On the thickness of graphs with genus 2

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ABSTRACT. By a graph we mean an undirected simple graph. The *genus* $\gamma(G)$ of a graph G is the minimum genus of the orientable surface on which G is embeddable. The *thickness* $\Theta(G)$ of G is the minimum number of planar subgraphs whose union is G.

In [1], it is proved that, if $\gamma(G) = 1$, then $\Theta(G) = 2$. If $\gamma(G)=2$, the known best upper bound on $\gamma(G)$ is 4 and, as far as the author knows, the known best lower bound is 2. In this paper, we prove that, if $\gamma(G) = 2$, then $\Theta(G) \leq 3$.

1 Introduction

By a graph we mean an undirected simple graph. The genus $\gamma(G)$ of a graph G is the minimum genus of the orientable surface on which G is embeddable. The thickness $\Theta(G)$ of G is the minimum number of planar subgraphs whose union is G. The *arboricity* of G is the minimum number of forests whose union is G.

A graph with genus g has the arboricity at most $2 + \sqrt{3g}$ by [5]. Hence the thickness of a graph with fixed genus g has an upper bound. Recently, Dean and Hutchinson [3] proved the inequality,

$$\Theta(G) \leq 5 + \sqrt{2\gamma(G) - 2},$$

which is best possible up to a constant.

In [1], it is proved that, if $\gamma(G) = 1$, then $\Theta(G) = 2$. If $\gamma(G) = 2$, the known best upper bound on $\Theta(G)$ is 4 and, as far as the author knows, the known best lower bound is 2. For example, the complete graph K_n and the complete bipartite graph K_{mn} with genus 2 has the thickness 2, since

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$\Theta(K_8) = \Theta(K_{4,6}) = \Theta(K_{3,10}) = 2$, see [6]. In this paper, we prove that, if $\gamma(G) = 2$, then $\Theta(G) \leq 3$.

Let Σ be an orientable surface of genus g and G a graph of genus g embedded in Σ. The surface Σ' obtained from Σ by cutting along a nonseparating cycle C in G and pasting two disks along the boundaries is of genus $g - 1$. In § 3, we show that there is a subgraph H with $\Theta(H) \leq 2$, which is called a *collar* of C, such that $G - E(H)$ is embeddable in Σ', where $E(H)$ is the edge set of H.

In § 4, we consider the embedding of G with $\gamma(G) = 2$, and, show that, if a collar H of a nonseparating cycle is nonplanar, $G - E(H)$ is planar.

2 Preliminaries

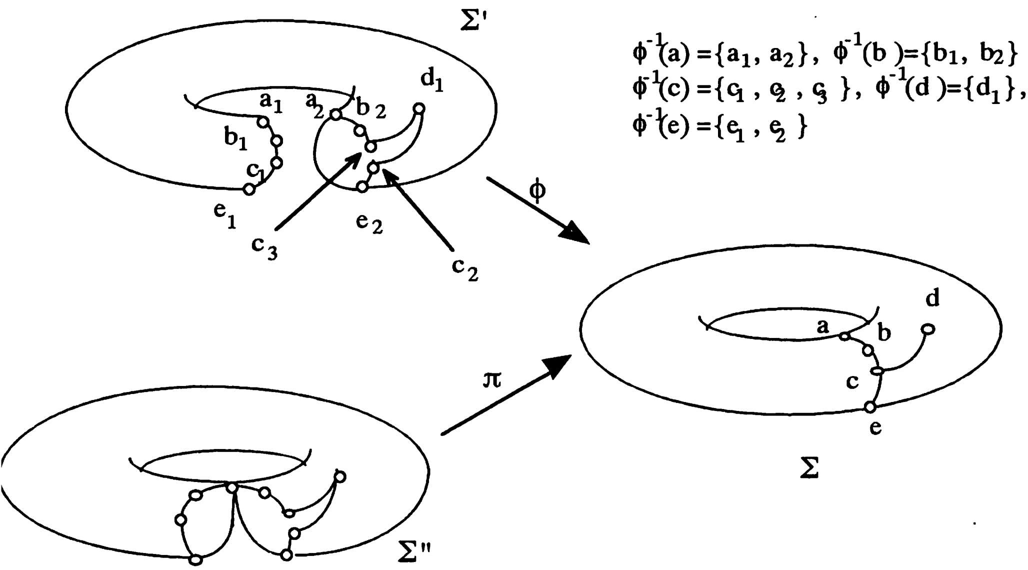
In this article, a graph is considered as a one-complex. A *surface* is a two-complex in which a neighborhood of each point is homemorphic to the Euclidean plane R^2 or $R^2_+ = \{(x, y) \in R^2 : y \geq 0\}$.

Let G be a graph embedded in the surface Σ with the vertex set $V(G)$ and the edge set $E(G)$. Cutting Σ along a subgraph H of G in which the degree $d_H(v)$ of any vertex v is at least 1, we obtain a surface Σ'. Then Σ can be considered to be an identification space of Σ'. The preimages of $e \in E(H)$ and $v \in V(H)$ under the identification map $\phi\colon \Sigma' \to \Sigma$ consistof two edges and $d_H(v)$ vertices, respectively.

Let V' be a subset of $V(H)$. By Σ'', we denote the complex obtained from Σ' by identifying the vertices in $\phi^{-1}(V')$ under $\phi|\phi^{-1}(V')$, see figure 1, for example. If there is a vertex v in V' with $d_H(v) \geq 2$, Σ'' is not a surface. Using the map ϕ, we obtain a map π from Σ'' onto Σ such that $\pi|\pi^{-1}(\Sigma - ((V(H) - V') \cup E(H)))$ is one-to-one. We call Σ'' and $\phi^{-1}(G)$ the *results of cutting* Σ and G *along* $(V(H) - V') \cup E(H)$, respectively. For $v \in V(H) - V'$, the vertices in $\pi^{-1}(v)$ will be denoted by $v^{(1)}, v^{(2)}, \ldots, v^{(n)}$. For convenience, we write v for $\pi^{-1}(v)$ of a vertex $v \in V(G)$, if $|\pi^{-1}(v)| = 1$.

The length of a path or a cycle P is denoted by $\ell(P)$. A *chord* of a cycle C in G is an edge which does not belong to C but joins vertices of C. For distinct vertices u and v, we will denote the section of C from u to v which follows the orientation of C by $C[u, v]$. If a cycle C in $G \subset \Sigma$ separates Σ, we say that C is *separating*. Otherwise, C is nonseparating. A path P connecting two vertices u and v in C is said to be *separating relative* to C, if $P \cup C[u, v]$ or $P \cup C[v, u]$ is separating.

A *bridge* B of C in G is either a chord together with both ends, or a connected component B' of $G - V(C)$ together with all edges from B' to C and all ends of these edges. For the definition of a bridge, we refer to [2]. The vertices in $V(B) \cap V(C)$ are called *attachments* of B. A bridge with k attachments is called a *k-bridges*. A 1-bridge B is said to be *trivial*, if $|E(B)| = 1$.



V'={a} $\quad \pi^{-1}(a) = \{a^{(1)}\}$, $\pi^{-1}(b) = \{b^{(1)}, b^{(2)}\}$
$\pi^{-1}(c) = \{c^{(1)}, c^{(2)}, c^{(3)}\}$, $\pi^{-1}(d) = \{d^{(1)}\}$,
$\pi^{-1}(e) = \{e^{(1)}, e^{(2)}\}$

Figure 1

We say that two bridges B_1, and B_2 *overlap* if at least one of the following two conditions holds:

(1) There are two attachments v_1 and v_2 of B_1 and two attachments v_3 and v_4 of B_2 such that all of four are distinct and they appear on C in the order v_1, v_3, v_2, v_4.

(2) There are three attachments common to B_1, and B_2. B_1 *avoids* B_2, if all the attachments of B_1 lie between two consecutive attachments of B_2. If B_1 and B_2 do not avoid, they overlap [2].

3 Collar of a cycle

Let G be a graph embedded in the closed surface Σ. Suppose that a cycle C in G has no chord. We denote the results of cutting Σ and G along $V(C) \cup E(C)$ by $\tilde{\Sigma}$ and $\tilde{G}$, respectively. Then $\tilde{\Sigma}$ has two boundaries $C^{(0)}$

and $C^{(1)}$. Let E' be the image of $\{e \in E(\tilde{G})\colon e$ is incident to a vertex in $V(C^{(0)})\}$ under the identification map $\pi\colon \tilde{\Sigma} \to \Sigma$. The subgraph H of G induced by E' is called a collar of C.

From the definition, a bridge of C in a collar H has exactly one vertex which does not belong to C. Hence H is planar, if all the bridges in H avoid each other.

The embedding of G in the orientable surface Σ of $g(\Sigma)$ is said to be minimal, if $\gamma(G) = g(\Sigma)$, where $g(\Sigma)$ denotes the genus of Σ. If $G \subset \Sigma$ is minimal, there is a nonseparating cycle in G [4]. We cut Σ along C and paste two disks along the boundaries and denote the resulting surface by Σ'. Let H be a collar of C. Then $g(\Sigma') = g(\Sigma) - 1$ and $G - E(H)$ is embeddable in Σ'. Hence we have the following.

Theorem 1. *Let $G \subset \Sigma$ be a minimal embedding. Then there exists a sequence of minimal embedding*

$$G_0 \subset \Sigma_0, G_1 \subset \Sigma_1, \dots G_n \subset \Sigma_n,$$

having the following properties.

(1) $G_0 = G$, $\Sigma_0 = \Sigma$.

(2) $g(\Sigma_{i+1}) < g(\Sigma_i)$ and $g(\Sigma_n) = 0$.

(3) For $0 \leq i \leq n-1$, there is a shortest nonseparating cycle C_i in G_i such that $G_{i+1} = G_i - E(H_i)$, where H_i is a collar of C_i.

In [1], we proved H is planar, if G has no triangle. However a collar is not always planar if G has a triangle, for example see Remark 1 in [1]. We next consider the properties of a collar of a shortest nonsepararing cycle. For nonnegative integers p and q, we define graph G_{pq} to be the union of a 4-cycle $v_0v_1v_2v_3v_0$, p (v_0, v_2)-paths and q (v_1, v_3)-paths of length 2. Let $G'_{pq} = G_{pq} \cup B$ and $G''_{pq} = G'_{pq} \cup B'$, where B is a 3-bridge with attachments $\{v_0, v_1, v_2\}$ and B' a 3-bridge with attachments $\{v_0, v_2, v_3\}$. Then G_{pq}, G'_{pq}, and G''_{pq} are planar, see figure 2.

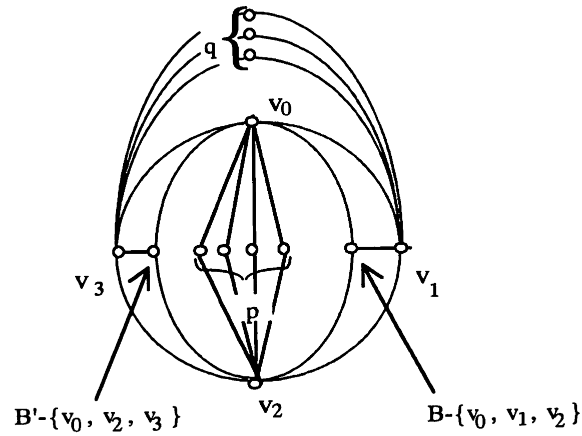
Theorem 2. *Let C be a shortest nonseparating cycle in a minimal embedding $G \subset \Sigma$. Then a collar H of C has one of the following properties.*

(1) All the bridges of H avoid each other.

(2) H is isomorphic to G_{pq}, G'_{pq}, or G''_{pq} with trivial bridges.

(3) $\ell(C) = 3$ and there are at least two 3-bridges of C in H.

It is easy to see that H is planar if H satisfies (1) or (2). Before proving Theorem 2, we will show that $\Theta(H) = 2$, if H satisfies (3). Let v be a



G''_{pq}

Figure 2

vertex of $V(C)$ and H_v the subgraph of H induced by the set of all edges incident to v. Then both H_v and $H - E(HV)$ are planar, hence, $\Theta(H) \leq 2$.

From now on, we assume $G \subset \Sigma$ is a minimal embedding, and H is a collar of a shortest nonseparating cycle C. The following two lemmas are proved in [1].

Lemma 3 [1, Lemma 4]. *Let u_1, u_2, v_1, v_2 be distinct vertices in C which appear on C in the order u_1, u_2, v_1, v_2.*

Suppose that there is a (u_i, v_i)-path P_i with $P_i \cap C = \{u_i, v_i\}$, $i = 1, 2$, and $P_1 \cap P_2 = \emptyset$. Then both P_1 and P_2 are nonseparating relative to C.

Lemma 4 [1. Lemma 5]. *Suppose that a bridge of H contains a (u, v)-path P of length 2. If $P \cup C[u, v]$ is nonseparating, then $\ell(C[v, u]) \leq 2$.*

Proof of Theorem 2: Since C is a shortest nonseparating cycle, there is no chord of C. Let $\{B_1, \ldots, B_n\}$ be the set of all bridges in H and let $\{x_i\} = V(B_i) - V(C)$, $1 \leq i \leq n$.

First we suppose that every path in every bridge joining two vertices in C is separating relative to C. In this case, we will show that the bridges of H avoid each other. To do this, we assume that B_i and B_j overlap. From Lemma 3, there are three attachments u_1, u_2, and u_3 common to B_i and B_j. Then two of the three cycles $x_i[u_1, u_2]x_i$, $x_iC[u_2, u_3]x_i$, and $x_iC[u_3, u_1]x_i$ are separating. We may assume $x_iC[u_1, u_2]x_i$ and $x_iC[u_2, u_3]x_i$ bound submanifolds D_1 and D_2 of Σ, respectively. From the construction of H, the edge x_ju_2 is contained in $D_1 \cup D_2$. This contradicts the fact that B_j has three attachments u_1, u_2, and u_3.

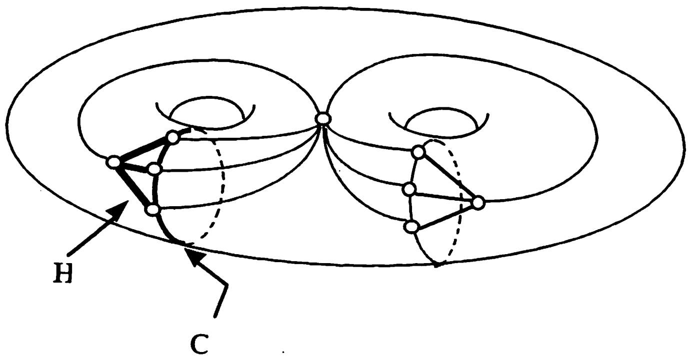


Figure 3

Second we suppose that there is a (u, v)-path P in a bridge such that $P \cap C = \{u, v\}$ and P is nonseparating relative to C. Then, since both $P \cup C[u, v]$ and $P \cup C[v, u]$ are nonseparating, we have $\ell(C) \leq 4$ from Lemma 4.

For the case $\ell(C) = 4$, we let $C = v_1v_2v_3v_4v_1$. If there is a 4-bridge B_i, it can be shown that the 3-cycles $x_iv_1v_2x_i$, $x_iv_2v_3x_i$, $x_iv_3v_4x_i$ and $x_iv_4v_1x_i$ are separating from the minimality of C. This contradicts the fact that C is nonseparating. Hence C has no 4-bridge.

Assume there are two 3-bridges B_i and B_j. Suppose that $\{v_1, v_2\} \subset V(B_i) \cap V(B_j)$. Then $x_iv_1v_2x_i$ and $x_jv_1v_2x_j$ are separating. Since there is a path from v_3 to x_i not intersectiong $x_jv_1v_2x_j$, and a path from v_3 to x_j not intersection $x_iv_1v_2x_i$, we have a contradiction. Hence $V(B_i) \cap V(B_j)$ is a pair of nonconsecutive vertices in C. Thus there are at most two 3-bridges in H, and H satisfies (1) or (2).

For the case $\ell(C) = 3$, it can be seen easily that H satisfies (1) or (3). This completes the proof of Theorem 2.

4 Graphs of Genus 2

In this section we consider a minimal embedding $G \subset \Sigma$ of a graph G of genus 2. If H is a collar of a shortest nonseparating cycle C, we have $\gamma(G - E(H)) \leq 1$ by Theorem 1. It is easy to see that $\gamma(G - E(H)) = 1$ for G and C in Figure 3.

In Figure 3, H is planar, however a collar is not always planar, for example, see Remark 1[1]. We will prove

Theorem 5. *Let $G \subset \Sigma$ be minimal embedding of graph G of genus 2. If a collar H of a shortest nonseparating cycle C is nonplanar, $G - E(H)$ is* *planar.*

Proof: Cutting Σ and G along $V(C) \cup E(C)$, we obtain the surface $\tilde{\Sigma}$ and the graph G'. Let π be the projection. From the construction of a collar H, there is the subgraph H' of G' such that $\pi(H') = H$ and H' is isomorphic to H. $\pi^{-1}(C)$ consists of two cycles C_0 and C_1, where C_0 is contained in H' and C_1 is disjoint from H'. By Σ', we denote the surface obtained from $\tilde{\Sigma}$ by pasting two disks D_0 and D_1, along C_0 and C_1.. The subgraph $G' - E(H') - V(C_0) - E(C_1)$, which is denoted by K, is isomorphic to $G - E(H)$.

Suppose that H is nonplanar. Then the collar H satisfies (3) in Theorem 2 and $\Theta(H) \leq 2$. Since H' is isomorphic to H, H' has three 3-bridges B_1, B_2 and B_3. Let $\{v_1, v_2, v_3\} = V(C_0)$ and $\{x_i\} = V(B_i) - V(C_0)$, $i = 1, 2, 3$. We consider the rotation scheme for the graph $C_0 \cup B_1 \cup B_2 \cup B_3$ in Σ'. (For the definition of the rotation scheme, we refer to [6].) For this purpose, we choose an orientation for Σ', which will be called the counter clockwise orientation, i.e., the rotation $\rho(v)$ for each vertex v is the cyclic permutation $(u_1, u_2, \ldots, u_p)$ of the adjacent vertices such that the edges $vu_1, vu_2, \ldots, vu_p$ appear in the counter clockwise order around v. We assume that the vertices of $V(C_0)$ appear on C_0 in the order v_1, v_2, v_3, if we follow C_0 in the counter clockwise direction with respect to D_0.

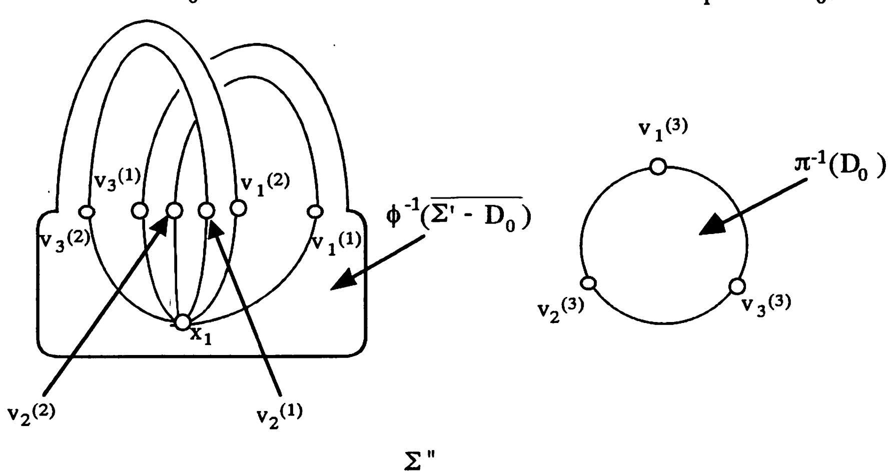


Figure 4

First, we consider the case that $\rho(x_i) = (v_1, v_2, v_3)$, for $i = 1, 2$ or 3. Suppose that $\rho(x_1) = (v_1, v_2, v_3)$. Let Σ'' be the result of Σ' cutting along $V(C_0) \cup E(C_0 \cup B_1)$ and $\phi\colon \Sigma'' \to \Sigma'$ the identification map. The vertices in $\phi^{-1}(V(C_0))$ are denoted by $v_i^{(j)}$ as shown in Figure 4. By Euler's formula, the region $\phi^{-1}\overline{(\Sigma' - D_0)}$ must be a disk for Σ' to have genus 1.

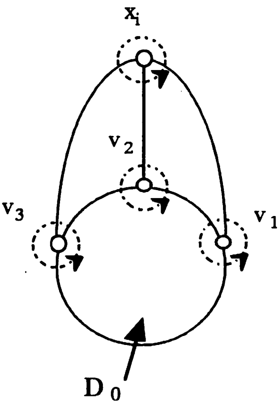


Figure 5

Since K is disjoint from $V(C_0) \cup E(C_0 \cup B_1)$, $\phi^{-1}(K)$ is isomorphic to K. We paste three disks along three cycles $x_i v_1^{(1)} v_2^{(2)} x_1$, $x_1 v_1^{(2)} v_3^{(1)} x_1$ and $x_1 v_2^{(1)} v_3^{(2)} x_1$. Then the resulting surface, in which $\phi^{-1}(K)$ is embedded, is the disjoint union of a 2-sphere and a disk $\phi^{-1}(D_0)$. Therefore K is planar.

Second we consider the case that $\rho(x_i) = (v_1, v_3, v_2)$ for $i = 1, 2, 3$.

As it can be seen in Figure 5, $\{v_2, v_3\}$, $\{v_3, v_1\}$ and $\{v_1, v_2\}$ appear consecutively in the order v_2, v_3 in $\rho(v_1)$, v_3, v_1 in $\rho(v_2)$ and v_1, v_2 in $\rho(v_3)$. Hence we may assume $\rho(v_1) = (x_1, x_2, x_3, v_2, v_3)$, without loss of generality. Then there are 36 possibilities for the rotation scheme of $C_0 \cup B_1 \cup B_2 \cup B_3$ in Σ'. Using the fact that $g(\Sigma') = 1$, we can see that the only possibility is

$$\begin{aligned}
\rho(x_1) &= (v_1, v_3, v_2), \\
\rho(x_2) &= (v_1, v_3, v_2), \\
\rho(x_3) &= (v_1, v_3, v_2), \\
\rho(v_1) &= (x_1, x_2, x_3, v_2, v_3), \\
\rho(v_2) &= (x_3, x_1, x_2, v_3, v_1), \\
\rho(v_3) &= (x_2, x_3, x_1, v_1, v_2).
\end{aligned}$$

Actually, this scheme has six orbits $O_1 = v_1 v_3 v_2 v_1$, $O_2 = x_1 v_3 v_1 x_1$, $O_3 = x_2 v_2 v_3 x_2$, $O_4 = x_3 v_1 v_2 x_3$, $O_5 = x_1 v_1 x_2 v_3 x_3 v_2 x_1$ and $O_6 = x_1 v_2 x_2 v_1 x_3 v_3 x_1$, and each of these orbits must bound a disk for Σ' to have genus 1. Every other scheme has two or four orbits.

Let F_i be the face of $C_0 \cup B_1 \cup B_2 \cup B_3$ in Σ' whose boundary is O_1, $1 \le i \le 6$. Define K_i to be a $K \cap \overline{F_i}$ for $1 \le i \le 6$. Then each K_i is planar, K_1 is empty, and K_2, K_3 and K_4 are disjoint, with each joined to $K_5 \cup K_6$ at a single vertex. Thus, it suffices to prove that $K_5 \cup K_6$ is planar.

$V(K_5) \cap V(K_6) = \{x_1, x_2, x_3\}$, and each K_5 and K_6 has a planar embedding with three vertices on the boundary of the outer face, so $K_5 \cup K_6$ is planar.

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