_{n-1}$ where $0 \leq m_0 < m_1 < m_2 < \cdots < m_{n-1}$ and $m_i = i(m_1 - m_0) + m_0$. We refer to this definition for n-ary designs as Definition 2. Every Definition 1 n-ary design is a Definition 2 n-ary design.

Example 2:

$$N = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

is the incidence matrix of a Definition 2 ternary design with parameters (3,3,9,9,26).

The definition of n-ary design is sometimes taken even one step further (see for example [1] or [3]) and any set of n distinct nonnegative integers is allowed as the entries of the incidence matrix. We refer to n-ary designs of this type as Definition 3 n-ary designs. Every Definition 2 n-ary design is a Definition 3 n-ary design.

Example 3:

$$N = \begin{bmatrix} 3 & 5 & 2 \\ 5 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

is the incidence matrix of a Definition 3 ternary design with parameters (3,3,10,10,31).

The purpose of this paper is to show how the two "more general" definitions of n-ary designs (Definitions 2 and 3) relate to the standard definition (Definition 1).

Throughout the remainder of the paper n-ary design refers to any design whose corresponding incidence matrix contains exactly n distinct entries and design implies an incidence matrix with constant row sum, constant column sum and constant inner product of any two distinct rows.

We say two n-ary designs D_1, with incidence matrix N_1, and D_2, with incidence matrix N_2, are *equivalent* if there exists a one to one map f, from the entries of N_1 to the entries of N_2 such that a permutation of the rows and columns of $f(N_1)$ yields N_2. Equivalent n-ary designs have the same number of treatments and blocks, but their respective incidence matrices may well have different row sums, column sums and row inner product value. Examining the examples above we see that all examples are equivalent To show Example 1 equivalent to Example 2 let $f(0) = 4$, $f(1) = 3$, and $f(2) = 2$. To show Example 1 equivalent to Example 3 let $f(0) = 2$, $f(1) = 5$, and $f(2) = 3$.

2 The relationship of Definition 1 and 2 designs

The following theorems show that for every Definition 2 n-ary design there exists an equivalent Definition 1 n-ary design and vice versa. Thus, Definition 2 is not a true generalization of Definition 1.

Theorem 2.1. *Every Definition 1 n-ary design is equivalent to an infinite number of Definition 2 n-ary designs.*

Proof: Let D be a Definition 1 n-ary design with parameters (V, B, R, K, Λ) and incidence matrix $N_1 = [n_{ij}]$. Assume a is a positive integer and b a nonnegative integer. For $i = 0, 1, \ldots, n-1$ let $m_{ij} = n_{ij}a + b$. We claim $N_2 = [m_{ij}]$ is the incidence matrix of a Definition 2 n-ary design with parameters $(V, B, aR + bB, aK + bV, a^2\Lambda + 2abR + b^2B)$.

The replication number:

Since for $i = 1, 2, \ldots, V, \sum_{j=1}^{B} n_{ij} = R$, we get for

$$i = 1, 2, \ldots, V, \sum_{j=1}^{B}(n_{ij}a + b) = a\left(\sum_{j=1}^{B} n_{ij}\right) + bB = aR + bB.$$

The block size:

Since for $j = 1, 2, \ldots, B, \sum_{i=1}^{V} n_{ij} = K$, we get for

$$j = 1, 2, \ldots, B, \sum_{i=1}^{V}(n_{ij}a + b) = a\left(\sum_{i=1}^{V} n_{ij}\right) + bV = aK + bV.$$

The index:

Since for $i \neq j$ in $\{1, 2, \ldots, V\}$ we have $\sum_{m=1}^{B} n_{im} n_{jm} = \Lambda$, we get for

$$i \neq j \text{ in } \{1, 2, \ldots, V\} \sum_{m=1}^{B} (an_{im} + b)(an_{jm} + b) = a^2 \left(\sum_{m=1}^{B} n_{im} n_{jm} \right)$$

$$+ ab \left(\sum_{m=1}^{B} n_{im} \right) + ab \left(\sum_{m=1}^{B} n_{jm} \right) + b^2 B = a^2 \Lambda + 2abR + a^2 B.$$

□

Theorem 2.2. *Every Definition 2 n-ary design is equivalent to a Definition 1 n-ary design.*

Proof: Let D be a Definition 2 n-ary design with parameters (V, B, R, K, Λ) and incidence matrix N_1. From the definition of a Definition 2 n-ary design we see that N_1 can be written uniquely as $[n_{ij} a + b]$, where we are assuming that incidence matrix N_1 has entries $0 \leq b < b + a < \cdots < b + (n-1)a$ and that n_{ij} is in $\{0, 1, \ldots, n-1\}$. We claim $N_2 = [n_{ij}]$ is the incidence matrix of a Definition 1 n-ary design with parameters $(V, B, (R - bB)/a, (K - bV)/a, (\Lambda - 2abR - b^2 B)/a^2)$.

The replication number:

Since for $i = 1, 2, \ldots, V, \sum_{j=1}^{B} (n_{ij} a + b) = R$, we get for

$$i = 1, 2, \ldots, V, \sum_{j=1}^{B} n_{ij} = (R - bB)/a.$$

The value $(R - bB)/a$ is a positive integer since each n_{ij} is a nonnegative integer with at least one of them being greater than 0.

The block size:

Since for $j = 1, 2, \ldots, B, \sum_{i=1}^{V} (n_{ij} a + b) = K$, we get for

$$j = 1, 2, \ldots, B, \sum_{i=1}^{V} n_{ij} = (K - bV)/a.$$

The value $(K - bV)/a$ is a positive integer since each n_{ij} is a nonnegative integer with at least one of them being greater than 0.

The index:

Since for $i \neq j$ in $\{1, 2, \ldots, V\}$ we know $\sum_{m=1}^{B} (an_{im} + b)(an_{jm} + b) = \Lambda$, we get for

$$i \neq j \text{ in } \{1, 2, \ldots, V\} \sum_{m=1}^{B} n_{im} n_{jm} = (\Lambda - 2abR - b^2 B)/a^2.$$

This value is a nonnegative integer since each n_{im} and n_{jm} is a nonnegative integer. Since $n \geq 3$ and since we have shown above that column sums in N_2 are constant and positive, it must be the case that some column of N_2 has at least two nonzero entries. Thus, $(\Lambda - 2abR - b^2B)/a^2$ is in fact positive. □

3 The relationship of Definition 1 and 3 designs

We use Example 4 to show that Definition 3 is a true generalization of Definition 1 (i.e. we exhibit a Definition 3 design that is not equivalent to any Definition 1 design).

Example 4:

$$N = \begin{bmatrix} 2 & 3 & 3 & 2 & 3 & 3 & 2 & 0 & 3 & 3 & 3 & 0 \\ 2 & 3 & 3 & 2 & 0 & 3 & 3 & 3 & 0 & 2 & 3 & 3 \\ 3 & 3 & 0 & 2 & 3 & 3 & 2 & 3 & 3 & 2 & 0 & 3 \\ 2 & 0 & 3 & 3 & 3 & 0 & 2 & 3 & 3 & 2 & 3 & 3 \end{bmatrix}$$

is the incidence matrix of a Definition 3 ternary design with parameters (4,12,27,9,56). If there existed a one to one map f from $\{0,2,3\}$ to $\{0,1,2\}$ such that $f(N)$ were the incidence matrix of a Definition 1 ternary design we would have $3f(2) + f(3) = 3f(3) + f(0)$ as the block size in the Definition 1 design (for example see block 1 and block 2 of the original design), but $3f(2) + f(3) = 3f(3) + f(0)$ implies $3f(2) - f(0) = 2f(3)$. An exhaustive search of all possible f's shows that this latter equation can never be satisfied.

Before we show a class of Definition 3 designs which is equivalent to Definition 1 designs we introduce the concept of design regularity.

An n-ary design D with incidence matrix entries $a_0, a_1, \ldots, a_{n-1}$ is said to be *row regular* if there exist constants $\rho_0, \rho_1, \ldots, \rho_{n-1}$ such that each treatment v appears in exactly ρ_0 blocks a_0 times, in ρ_1 blocks a_1 times, ..., and in ρ_{n-1} blocks a_{n-1} times. Examples 1, 2, 3 and 4 are all row regular.

Similar to row regular we define an n-ary design D with incidence matrix entries $a_0, a_1, \ldots, a_{n-1}$ to be *column regular* if there exist constants $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that each block b consists of α_0 treatments that appear α_0 times, α_1 treatments that appear α_1 times, ..., and α_{n-1} treatments that appear α_{n-1} times. Examples 1, 2 and 3 are column regular; Example 4 is not.

Finally we say an n-ary design D with incidence matrix entries $a_0, a_1, \ldots,$ a_{n-1} is *index regular* if there exist constants $\beta_{0,0}, \beta_{0,1}, \ldots, \beta_{0,n-1}$; $\beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,n-1}$; $\ldots \beta_{n-1,n-1}$ such that for each inner product of distinct rows the inner product sum consists of a_0a_0 appearing $\beta_{0,0}$ times, a_0a_1 or a_1a_0 appearing $\beta_{0,1}$ times, ..., and $a_{n-1}a_{n-1}$ appearing $\beta_{n-1,n-1}$ times. That is,

β_{ij} denotes the number of times $a_i a_j$and $a_j a_i$ appear in the inner product of two distinct rows. Examples 1, 2, 3 and 4 are all index regular.

Example 5, given below, shows that an incidence matrix of a design can be row and column regular without being index regular. Example 6, given below, shows that an incidence matrix of a design can be row and index regular without being column regular. A design cannot be index regular without also being row regular (Proposition 3.1).

Example 5:

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 & 1 \end{bmatrix}$$

is the incidence matrix of a Definition 1 ternary design with parameters (5,5,5,5,4) that is row and column regular but not index regular (for example see the inner product of rows 1 & 2 and rows 3 & 4).

Example 6:

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

is the incidence matrix of a Definition 1 ternary design with parameters (3,4,4,3,3) that is row and index regular but not column regular (see columns 1 and 2).

Proposition 3.1. *If a Definition 3 n-ary design is index regular, then it is row regular.*

Proof: Let N be the incidence matrix of an n-ary design with parameters (V, B, R, K, Λ) which is index regular. Assume that $a_0, a_1, \ldots, a_{n-1}$ are the entries of N, and let β_{ij} denote the number of times $a_i a_j$ or $a_j a_i$ occurs in the inner product of any two distinct rows of N. Also, for $m = 1, 2, \ldots, V$ let τ_m denote the number of times a_0 occurs in the mth row of N; then for distinct r and s in $\{1, 2, \ldots, V\}$, $\sum_{j=1}^{n-1} \beta_{0j} = \tau_r + \tau_s - 2\beta_{00}$. Hence if r, s, t are distinct elements of $\{1, 2, \ldots, V\}$, then $\tau_r + \tau_s - 2\beta_{00} = \sum_{j=1}^{n-1} \beta_{0j} = \tau_s + \tau_t - 2\beta_{00}$ which implies that $\tau_r = \tau_t$. Thus, a_0 appears the same number of times in each row of N. In arguments similar to the one given above we can show each a_m for $m = 0, 1, \ldots, n-1$ appears the same number of times in each row of N. Therefore, N is row regular. □

We say a design is *triply regular* if it is row, column and index regular.

Theorem 3.2. *If D is a triply regular n-ary design and $\{b_0, b_1, \ldots, b_{n-1}\}$ is any set of n distinct nonnegative integers, then D is equivalent to an n-ary design whose incidence matrix entries are the elements of $\{b_0, b_1, \ldots, b_{n-1}\}$.*

Proof: Let D be a triply regular n-ary design with parameters (V, B, R, K, Λ) and incidence matrix N, where the entries of N are elements of the set $\{a_0, a_1, \ldots, a_{n-1}\}$. Since D is row regular there exist constants $\rho_0, \rho_1, \ldots, \rho_{n-1}$ such that each treatment appears in exactly ρ_0 blocks a_0 times, ρ_1 blocks a_1 times, ..., and ρ_{n-1} blocks a_{n-1} times. Similarly since D is also column and index regular we can find constants $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ that correspond to the column regularity of D and constants $\beta_{0,0}, \beta_{0,1}, \ldots, \beta_{0,n-1}$; $\beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,n-1}$; $\ldots \beta_{n-1,n-1}$ that correspond to the index regularity of D.

Let V and B be as defined for D and define constants R_1, K_1, and Λ_1 as follows:

$$R_1 = \sum_{i=0}^{n-1} \rho_i b_i, K_1 = \sum_{i=0}^{n-1} \alpha_i b_i, \text{ and } \Lambda_1 = \sum_{i=0}^{n-1} \sum_{j \geq i}^{n-1} \beta_{ij} b_i b_j.$$

For $i = 0, 1, \ldots, n-1$ let $f(a_i) = b_i$. The entries of the matrix $f(N) = [f(n_{ij})]$ are elements of the set $\{b_0, b_1, \ldots, b_{n-1}\}$. Since N is row regular, each row sum of $f(N)$ will be $R_1 = \sum_{i=0}^{n-1} \rho_i b_i$. Since N is column regular, each column sum of $f(N)$ will be $K_1 = \sum_{i=0}^{n-1} \alpha_i b_i$. Since N is index regular, the inner product of any two distinct rows of $f(N)$ will be $\Lambda_1 = \sum_{i=0}^{n-1} \sum_{j \geq i}^{n-1} \beta_{ij} b_i b_j$. Thus, $f(N)$ is the incidence matrix of an n-ary design with parameters $(V, B, R_1, K_1, \Lambda_1)$. □

Corollary 3.3. *Every triply regular Definition 1 n-ary design is equivalent to an infinite number of Definition 3 n-ary designs.*

Corollary 3.4. *Every triply regular Definition 3 n-ary design is equivalent to a Definition 1 n-ary design.*

As illustrated below in Example 7, Definition 3 designs do not have to be triply regular to be equivalent to Definition 1 designs.

Example 7:

$$M = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & 5 \\ 5 & 0 & 2 & 3 \\ 3 & 5 & 0 & 2 \\ 2 & 3 & 5 & 0 \end{bmatrix}$$

and

$$N = \begin{bmatrix} r_1r_1r_1 & r_2r_2r_2 & r_3r_3r_3 & r_4r_4r_4 & 555000 \\ r_2r_3r_4 & r_1r_3r_4 & r_2r_1r_4 & r_2r_3r_1 & 500550 \\ r_3r_4r_2 & r_3r_4r_1 & r_1r_4r_2 & r_3r_1r_2 & 050505 \\ r_4r_2r_3 & r_4r_1r_3 & r_4r_2r_1 & r_1r_2r_3 & 005055 \end{bmatrix}.$$

N is the incidence matrix of a 4-ary Definition 3 design. N is not column regular (see columns 1 and 49).

Let f be the map from $\{0,2,3,5\}$ to $\{0,1,2,3\}$ such that $f(0) = 0$, $f(2) = 1$, $f(3) = 2$ and $f(5) = 3$. The matrix $f(N)$ is the incidence matrix of a 4-ary Definition 1 design. Thus, N and $f(N)$ are two non-triply regular 4-ary designs that are equivalent.

4 Some ternary results

We conclude by presenting several interesting results about regularity that are true for ternary designs. The first is a generalization of a result of Billington [2]. The others strongly tie row and column regularity to equivalence. We begin with a definition. Although we will only use the definition as applied to ternary designs we present it in its general form.

We say a Definition 3 n-ary design is *reduced* if zero is one of the n distinct elements of the incidence matrix of the design. Given the incidence matrix of an n-ary design with parameters (V, B, R, K, Λ) we can "reduce" the design by simply subtracting the minimum element b of the incidence matrix from each element of the incidence matrix. The reduced design will have row sum $R - bB$, column sum $K - bV$, and index $\Lambda - 2bR - b^2B$. We do not claim that each Definition 3 n-ary design is equivalent to a reduced Definition 3 n-ary design. The reason for this is that in "reducing" a design we might be changing a row or column of the original incidence matrix to the zero vector.

Theorem 4.1. *Every reduced Definition 3 ternary design is row regular.*

Proof: Let D be a reduced Definition 3 ternary design with parameters (V, B, R, K, Λ) and with incidence matrix $N = [n_{ij}]$ where each n_{ij} is an element of $\{0, a_1, a_2\}$.

Assume x is a treatment of D and assume x appears a_1 times in ρ_{x1} blocks and a_2 times in ρ_{x2} blocks. Then $\sum_{j=1}^{B}(n_{xj})^2 = a_1^2\rho_{x1} + a_2^2\rho_{x2}$.

Counting the number of ordered pairs in which x appears we get

$$\begin{aligned}\Lambda(V-1) &= \sum_{i=1}^{2} a_i\rho_{xi}(K - a_i) \\ &= \sum_{i=1}^{2} a_i\rho_{xi}K - \sum_{i=1}^{2} a_i^2\rho_{xi} = RK - \sum_{i=1}^{2} a_i^2\rho_{xi},\end{aligned}$$

which implies that

$$RK - \Lambda(V-1) = \sum_{i=1}^{2} a_i^2\rho_{xi}.$$

Hence,

$$RK - \Lambda(V-1) = a_1 R = \sum_{i=1}^{2} a_i^2 \rho_{xi} - a_1 R$$
$$= \sum_{i=1}^{2} a_i^2 \rho_{xi} - a_1(a_1\rho_{x1} + a_2\rho_{x2}) = (a_2^2 - a_1 a_2)\rho_{x2}.$$

Solving for ρ_{x2} we see that ρ_{x2} is the constant $(RK - \Lambda(V-1) - a_1 R)/(a_2^2 - a_1 a_2)$. The result follows. □

Theorem 4.2. *If a ternary Definition 3 design is equivalent to a Definition 1 design, then the Definition 3 design is either row regular or a Definition 2 design.*

Proof: Let D_3 be a ternary Definition 3 design with incidence matrix entries a_0, a_1, a_2. Assume $f(a_i) = i$, $i = 0, 1, 2$ is an equivalence map from D_3 to a Definition 1 design.

Let each treatment occur R_3 times in D_3 and let x and y be two distinct treatments of D_3; then there exist constants $\rho_{x0}, \rho_{x1}, \rho_{x2}$ and $\rho_{y0}, \rho_{y1}, \rho_{y2}$ such that

$$R_3 = \sum_{i=0}^{2} \rho_{xi} a_i = \sum_{i=0}^{2} \rho_{yi} a_i$$

which implies

$$0 = \sum_{i=0}^{2} (\rho_{yi} - \rho_{xi}) a_i. \tag{1}$$

Since f is an equivalence map it is then also the case that

$$0 = \sum_{i=0}^{2} (\rho_{yi} - \rho_{xi}) f(a_i).$$

Thus,

$$(\rho_{x1} - \rho_{y1}) = 2(\rho_{y2} - \rho_{x2}). \tag{2}$$

If B_3 is the number of blocks in D_3, then

$$B_3 = \sum_{i=0}^{2} \rho_{xi} = \sum_{i=0}^{2} \rho_{yi}.$$

Combining this fact with (2) we get

$$(\rho_{x0} - \rho_{y0}) = -(\rho_{y2} - \rho_{x2}). \tag{3}$$

Substituting (2) and (3) into (1) gives us $(\rho_{y2} - \rho_{x2})(-a_0 + 2a_1 - a_2) = 0$. Hence, $(\rho_{y2} - \rho_{x2}) = 0$ or $(-a_0 + 2a_1 - a_2) = 0$.

If $\rho_{y2} - \rho_{x2}$ is zero, then $\rho_{y0} = \rho_{x0}$ and $\rho_{y1} = \rho_{x1}$ and it follows that D_3 is row regular.

If $(-a_0 + 2a_1 - a_2) = 0$, then $a_0 + a_2 = 2a_1$, and it follows that D_3 is a Definition 2 design. □

Theorem 4.3. *If a ternary Definition 3 design is equivalent to a Definition 1 design, then the definition 3 design is either column regular or a Definition 2 design.*

Proof: Similar to the proof of Theorem 4.2. □

Corollary 4.4. *If a ternary Definition 3 design which is not a Definition 2 design is equivalent to a Definition 1 design, then both the designs are row and column regular.*

Proof: See Theorem 4.2 and 4.3. □

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