PERMUTATIONS WITH UNIQUE FIXED AND REFLECTED POINTS

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Abstract: We examine permutations having a unique fixed point and a unique reflected point; such permutations correspond to permutation matrices having exactly one 1 on each of the two main diagonals. The permutations are of two types according to whether or not the fixed point is the same as the reflected point. We also consider permutations having no fixed or reflected points; these have been enumerated using two different methods, and we employ one of these to count permutations with unique fixed and reflected points.

Introduction

The original inspiration for this work was a coin problem published by Henry Ernest Dudeney, which we discovered on page 249 of [2]. Dudeney challenges the reader to arrange twenty pennies in a square so that there is the same number of pennies in each row and each column and along each of the two main diagonals. The solution is obtained by placing sixteen pennies in a square and making an appropriate choice of four of them on top of which to put the remaining four. Evidently an appropriate choice corresponds to a 4×4 permutation matrix with exactly one 1 on the main diagonal and one 1 on the main antidiagonal. We consider the problem of enumerating the $n \times n$ permutation matrices having exactly one 1 on each of the main diagonals. We also examine a related problem, that of enumerating permutation matrices with no 1 on either of the main diagonals; this has been solved using at least two different methods, one of which we shall apply to the former problem.

Let $[n]$ denote the set $\{1, \ldots, n\}$ and $\mathfrak{S}_n$ the set of permutations of $[n]$. A *fixed point* of $\sigma \in \mathfrak{S}_n$ is an $i \in [n]$ such that $\sigma(i) = i$; a *reflected point* of σ is a $j \in [n]$ such that $\sigma(n-j+1) = j$. In other words, i is fixed if it is the ith letter of the word $\sigma(1)\sigma(2)\ldots\sigma(n)$ read left-to-right; j is reflected if it is the jth letter of this word read right-to-left. Notice that if $P = [p_{ij}]$ is the permutation matrix corresponding to σ, i.e., if $p_{ij} = \delta_{j\sigma(i)}$, then each fixed point of σ corresponds to a 1 on the main diagonal of P, and each reflected point corresponds to a 1 on the main antidiagonal. We denote by

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X_n the number of $\sigma \in \mathfrak{S}_n$ having no fixed points and no reflected points, and by Σ_n the number of $\sigma \in \mathfrak{S}_n$ having exactly one fixed point and one reflected point.

Our purpose is to determine X_n and Σ_n for arbitrary $n \geq 0$. We begin with X_n, which counts permutation matrices having no 1 on either the main diagonal or the main antidiagonal. Formulas for X_n have been derived by S. Hertzsprung [1], using derangement numbers, and by J. Riordan [4], who derives a recurrence using rook theory. We shall look at Hertzsprung's approach first.

The Work of Hertzsprung

Hertzsprung's paper appeared in 1879, well before the development of rook theory as a method for enumeration of permutations with restricted positions. The paper begins with the solution of the problem: How many terms appear in the formula for the determinant of an $n \times n$ matrix which has only 0s on its main diagonal? The answer, as is well-known now, is the derangement number $D_n := n! \sum_{k=0}^{n} (-1)^k/k!$, which also happens to be the integer nearest to $n!/e$. Hertzsprung then asks: How many terms appear in the formula for the determinant of an $n \times n$ matrix with only 0s on both the main diagonal and the main antidiagonal? Clearly, the number of terms in this formula is X_n.

The key observation allowing Hertzsprung to arrive at his formula is that for the permutations being counted, "no position is forbidden to more than two numbers, and two [distinct] positions forbidden to the same number are also both forbidden to a certain other number. When these conditions hold, it is immaterial whether the forbidden positions are stated for one number or for the other." He considers permutations with each position forbidden to either one or two numbers, and finds a formula for the number of such permutations in terms of the derangement numbers. Next he observes that if i and j have a forbidden position in common, then $i + j = n + 1$. The problem then splits into two cases, because if n is odd there is a position with only one forbidden number, but if n is even all positions are forbidden to two numbers. His formulas for X_n are as follows:

$$X_{2m} = \sum_{j=0}^{\lfloor m/2 \rfloor} 2^{m-2j} \frac{m!}{j!^2(m-2j)!} (\delta^j D_{m-2j})^2;$$

$$X_{2m+1} = \sum_{j=0}^{\lfloor m/2 \rfloor} 2^{m-2j+1} \frac{m!}{j!^2(m-2j)!} \delta^j D_{m-2j} \delta^j D_{m-2j+1},$$

where $\delta^j D_k := \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i} D_{k+2i}$.

Beginning with $n = 0$, we find the first few values of X_n to be 1, 0, 0, 0, 4, 16, 80, 672, 4752, 48768, 440192. We may also obtain these values from the following recurrence formula, stated by Hertzsprung:

$$X_n = (n-1)X_{n-1} + \begin{cases} 2(n-2)X_{n-4} & (n \text{ even}); \\ 2(n-1)X_{n-2} & (n \text{ odd}). \end{cases}$$

This formula is mentioned by Muir in [3], but with the index $n-3$ appearing where there should be $n-4$. The values given by Muir (through 4752) agree with those given by Hertzsprung, but the first few values given by the misstated formula are 1, 0, 0, 0, 4, 16, 80, 672, 4896, 49920, 460032. The latter sequence appears as sequence 1432 in [5] with a reference to Muir. Hertzsprung does not show how the recurrence may be derived from his original formula: he claims that this would take too much space. We shall see that Hertzsprung's recurrence formula is valid, but will derive it using rook theory rather than derangements. We will also use rook theory to compute Σ_n.

Rook Theory; Formulas for X_n and Σ_n

Suppose we have a positive integer n and a subset $\mathfrak{B}$ of $[n] \times [n]$. Let $R(x) = \sum r_k x^k$ and $H_n(t) = \sum h_{nk} t^k$, where for each $k \geq 0$ we define

$$\begin{aligned} r_k &= |\{P \subseteq \mathfrak{B} : |P| = k \text{ and no two elements of } P \\ &\qquad \text{have a common coordinate}\}|, \\ h_{nk} &= |\{\sigma \in \mathfrak{S}_n : (i, \sigma(i)) \in \mathfrak{B} \text{ for exactly } k \text{ values of } i \in [n]\}|. \end{aligned}$$

Evidently $R(x)$ and $H_n(t)$ are polynomials of degree at most n. Observe also that $R(x)$ depends only on $\mathfrak{B}$ (so long as n is large enough that $[n]\times[n]$ contains $\mathfrak{B}$), but $H_n(t)$ depends on $\mathfrak{B}$ and n.

We often call $\mathfrak{B}$ a *board*. This is because we think of it as a sort of chessboard, upon which we try to place rooks in such a way that none can attack another. The number of ways to do this with k rooks is r_k; therefore we call $R(x)$ the *rook polynomial* of $\mathfrak{B}$. The function $H_n(t)$ is the *hit polynomial* of $\mathfrak{B}$ and n; the number of ways to place n rooks on the board $[n] \times [n]$ so that none can attack another and exactly k of them 'hit' $\mathfrak{B}$ is h_{nk}. We think of $\mathfrak{B}$ as defining restricted positions for permutations of $[n]$: if $(i, j) \in \mathfrak{B}$, we have the restriction "$\sigma(i) \neq j$", or equivalently, "j cannot be in the ith position of the word of σ". From this point of view h_{nk} is the number of $\sigma \in \mathfrak{S}_n$ which violate exactly k of the restrictions imposed by

$\mathfrak{B}$. We are most often interested in finding $h_{n0} = H_n(0)$, the number of σ which do not violate any restrictions.

Although we are more interested in the hit polynomial, it is the rook polynomial that is usually easier to compute for a given $\mathfrak{B}$. The idea of rook theory is to establish connections between the two polynomials and to use properties of one to determine properties of the other. The following theorem, whose proof may be found in [4], is of fundamental importance.

> **Theorem.** Let $\mathfrak{B} \subseteq [n] \times [n]$. If $R(x) = \sum r_k x^k$ is the rook polynomial of $\mathfrak{B}$, then the hit polynomial $H_n(t)$ of $\mathfrak{B}$ and n is $\sum r_k (n-k)!(t-1)^k$.

In applying this theorem, we shall make use of the symbol E, which we define to have the property $E^i = i!$ for all nonnegative integers i. We may substitute E for indeterminates in a polynomial: e.g., if $P(x) = \sum p_k x^k$, then $P(E) = \sum p_k E^k = \sum p_k k!$. More generally, if q is another indeterminate, then $P(qE) = \sum p_k q^k k!$. Similarly, if $P(x,y) = \sum p_{kl} x^k y^l$, then $P(qE, rE) = \sum p_{kl} q^k r^l (k+l)!$. (We may also describe E using integrals. For instance, $P(qE) = \int_0^\infty P(qx)e^{-x}dx$ and $P(qE,rE) = \int_0^\infty P(qx,rx)e^{-x}dx$.)

If R is any commutative ring with identity, then it is clear that $P(x) \mapsto P(E)$ defines a linear map from $R[x]$ to R. On the other hand, E^m times E^n is clearly not equal to E^{m+n}. Given $P(x)$ and $Q(x)$, we obtain $P(E)Q(E)$ by first multiplying together $P(x)$ and $Q(x)$, then replacing x with E. For example, $(E^2-1)^2$ is $4! - 2(2!) + 0! = 21$, not $(2!-0!)^2 = 1$. This definition ensures that if $P_1(x)Q_1(x) = P_2(x)Q_2(x)$, then $P_1(E)Q_1(E) = P_2(E)Q_2(E)$. It also allows us to make sense of expressions such as $P(E)Q(E^{-1})$ whenever $P(x)$ and $Q(x)$ are such that $P(x)Q(x^{-1})$ is a polynomial in x, i.e., has no terms of negative degree. In particular, if the degree of $P(x) = \sum p_k x^k$ is at most n, then $E^n P(E^{-1}) = \sum p_k (n-k)!$ makes sense.

We now observe that the theorem above may be restated in terms of E. Namely, we have that if $R(x)$ is the rook polynomial of $\mathfrak{B} \subset [n] \times [n]$, then the corresponding hit polynomial $H_n(t)$ is equal to $E^n R(E^{-1}(t-1))$. When we set t equal to 0, we obtain the symbolic expression $h_{n0} = E^n R(-E^{-1})$.

Suppose $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are boards with no common coordinates; i.e., for any $(i_1, j_1) \in \mathfrak{B}_1$ and $(i_2, j_2) \in \mathfrak{B}_2$ we have $i_1 \neq j_1$ and $i_2 \neq j_2$. Then we shall say $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are *disjoint*. If $\mathfrak{B}$ is the union of disjoint boards $\mathfrak{B}_1$ and $\mathfrak{B}_2$, then any placement of j non-attacking rooks on $\mathfrak{B}_1$ and of k non-attacking rooks on $\mathfrak{B}_2$ will correspond to a placement of $j+k$ non-attacking rooks on $\mathfrak{B}$. This observation leads directly to the useful result that if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are disjoint, with $R_1(x)$ and $R_2(x)$ their respective rook polynomials, then the rook polynomial of $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is $R_1(x)R_2(x)$.

Now we are prepared to apply rook theory to the problems of computing X_n and Σ_n. Let $\mathfrak{A}_n = \{(i,i),(i,n-i+1) : 1 \leq i \leq n\} \subset [n] \times [n]$. We find that $\mathfrak{A}_n$ is the union of either $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$ disjoint boards, according as n is even or odd: in each case, we have the boards $\{(i,i),(i,n-i+1),(n-i+1,i),(n-i+1,n-i+1)\}$ for $1 \leq i \leq \lfloor n/2 \rfloor$, and if n is odd we also have the board $\{((n+1)/2,(n+1)/2)\}$.

It is easy to see that any board of the form $\{(i,k),(i,l),(j,k),(j,l)\}$, with $i \neq j$ and $k \neq l$, has rook polynomial $1+4x+2x^2$, and that any board consisting of a single point has rook polynomial $1+x$. It follows that for $m \geq 0$, the rook polynomials of $\mathfrak{A}_{2m}$ and $\mathfrak{A}_{2m+1}$ are respectively $(1+4x+2x^2)^m$ and $(1+4x+2x^2)^m(1+x)$. Now observe that the constant terms of the corresponding hit polynomials are X_{2m} and X_{2m+1}. By virtue of the theorem above, we have the following symbolic formulas:

$$X_{2m} = (E^2 - 4E + 2)^m, \tag{1}$$

$$X_{2m+1} = (E^2 - 4E + 2)^m (E-1). \tag{2}$$

The situation is slightly more complicated in the case of Σ_n. There are two kinds of permutations enumerated by Σ_n: those for which the fixed point is different from the reflected point, and those for which the two points coincide. The second kind exist only for odd n, as a point i that is both fixed and reflected must satisfy $i = n - i + 1$.

The following figure will help to illustrate our strategy for computing Σ_n. It depicts $[n]\times[n]$, but with the rows and columns indexed in an unusual way; we can see that each 2×2 'block' of horizontally- and vertically-shaded squares corresponds to a factor $1+4x+2x^2$ in the rook polynomial of $\mathfrak{A}_n$, while the doubly-shaded square corresponds to the factor $1+x$, which appears iff n is odd. The shading describes whether a point in $\mathfrak{A}_n$ is of the form (i,i), $(i,n-i+1)$, or both.

We will compute Σ_n by counting placements of n non-attacking rooks on $[n] \times [n]$ such that either (a) there is one rook on a horizontally-shaded square, one rook on a vertically-shaded square, and $n-2$ rooks on unshaded squares, or (b) there is one rook on the doubly-shaded square and $n-1$ rooks on unshaded squares.

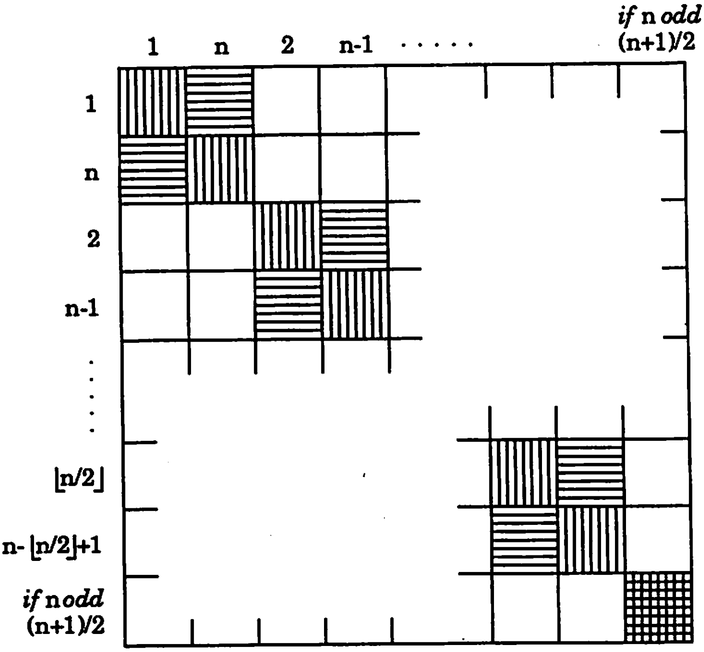

FIGURE 1

In case (a), we shall have the same number of placements for each possible choice of the two shaded squares. These two squares must not be in the same row or the same column. This condition is equivalent to their being in different 2×2 blocks in Figure 1. The number of ways to choose the blocks is $\lfloor n/2 \rfloor (\lfloor n/2 \rfloor - 1)$, and once they have been chosen, we have two horizontally-shaded squares in the first block and two vertically-shaded squares in the second block from which to choose. We have shown that there are $4\lfloor n/2 \rfloor (\lfloor n/2 \rfloor - 1)\psi_n$ placements of the type described by (a), where ψ_n denotes the number of such placements with rooks on the shaded squares $(1,1)$ and $(2, n-1)$.

Now to evaluate ψ_n, we place rooks on $(1,1)$ and $(2, n-1)$. We cannot put anything else in the 1st or 2nd rows or the 1st or $(n-1)$st columns, so we remove them from Figure 1, which gives us the following:

The placements enumerated by ψ_n correspond to placements of $n-2$ rooks on the board in Figure 2 such that none of the rooks is on a shaded square. We see that the rook polynomial associated with Figure 2 is $(1 + 4x + 2x^2)^{m-2}(1+x)^2$ if $n = 2m$ is even, $(1+4x+2x^2)^{m-2}(1+x)^3$ if $n = 2m+1$ is odd. Therefore we have $\psi_{2m} = (E^2 - 4E + 2)^{m-2}(E-1)^2$ and $\psi_{2m+1} = (E^2 - 4E + 2)^{m-2}(E-1)^3$.

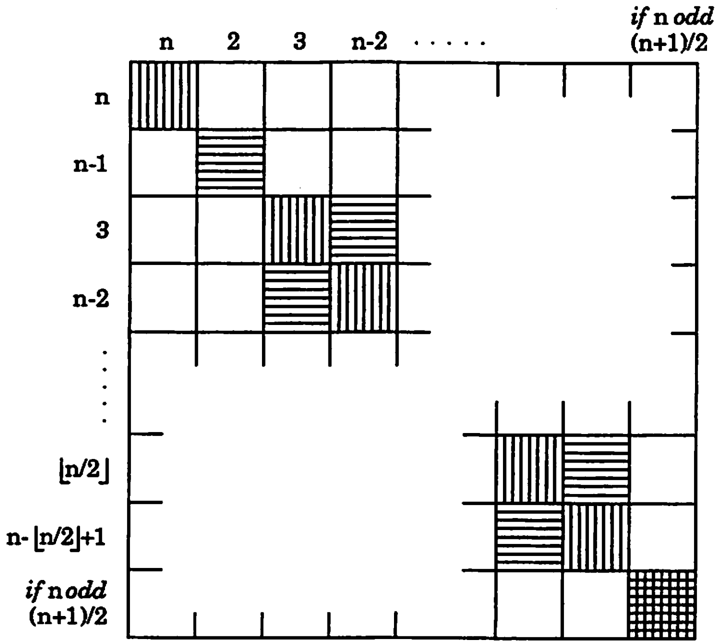

FIGURE 2

We now consider case (b), which occurs only if $n = 2m + 1$ is odd. By removing the row and column containing the doubly-shaded square, we find that the placements described by (b) correspond to placements of $2m$ rooks on $[2m] \times [2m]$ with none on $\mathfrak{A}_{2m}$; the number of such placements is just X_{2m}. (This is why the problem of enumerating permutations with unique fixed and reflected points led us to consider permutations with no fixed or reflected points.)

Combining the two cases, we can give formulas for Σ_n in terms of the symbol E:

$$\Sigma_{2m} = 4m(m-1)(E^2 - 4E + 2)^{m-2}(E-1)^2; \tag{3}$$

$$\Sigma_{2m+1} = 4m(m-1)(E^2 - 4E + 2)^{m-2}(E-1)^3 + (E^2 - 4E + 2)^m. \tag{4}$$

We shall now derive several recurrence formulas involving X_n and Σ_n, including Hertzsprung's formulas for X_n. Our strategy is a simplified version of what Riordan does in [4], pp. 186–187; indeed, he computes recurrences for the hit polynomials of $\mathfrak{A}_n$, and when we put $t = 0$ in these, we get recurrences for X_n. We will derive recurrences that involve only X_n and Σ_n.

Recall that if $\mathfrak{B} \subset [n]\times[n]$ has rook polynomial $R(x)$, then $E^n R(-E^{-1})$ is the constant term of the corresponding hit polynomial. We have that $E^i = iE^{i-1}$ for any $i > 0$; by linearity of differentiation, this implies that if $P(x)$ is any polynomial and $P'(x)$ its derivative with respect to x, then

$$P(E) = P'(E) + P(0). \tag{5}$$

(If we write $P(E) = \int_0^\infty P(x)e^{-x}dx$, then (5) follows from integration by parts.)

For X_n, the polynomials to which we apply (5) are $(x^2-4x+2)^m$ and $(x^2-4x+2)^m(x-1)$, for $n = 2m$ and $n = 2m+1$ respectively. Considering the first of these, we see that

$$\begin{aligned}\frac{d}{dx}(x^2 - 4x + 2)^m &= m(x^2 - 4x + 2)^{m-1}(2x - 4)\\ &= 2m(x^2 - 4x + 2)^{m-1}(x - 1) - 2m(x^2 - 4x + 2)^{m-1};\end{aligned}$$

this gives us

$$X_{2m} = 2m(X_{2m-1} - X_{2m-2}) + 2^m. \tag{6}$$

Similarly, one can write

$$\begin{aligned}\frac{d}{dx}[(x^2 - 4x + 2)^m(x - 1)] &= (2m + 1)(x^2 - 4x + 2)^m\\ &\quad + 2m((x^2 - 4x + 2)^{m-1}(x - 1) + (x^2 - 4x + 2)^{m-1}),\end{aligned}$$

and this gives us

$$X_{2m+1} = (2m + 1)X_{2m} + 2m(X_{2m-1} + X_{2m-2}) - 2^m.$$

Now using (6), we can rewrite the latter equation to eliminate the 2^m-term, and after simplifying we obtain

$$X_{2m+1} = 2m(X_{2m} + 2X_{2m-1}), \tag{7}$$

which is one of Hertzsprung's formulas and which also appears in [4].

We can also use (6) to rewrite itself, by observing that $2^m = 2(2^{m-1}) = 2[X_{2m-2} - (2m - 2)(X_{2m-3} - X_{2m-4})]$; substituting this into (6) gives us

$$X_{2m} = 2mX_{2m-1} - (2m-2)X_{2m-2} - 2(2m-2)X_{2m-3} + 2(2m-2)X_{2m-4}.$$

We substitute into the latter equation (7), with m replaced by $m-1$, to obtain

$$X_{2m} = (2m-1)X_{2m-1} + 2(2m-2)X_{2m-4}, \tag{8}$$

the second of Hertzsprung's formulas.

Our last recurrence for X_n involves only even values of n and, like (7), is a special case of a recurrence given in [4] for hit polynomials. (Formula (8) does not appear in Riordan's work.) If we iterate (5), we obtain the identity $P(E) = P''(E) + P'(0) + P(0)$. We combine this with the identity

$$\frac{d^2}{dx^2}(x^2-4x+2)^m = 2m(2m-1)(x^2-4x+2)^{m-1}+8m(m-1)(x^2-4x+2)^{m-2}$$

to derive

$$X_{2m} = 2m(2m-1)X_{2m-2} + 8m(m-1)X_{2m-4} + 2^m - m2^{m+1}. \tag{9}$$

We now consider the case of Σ_n. The formulas we give here are not recurrences, because they do not express Σ_n in terms of Σ_m for certain $m < n$; rather, they express Σ_n in terms of X_m for $m \leq n$. We have not succeeded in coming up with recurrence formulas for Σ_n.

Let $n = 2m \geq 4$ be even, and observe that we may write

$$\begin{aligned}(E^2-4E+2)^{m-2}(E-1)^2 &= (E^2-4E+2)^{m-1} \\ &\quad + 2(E^2-4E+2)^{m-2}(E-1) + (E^2-4E+2)^{m-2};\end{aligned}$$

thus we have

$$\begin{aligned}\Sigma_{2m} &= 4m(m-1)(X_{2m-2} + 2X_{2m-3} + X_{2m-4}) \\ &= 2mX_{2m-1} + 2m(2m-2)X_{2m-4} \\ &= m(X_{2m} - (2m-3)X_{2m-1}).\end{aligned} \tag{10}$$

In case n is odd, $n = 2m+1 \geq 5$, we write

$$\begin{aligned}(E^2-4E+2)^{m-2}(E-1)^3 &= (E^2-4E+2)^{m-1}(E-1) \\ &\quad + 2(E^2-4E+2)^{m-1} + 5(E^2-4E+2)^{m-2}(E-1) \\ &\quad + 2(E^2-4E+2)^{m-2},\end{aligned}$$

and go on to obtain

$$\begin{aligned}\Sigma_{2m+1} &= X_{2m} + 4m(m-1)(X_{2m-1} + 2X_{2m-2} + 5X_{2m-3} + 2X_{2m-4}) \\ &= X_{2m} + 2m((2m-1)X_{2m-1} + 2(2m-2)X_{2m-4}) - 2mX_{2m-1} \\ &\quad + 5m(2m-2)(X_{2m-2} + 2X_{2m-3}) - 2m(m-1)X_{2m-2} \\ &= (2m+1)X_{2m} + 3mX_{2m-1} - 2m(m-1)X_{2m-2}.\end{aligned} \tag{11}$$

We conclude this section with a table of some of the first few values of X_n and Σ_n.

n	X_n	Σ_n
0	1	0
1	0	1
2	0	0
3	0	0
4	4	8
5	16	20
6	80	96
7	672	656
8	4752	5568
9	48768	48912
10	440192	494080
11	5377280	5383552
12	59245120	65097600

Concluding Remarks

We have found formulas that will tell us, for any n, how many permutations of n have exactly one fixed and one reflected point. We now consider ways in which our work could be improved upon or extended.

Concerning X_n, we used rook theory to prove Hertzsprung's recurrence formulas; he could not have done it this way, and he does not give any clues as to how he did it. Also, the techniques we have employed do not give a clear indication of how one could prove (7) and (8) combinatorially, although the simplicity of these formulas suggests that they should have combinatorial interpretations. In particular, Hertzsprung notes the similarity between the formula $X_n = (n-1)(X_{n-1} + 2X_{n-2})$, which holds for odd n, and the formula $D_n = (n-1)(D_{n-1} + D_{n-2})$ involving derangement numbers. The latter has a simple combinatorial proof; perhaps the former does too.

The machinery we have developed gives us concise formulas for X_n and Σ_n in terms of the symbol E, and it allows us to derive recurrence formulas for X_n. But it is not so helpful when we try to find recurrences satisfied by Σ_n. We expected that Σ_n, like X_n, could be given by a homogeneous recurrence relation of constant order, whose coefficients were polynomials in n; but the data do not seem to suggest such a relation. Nor do the data suggest how we could express X_n in terms of Σ_m for $m \leq n$ (if we could do this, we could combine it with (10) and (11) to get recurrences for Σ_n). Perhaps a new approach is called for to decide whether there are recurrence formulas for Σ_n. We now offer one possibility, although it is not immediately clear how useful it will be.

Recall that we began by considering permutations of $[n]$ that 'hit' the board $\mathfrak{A}_n$ exactly once on each diagonal. We could not use rook theory directly, because it does not distinguish between diagonals. Some permutations that hit $\mathfrak{A}_n$ twice hit it once on each diagonal, some hit it twice on a single diagonal and miss the other. If n is odd, it is possible for a permutation that hits $\mathfrak{A}_n$ only once to hit both diagonals. This suggested the possibility of rook polynomials and hit polynomials in several variables, each variable corresponding to some sub-board of a given board. We shall show how this can be done in the two-variable case, and it will generalize readily to more variables.

Suppose we are given $\mathfrak{B}_x, \mathfrak{B}_y \subseteq [n] \times [n]$; then we define $R(x,y) = \sum r_{ij} x^i y^j$, where r_{ij} is the number of ways to place i rooks on $\mathfrak{B}_x$ and j rooks on $\mathfrak{B}_y$ such that none can attack another. We also define $H_n(s,t) = \sum h_{nij} s^i t^j$, where h_{nij} is the number of ways to place n non-attacking rooks on $[n] \times [n]$ with exactly i on $\mathfrak{B}_x$ and exactly j on $\mathfrak{B}_y$. We shall call $R(x,y)$ the rook polynomial of $\mathfrak{B}_x$ and $\mathfrak{B}_y$, and $H_n(s,t)$ the hit polynomial of n, $\mathfrak{B}_x$, and $\mathfrak{B}_y$. We have the following theorems, which may be proved in much the same way as the corresponding theorems for the single-variable case:

Theorem A. Suppose $\mathfrak{B}_{1x} \cup \mathfrak{B}_{1y}$ and $\mathfrak{B}_{2x} \cup \mathfrak{B}_{2y}$ have no common coordinate, and let $R_i(x,y)$ be the rook polynomial of $\mathfrak{B}_{ix}$ and $\mathfrak{B}_{iy}$ for $i = 1, 2$. Then $R_1(x,y)R_2(x,y)$ is the rook polynomial of $\mathfrak{B}_{1x} \cup \mathfrak{B}_{2x}$ and $\mathfrak{B}_{1y} \cup \mathfrak{B}_{2y}$.

Theorem B. Let $\mathfrak{B}_x, \mathfrak{B}_y \subseteq [n] \times [n]$ and $\mathfrak{B}_x \cap \mathfrak{B}_y = \emptyset$. If $R(x,y) = \sum r_{ij} x^i y^j$ is the rook polynomial of $\mathfrak{B}_x$ and $\mathfrak{B}_y$, then the hit polynomial $H_n(s,t)$ of $\mathfrak{B}_x$, $\mathfrak{B}_y$, and n is $\sum r_{ij}(n-i-j)!(s-1)^i(t-1)^j$.

Another way of stating Theorem B is that if $\mathfrak{B}_x \cap \mathfrak{B}_y = \emptyset$, then $H_n(s,t) = E^n R(E^{-1}(s-1), E^{-1}(t-1))$. If $\mathfrak{B}_x \cap \mathfrak{B}_y$ is not empty, this result need not hold; for $\mathfrak{B}_x = \mathfrak{B}_y = \{(i,j)\} \subset [n] \times [n]$, we have rook polynomial $1 + xy$ and hit polynomial $E^n + E^{n-1}(st-1)$. (Theorem A does not require $\mathfrak{B}_{ix} \cap \mathfrak{B}_{iy}$ to be empty.)

Now to see what this has to do with Σ_n, we define $\mathfrak{A}_{nx} = \{(i,i) : 1 \leq i \leq n\}$ and $\mathfrak{A}_{ny} = \{(i, n-i+1) : 1 \leq i \leq n\}$. With $[n] \times [n]$ depicted as in Figure 1, we have that $\mathfrak{A}_{nx}$ consists of all vertically-shaded squares and $\mathfrak{A}_{ny}$ consists of all horizontally-shaded squares. (The doubly-shaded square is both horizontally- and vertically-shaded.) Evidently the rook polynomial of each 2×2 block of shaded squares is $1 + 2x + 2y + x^2 + y^2$, while the doubly-shaded square has rook polynomial $1 + xy$. From the theorems above, we see that the hit polynomial $H_n(s,t)$ of n, $\mathfrak{A}_{nx}$, and $\mathfrak{A}_{ny}$ is given by the following formulas:

$$H_{2m}(s,t) = (E^2 + 2E((s-1)+(t-1)) + (s-1)^2 + (t-1)^2)^m;$$

$$H_{2m+1}(s,t) = (E^2 + 2E((s-1)+(t-1)) + (s-1)^2 + (t-1)^2)^m(E + st - 1).$$

From the definition of $\mathfrak{A}_{nx}$ and $\mathfrak{A}_{ny}$, we see that the coefficient of st in $H_n(s,t)$ will be the number of permutations of $[n]$ having exactly one fixed and one reflected point—i.e., Σ_n. It is not hard to compute the coefficient of st from the formulas above and to verify that it agrees with our previously obtained formula for Σ_n.

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