Inequalities for Total Matchings of Graphs

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Abstract. As a generalization of a matching consisting of edges only Alavi *et al* in [1] define a total matching which may contain both edges and vertices.

Using total matchings [1] defines a parameter $\beta_2'(G)$ and proves that $\beta_2'(G) \leq p-1$ holds for a connected graph of order $p \geq 2$.

Our main result is to improve this inequality to $\beta_2'(G) \leq p-2\sqrt{p}+2$ and we give an example demonstrating this bound to be best possible.

Relations of several other parameters to β_2' are demonstrated.

Notation.

For $x \in \mathcal{R}$ we let $\lceil x \rceil$, the ceiling of x, denote the least integer not less than x and $\lfloor x \rfloor$, the floor of x, denotes the largest integer not larger than x. For a graph G we use p and q to denote its number of vertices and edges; $p = |V(G)|$, $q = |E(G)|$. By $A \subseteq G$ we mean $A \subseteq V(G) \cup E(G)$ and we let $V(A)$ denote the vertices of G which either belong to A or are incident with an edge of A. Let $\langle A \rangle$ denote the graph induced by A in G, that is, $V(A)$ together with those edges of G which join two vertices of $V(A)$. By K_p we denote the complete graph on p vertices, P_p denotes the path with p vertices, β_0 and β_1, are the vertex- and edge-independence numbers, that is, the maximum number in G of independent vertices and edges, respectively. For undefined terms, the reader is referred to [3].

Total matching, cover and β_2'.

An element of a graph is either a vertex or an edge. For a graph G with vertex set $V(G)$ and edge set $E(G)$ we say that a subset $M \subseteq G$ is a **total matching** if no distinct pair of elements of M are adjacent nor incident.

A total matching M of G is called **maximal** if no other total matching M' of G containing M as a proper subset exists.

A subset C of $V(G) \cup E(G)$ is said to **cover** G if each element of G not in C is adjacent to or incident to an element in C.

A total matching is maximal if and only if it covers G. For any total matching there exists a maximal total matching containing it as a subset.

ARS COMBINATORIA 39(1995), pp. 109-119

A set which covers G contains a total matching, but not necessarily a maximal total matching. As an example let G be the path $abcd$. The set $\{b, c\}$ covers G and contains the total matching $\{b\}$, but contains no maximal total matching.

We define $\beta_2'(G)$ to be the minimum cardinality among all maximal total matchings of G. For a fixed graph G, thus, $\beta_2'(G)$ denotes the number of elements in a total matching M which firstly is maximal, that is, M cannot by addition of any element from $V(G) \cup E(G)$ be extended to a larger total matching of G, and which secondly is of minimum cardinality, that is, no other maximal total matching of G contains fewer elements than M.

Matchings as sets of edges have been extensively studied, a survey is given in [6]. Total matchings as sets, where both edges and vertices are permitted, was defined in [1] and a related concept of total cover was treated in [2].

Comparison with other graph parameters.

Proposition 1. *For any graph G with p vertices, $p \geq 1$, we have*

$$\left\lceil \frac{\beta_1(G)}{2} \right\rceil \leq \beta_2'(G) \leq p - \beta_1(G).$$

Proof: A set F of $\beta_1(G)$ independent edges in G is by definition a total matching and can be extended to a maximal total matching M in G.

M contains the original $\beta_1(G)$ edges plus the $p-2\beta_1(G)$ vertices from $V(G) \setminus V(F)$, thus, $|M| = p - \beta_1(G)$, and that by definition of $\beta_2'(G)$ proves $\beta_2'(G) \leq p - \beta_1(G)$. Next, let F be a set of $\beta_1(G)$ independent edges and let M be a maximal total matching with $\beta_2'(G)$ elements. A vertex from M covers at most one edge in F, and an edge from M covers at most two edges in F, thus, since M covers F, we have that $|M| \geq \frac{\beta_1(G)}{2}$. Hence, $\beta_2'(G) = |M| \geq \lceil \frac{\beta_1(G)}{2} \rceil$ and Proposition 1 is proven. ∎

Proposition 2. *For any graph G with p vertices we have*

$$\beta_2'(G) \leq \frac{p + \beta_0(G)}{2}.$$

Proof: A set S of $\beta_0(G)$ independent vertices can be extended to a maximal total matching M which contains $\beta_0(G)$ vertices plus possibly some edges, each of which must have both its ends in $V(G) \setminus S$, that is, plus at most $\frac{p-\beta_0(G)}{2}$ edges. Thus, $\beta_2'(G) \leq |M| \leq \frac{p+\beta_0(G)}{2}$. ∎

Proposition 3. *Let G be a graph with p vertices, q edges and maximum valency Δ. Then we have*

$$\frac{p + q}{2\Delta + 1} \leq \beta_2'(G).$$

Proof: Consider a maximal total matching M with $|M| = \beta_2'(G)$. Each element of M covers at most $2\Delta + 1$ elements of G, because a vertex covers itself plus at most Δ edges leading to Δ neighbours, and an edge covers itself, two ends and at most $\Delta - 1$ incident edges at each end.

The $\beta_2'(G)$ elements of M can, thus, altogether cover at most $\beta_2'(G)$ $(2\Delta + 1)$ elements. Since M covers all $p + q$ elements of G we obtain $p + q \leq \beta_2'(G)$ $(2\Delta + 1)$. ∎

Let $i(G)$ be the independent dominance number of G, that is, $i(G)$ is the minimum order of an independent set $I \subseteq V(G)$ such each vertex in $V(G)\setminus I$ is adjacent to a vertex in I.

Proposition 4. *For any graph G we have*

$$i(G) \leq \beta_2'(G).$$

Proof: Let M be a maximal total matching with $|M| = \beta_2'(G)$. M consists of edges e_i and vertices v_j, $M = \{e_1, e_2, \ldots, e_n, v_1, v_2, \ldots, v_m\}$. Denote the ends of e_i by x_i and y_i. Let $I = \{v_1, v_2, \ldots, v_m\}$.

Add to I vertex x_1 if x_1 is not dominated by any vertex already in I. Continue on with $x_2, x_3, \ldots, x_n$ and do the same with $y_1, y_2, \ldots, y_n$.

We note that when finished the set I cannot contain both ends of an edge e_i.

Thus, the final I dominates all vertices of G and has cardinality no more than $|M|$, so that $i(G) \leq |I| \leq |M| = \beta_2'(G)$. ∎

Lemma 5. *With the preceding notation $\beta_2'(K_p) = \lceil \frac{p}{2} \rceil$. In fact, any maximal total matching M of K_p has $\lceil \frac{p}{2} \rceil$ elements. Further*

(1) *If p is even then M consists of either $\frac{p}{2}$ edges or of $\frac{p}{2} - 1$ edges and one vertex.*

(2) *If p is odd then M consists of $\frac{p-1}{2}$ edges and one vertex.*

Proof: Let M be a maximal total matching of K_p. M contains at most one vertex. If M contains exactly one vertex v then $M\setminus\{v\}$ is a maximal set of independent edges in $K_p - v$, therefore, $M\setminus\{v\}$ consists of $\lfloor \frac{p-1}{2} \rfloor$ edges and $|M| = 1 + \lfloor \frac{p-1}{2} \rfloor = \lceil \frac{p}{2} \rceil$. If M contains no vertex, but only edges, then p is even, because otherwise M could be extended by a vertex; thus, $|M| = \frac{p}{2} = \lceil \frac{p}{2} \rceil$. All the claims of Lemma 5 now follow immediately.

Propositions 1–4 are best possible: With $G = K_p$ we obtain the right side equality in Proposition 1 because $\beta_2'(K_p) = \lceil \frac{p}{2} \rceil$ and $p - \beta_1(K_p) = p - \lfloor \frac{p}{2} \rfloor = \lceil \frac{p}{2} \rceil$.

With G consisting of k copies of a 5-circuit and one K_2, all disjoint except for exactly one common vertex, we obtain the left side equality in Proposition 1, because $\beta_1(G) = 2k + 1$, $\beta_2'(G) = k + 1$.

For p odd, we obtain with $G = K_p$ equality in Proposition 2, because $\beta_2'(K_p) = \lceil \frac{p}{2} \rceil$ and $\frac{p + \beta_0(G)}{2} = \frac{p+1}{2} = \lceil \frac{p}{2} \rceil$.

Proposition 3 is best possible: For a path P_p with $p = 5k + 3$ vertices we obtain equality in Proposition 3, because $\beta_2'(P_p) = \lceil \frac{1}{3} \lfloor \frac{6p}{5} \rfloor \rceil$ by [1] so that $\beta_2'(P_{2k+3}) = 2k + 1$ and $\frac{p+q}{2\Delta+1} = \frac{10k+5}{5} = 2k + 1$. That Proposition 4 is best possible is seen from $G = C_{2k}$, where $i(C_{2k}) = \beta_2'(C_{2k}) = k$. ∎

Monotonicity.

β_0, β_1 are not monotone in general, but consider the special case of a sequence of graphs $G_1 \subseteq G_2 \subseteq G_3 \ldots \subseteq G_i \subseteq \ldots$, all spanned by the same fixed set of vertices, then:

- β_1 is monotonically non-decreasing, because adding more edges to a graph does not decrease the number of independent edges.
- β_0 is monotonically non-increasing, because adding more edges may lower the number of independent vertices, but not increase it.
- β_2' is not monotone at all: Let G_1 consist of p isolated vertices, let $G_2 = K_{1,p-1}$ and let $G_3 = K_p$. Then we have $G_1 \subseteq G_2 \subseteq G_3$ while p, 1, $\lceil \frac{p}{2} \rceil$ are the corresponding values for $\beta_2'(G_i)$, $i = 1, 2, 3$.

Main result (Theorem 1).

We shall first prove a lemma.

Lemma 6. *Let G be a connected graph with p vertices. If $\beta_1(G) < \frac{p}{3}$ then $\beta_2'(G) \leq p - \beta_1(G) - \lceil \frac{p}{\beta_1(G)} \rceil + 2$.*

Proof: Let F be a set having the maximum number $\beta_1 = \beta_1(G)$ of independent edges in G and let S be the set $V(G) \backslash V(F)$ consisting of $p - 2\beta_1$ vertices. By maximality of F, we note $\langle S \rangle$ consists only of isolated vertices.

Since G is connected there exists by the pigeon hole principle an edge $e = (x, y)$ in F which is joined to at least $\frac{p-2\beta_1}{\beta_1}$ vertices in S, and, thus, to at least two vertices in S, because $p > 3\beta_1 \Rightarrow \frac{p-2\beta_1}{\beta_1} > 1$.

Assume $e = (x, y)$ is joined to $v_1, v_2, \ldots, v_k, k \geq 2$, in S and assume $(x, v_1) \in E(G)$. Then $(y, v_i) \notin E(G)$ for $2 \leq i \leq k$, because otherwise $\{F \backslash e\} \cup \{(x, v_1), (y, v_i)\}$ would be a set of $\beta_1 + 1$ independent edges in G, a contradiction. Hence, $(x, v_i) \in E(G), 1 \leq i \leq k$.

Define M to be the set $M = \{F \backslash e\} \cup \{S \backslash N(x)\} \cup \{x\}$. We can see that M is a total matching and that M covers G. Thus, M is a maximal total matching of G and we have that

$$\beta_2'(G) \leq |M| \leq \beta_1 + (p - 2\beta_1) - \frac{p - 2\beta_1}{\beta_1} = p - \frac{p}{\beta_1} - \beta_1 + 2.$$

This proves Lemma 6. ∎

We note that the function $-\frac{p}{x} - x$ attains its maximum for $x = \sqrt{p}$, therefore, we obtain by substituting $\beta_1 = \sqrt{p}$ that

$$\beta_2'(G) \leq p - 2\sqrt{p} + 2.$$

This inequality is, thus, implied by the hypothesis $\beta_1(G) < \frac{p}{3}$.

Theorem 1. *Let G be a connected graph with p vertices, $p \geq 1$, then $\beta_2'(G) \leq p-2\sqrt{p}+2$, unless G is spanned by one of the three exceptional graphs $G = G_1$, G_2, G_3, defined by:*

$$V(G_k) = \{v_i \mid 1 \leq i \leq 2k+1\},$$
$$E(G_k) = \{(v_1, v_i) \mid 2 \leq i \leq 2k+1\} \cup \{(v_{2i}, v_{2i+1}) \mid 1 \leq i \leq k\}, \quad k = 1,2,3.$$

In particular the inequality holds for $p \geq 8$.

Proof: If $\beta_1 < \frac{p}{3}$ then Lemma 6 proves Theorem 1.

In fact, with notation of Lemma 6, if just some edge $e = (x, y)$ in F is joined to at least two vertices in S, then we see from the proof of Lemma 6 that we can obtain $\beta_2'(G) \leq p - 2\sqrt{p} + 2$ as desired.

Hence, we may assume that $\beta_1 \geq \frac{p}{3}$ and that each edge $e \in F$ is joined to at most one vertex in S.

Let $v_1 \in S$ and assume that F contains an edge $e = (x, y)$ such that $(x, v_1) \in E(G), (y, v_1) \notin E(G)$. Then the set:

$$M = (F \backslash e) \cup (S \backslash v_1) \cup \{x\}$$

has $|M| = (\beta_1 - 1) + (p - 2\beta_1 - 1) + 1 = p - \beta_1 - 1$ and M is a maximal total matching. We then obtain:

$$\beta_2'(G) \leq |M| = p - \beta_1 - 1 \leq \frac{2}{3}p - 1 \leq p - 2\sqrt{p} + 2$$

because $0 \leq p - 6\sqrt{p} + 9$ holds for all p.

We may, hence, assume that each edge $e_i = (x_i, y_i)$ in F has both its ends joined to the same vertex $v_{j(i)}$ in S.

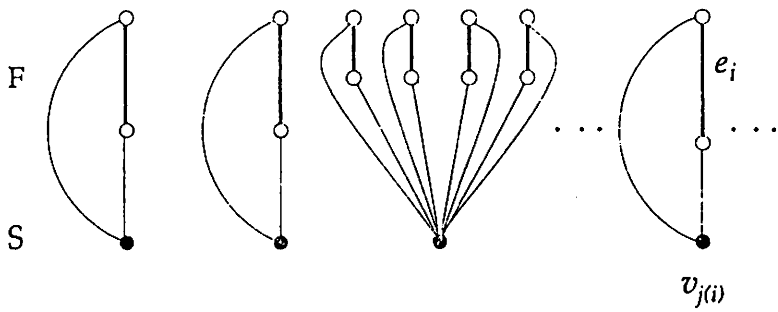


Figure 1: In the proof of Theorem 1 we find, first, this substructure to be contained in G, secondly, we see that G is spanned by only one of the components above which, thirdly, turns out to be either G_1, G_2, or G_3.

No edge of G can join two vertex-disjoint triangles x_i, y_i, v_j, and $x_{i'}, y_{i'}, v_{j'}$, otherwise G would contain $\beta_1 + 1$ independent edges, a contradiction.

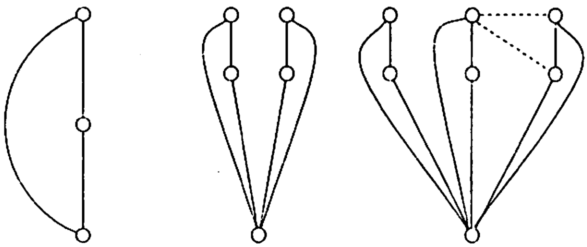
Since G is connected, it must be spanned by G_k for some $k \geq 1$, where G_k consists of k triangles pendant from the same vertex (Figure 1).

We shall now prove that $k \leq 3$. The set consisting of the vertex common to all k triangles and the opposite edges from each triangle is a maximal total cover of G with $k + 1$ elements, so $\beta_2'(G) \leq k + 1$.

Thus, when $k \geq 4$, G satisfies the inequality of Theorem 1, because $p = |V(G)| = 2k + 1, \beta_2'(G) \leq k + 1$ and then $k + 1 \leq 2k + 1 - 2\sqrt{2k + 1} + 2$ is implied by the fact that the function $x - 2\sqrt{2x + 1} + 2$ attains its minimum value for $x = \frac{3}{2}$, has value 0 for $x = 4$ and is increasing for $x \geq \frac{3}{2}$. The exceptional graphs, thus, are spanned by $G_k, 1 \leq k \leq 3$. This proves Theorem 1. ∎

The exceptions of Theorem 1 are listed in Figure 2.

The example below demonstrates that the inequality of Theorem 1 is best possible.



G_k	G_1	G_2	G_3
$\beta_2'(G_k)$	2	3	4
$p - 2\sqrt{p} + 2$	1.5	2.5	3.7

Figure 2: The exceptions of Theorem 1, the dotted edges do not belong to G_3, but may belong to G.

Example: We note that $\beta_2'(G_p) = p - 2\sqrt{p} + 2$ holds for an infinite family of graphs.

Proof: Let G_p denote a K_n, $n \geq 1$, with $n - 1$ pendant edges from each vertex, then $p = |V(G_p)| = n + n(n - 1) = n^2$. One can easily show that $\beta_2'(G_p) = n^2 - 2n + 2 = p - 2\sqrt{p} + 2$ by considering a maximal total matching M of G_p with $|M| = \beta_2'(G_p)$. Then M contains at most one vertex from the central graph K_n, and by examining each of the two cases, that M contains exactly one vertex or no vertex from K_n, the result $|M| = 1 + (n-1)(n-1) = n^2 - 2n + 2$ can be deduced. ∎

In general, connectivity is not needed to get the bound in Theorem 1. We have the somewhat stronger result below.

Theorem 1'. *With a finite number of exceptions, if G is a graph of order p with no isolated vertices, then $\beta_2'(G) \leq p - 2\sqrt{p} + 2$.*

Proof: Suppose $p \geq 23$. Consider the following two cases.

Case 1 Suppose $\beta_1(G) \geq \frac{p}{3}$. From Proposition 1 we know

$$\beta_2'(G) \leq p - \beta_1(G) \leq \frac{2p}{3}.$$

So long as $p \geq 23$, we note $\frac{2p}{3}$ is less than the desired expression.

Case 2 Suppose $\beta_1(G) < \frac{p}{3}$. Then we see from the argument used in the proof of Lemma 6 that the desired inequality holds. The argument of Lemma 6 remains valid if the assumption "G is connected" is relaxed to "G has no isolated vertex". ∎

An inequality for complementary graphs (Theorem 2).

We have for complementary graphs $G, \overline{G}$ that

$$\beta_1(K_p) = \left\lfloor \frac{p}{2} \right\rfloor \leq \beta_1(G) + \beta_1(\overline{G}).$$

The inequality follows from the fact that the $\lfloor \frac{p}{2} \rfloor$ edges of a matching in K_p will partition into two sets of independent edges in G and $\overline{G}$, respectively. For β_2' we can prove an analogous inequality:

$$\left\lceil \frac{p+2}{2} \right\rceil \leq \beta_2'(G) + \beta_2'(\overline{G})$$

which is stated in Theorem 2 below.

In Theorem 2 we shall in two steps prove that $\lceil \frac{p+2}{2} \rceil \leq \beta_2'(G) + \beta_2'(\overline{G})$. First we easily obtain the weaker result:

$$\left\lceil \frac{p}{2} \right\rceil = \beta_2'(K_p) \leq \beta_2'(G) + \beta_2'(\overline{G}).$$

Then we prove that equality cannot occur so that, in fact:

$$\left\lceil \frac{p+2}{2} \right\rceil = \left\lceil \frac{p}{2} \right\rceil + 1 \leq \beta_2'(G) + \beta_2'(\overline{G}).$$

For the first step we need Lemmas 7-9 below on covers of K_p.

Lemma 7. *For $p \geq 1$ we have*
If the set C covers K_p then $|C| \geq \lceil \frac{p}{2} \rceil$.

Proof: Assume that C covers K_p and that C consists of x edges and y vertices, $0 \leq x, y \leq |C|$, $x + y = |C|$.

The edges of C are altogether incident with at most $2x$ vertices in K_p so $|V(C)| \leq 2x + y \leq 2x + 2y = 2|C|$.

If $|C| < \lceil \frac{p}{2} \rceil$ and C has fewer than $\lfloor \frac{p}{2} \rfloor$ edges then $K_p \backslash V(C)$ contains at least two vertices a and b and, thus, the edge (a, b) is not covered by C, otherwise p is odd and C consists of $\lfloor \frac{p}{2} \rfloor$ edges so that $K_p \backslash V(C)$ consists of one vertex which is not covered by C. This proves Lemma 7. ∎

Lemma 8. *For even p, $p = 2m$, the following three statements are equivalent:*

(1) *C covers K_{2m} and $|C| = m$;*
(2) *either (i) C consists of m independent edges or (ii) C consists of $m - 1$ independent edges and one vertex not incident with any of them;*
(3) *C is a maximal total matching of K_{2m}.*

Proof: We shall now see that if C with $|C| = \lceil \frac{p}{2} \rceil$ covers K_p then C contains at least $\lfloor \frac{p}{2} \rfloor - 1$ edges. Assume that C consists of x edges and y vertices $0 \leq x$, $y \leq \lceil \frac{p}{2} \rceil$, $x + y = \lceil \frac{p}{2} \rceil$. If $x \leq \lfloor \frac{p}{2} \rfloor - 2$ then $|V(C)| \leq 2x + y \leq \lceil \frac{p}{2} \rceil + \lfloor \frac{p}{2} \rfloor - 2 = p - 2$ and as in the proof of Lemma 7 we see that $V(K_p) \backslash V(C)$ will contain two vertices a, b such that the edge (a, b) is not covered by C.

Thus, C must contain at least $\lfloor \frac{p}{2} \rfloor - 1$ edges. This statement and its proof applies not only to to this lemma, where p is even, but also to Lemma 9 below, where p is odd.

For $p = 2m$ we have, if $x = \lfloor \frac{p}{2} \rfloor = m$ then $y = 0$ and C consists $x = \frac{p}{2}$ edges which cover $p = 2x$ vertices. Thus, C is independent and 2(i) occurs.

If $x = \lfloor \frac{p}{2} \rfloor - 1 = m - 1$ then $y = 1$. Since C covers K_p we must have $p - 1 \leq |V(C)|$. This together with $|V(C)| \leq 2x + y = p - 1$ implies $|V(C)| = 2x + y$ so that C is independent and 2(ii) occurs. Thus, (1) implies (2). It is easily seen that (2) implies (3), and that (3) implies (1). This proves Lemma 8. ∎

Lemma 9. *For odd p, $p = 2m + 1$, the following two statements are equivalent:*

(1) *C covers K_{2m+1} and $|C| = m + 1$;*
(2) *either (i) C consists of m independent edges and one vertex or (ii) C consists of $m - 1$ independent edges plus one edge adjacent to exactly one of them plus one vertex incident with none of the m edges or (iii) C consists of $m - 1$ independent edges and two vertices not incident with any of them.*

Further, C is a maximal total matching of K_{2m+1} if and only if (i) above occurs and the vertex is not incident with any of the edges.

Proof: Assume that C with $|C| = m + 1$ covers K_{2m+1}. As noted in the proof of Lemma 8 we find that C must contain at least $\lfloor \frac{p}{2} \rfloor - 1$ edges. As previously we let x, y denote the respective numbers of edges and vertices in C.

If $x = \lfloor \frac{p}{2} \rfloor = m$ then $y = 1$.

If the m edges are independent then 2(i) occurs.

Otherwise, from

(a) C covers $K_p \Rightarrow p - 1 \leq |V(C)|$
(b) x edges cover $\leq 2x - 1$ vertices, and
(c) $|V(C)| \leq 2x = p - 1$

we obtain $V(C) = p - 1$; and $V(C) = p - 1$ implies that x edges cover $2x - 1$ vertices. Thus, C consists of $m - 1$ independent edges plus an mth edge adjacent to exactly one of them and 2(ii) occurs.

If $x = \lfloor \frac{p}{2} \rfloor - 1 = m - 1$ then $y = 2$ and $p - 1 \leq |V(C)| \leq 2x + 2 = p - 1$. Thus, $|V(C)| = p - 1$, or, equivalently, $|V(C)| = 2x + y$, which implies that C consists of $x = m - 1$ independent edges and $y = 2$ vertices not incident with any of them and case 2(iii) occurs.

This proves that (1) implies (2). Conversely, it is easy to verify that (2) implies (1). The last remark is easily seen and Lemma 9 is proven. ∎

Theorem 2. *For a graph G with p vertices, $p \geq 1$, we have*

$$\left\lceil \frac{p+2}{2} \right\rceil \leq \beta_2'(G) + \beta_2'(\overline{G}) \leq \left\lceil \frac{3p}{2} \right\rceil.$$

Proof of Theorem 2:

Left side inequality

Let M_1, M_2 be maximal total matchings in G, $\overline{G}$, respectively, with $|M_1| = \beta_2'(G)$, and $|M_2| = \beta_2'(\overline{G})$.

From the fact that M_1 covers G and M_2 covers $\overline{G}$ it follows that $M = M_1 \cup M_2$ covers K_p, and, hence, by Lemma 7 that $|M| \geq \lceil \frac{p}{2} \rceil$. Thus, we have:

$$\left\lceil \frac{p}{2} \right\rceil \leq |M| \leq |M_1| + |M_2| = \beta_2'(G) + \beta_2'(\overline{G}).$$

We cannot have equality above, for assume

$$\left\lceil \frac{p}{2} \right\rceil = \beta_2'(G) + \beta_2'(\overline{G}).$$

We shall then reach a contradiction.

From $M = M_1 \cup M_2$ we have $|M| \leq |M_1| + |M_2| = \lceil \frac{p}{2} \rceil$ which combined with $|M| \geq \lceil \frac{p}{2} \rceil$ gives $|M| = \lceil \frac{p}{2} \rceil$ and, hence, $M_1 \cap M_2 = \emptyset$. We shall, aided by Lemma 8 and Lemma 9, go through the possibilities for M:

Case 1 Suppose p is even and M consists of $\frac{p}{2}$ independent edges. Consider $e = (x, y)$, $e \in M$. We may assume $e \in M_1 \subseteq G$, then $e \notin M_2$ and $M_2 \cup \{x\}$ is independent, contradicting maximality of M_2.

Case 2 Suppose p is even and M consists of $\frac{p}{2}-1$ independent edges and one vertex x not incident with any of the edges. We may assume $x \in M_1$, then $x \notin M_2$ and $M_2 \cup \{x\}$ is independent, contradicting maximality of M_2.

Case 3 Suppose p is odd, M consists of $\lfloor \frac{p}{2} \rfloor$ independent edges and one vertex x. We may assume that $x \in M_1$, $x \notin M_2$. If x is not incident with any edge of M then $M_2 \cup \{x\}$ is independent. If x is incident with an edge of M, then for the unique vertex y in K_p not incident to any edge of M we obtain that $M_2 \cup \{y\}$ is independent. In both events we have a contradiction to maximality of M_2.

Case 4 Suppose p is odd, M consists of $\lfloor \frac{p}{2} \rfloor - 1$ independent edges plus one edge having exactly one vertex incident with this set plus a vertex x not incident with any of the edges. Denote the second vertex not incident with any of the edges by y, $y \notin M$.

If $x \in M_1$, then $M_2 \cup \{y\}$ is independent in contradiction to maximality of M_2.

Case 5 Suppose p is odd, M consists of $\lfloor \frac{p}{2} \rfloor - 1$ independent edges and two vertices x, y not incident with any of them. Denote the third vertex not incident with any of the edges by z, $z \notin M$.

If $x \in M_1$ then $\{x, z\} \cap M_2 = \emptyset$ and $M_2 \cup \{(x,z)\}$ is independent, a contradiction.

Thus, we cannot have equality in:

$$\left\lceil \frac{p}{2} \right\rceil \leq |M_1| + |M_2|$$

and we have proven

$$\left\lceil \frac{p+2}{2} \right\rceil \leq \beta_2'(G) + \beta_2'(\overline{G}).$$

Right side inequality

To prove the right inequality we use $\beta_0(G) + \beta_0(\overline{G}) \leq p+1$. Let S_1 and S_2 be sets of independent vertices in G and $\overline{G}$, respectively. In G the vertices S_2 span a complete graph $\langle S_2 \rangle$, which, thus, contains at most one vertex from S_1. Therefore, $|S_1| + |S_2| \leq p+1$. Proposition 2 applied to G and $\overline{G}$ gives $\beta_2'(G) + \beta_2'(\overline{G}) \leq p + \frac{\beta_0(G)+\beta_0(\overline{G})}{2} \leq p + \frac{p+1}{2} = \frac{3p+1}{2}$. The proof of Theorem 2 is complete. ∎

Theorem 2 is best possible.

For equality in the lower limit let p be odd and $G = K_{1,p-1}$, $\overline{G} = K_{p-1} \cup K_1$. Then $p = 2m+1$ and $\lceil \frac{p+2}{2} \rceil = m+2$ and $\beta_2'(K_{1,p-1}) = 1$ and $\beta_2'(\overline{G}) = \lceil \frac{p-1}{2} \rceil + 1 = m+1$. The equality $\beta_2'(G) + \beta_2'(\overline{G}) = \frac{3p+1}{2} = \lceil \frac{3p}{2} \rceil$ holds for the pair of graphs $G = K_p$, $\overline{G} = pK_1$.

Complexity.

Given G, let G' be G together with two pendant vertices attached to each vertex in G. We note that $\beta_2'(G') = 2p - \beta_0(G)$. From [3] we know computation of β_0 is an NP-complete problem. Hence, we have the following.

Proposition 5. *Computation of β_2' is an NP-complete problem.*

Added in proof: A parameter similar to, but slightly different from B_2' is treated in P. Erdős, A. Meir: "On Total Matching Numbers and Total Covering Numbers of Complementary Graphs". Discrete Mathematics 19 (1977), 229–233. Incorrectly, Theorem 3.2 of that paper has overseen the exceptions stated in our Theorem 1.

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