# A Dominating Property of $i$-center in $P_t$-free Graphs

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**Abstract.** The $i$-center $C_i(G)$ of a graph $G$ is the set of vertices whose distances from any vertex of $G$ are at most $i$. A vertex set $X$ $k$-dominates a vertex set $Y$ if for every $y \in Y$ there is a $x \in X$ such that $d(x,y) \leq k$. In this paper, we prove that if $G$ is a $P_t$-free graph and $i \geq \lfloor \frac{t}{2} \rfloor$, then $C_i(G)(q+1)$-dominates $C_{i+q}(G)$, as conjectured by Favaron and Fouquet [4].

## 1. Introduction

We consider only simple graphs, which have no loops and multiple edges, referred to as graphs for short. Any terminology not defined here will conform to the usage in [2].

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. For a positive integer $t$, $G$ is called **$P_t$-free** if it does not contain a path on $t$ vertices as an induced subgraph. The distance $d(x,y)$ between two vertices $x$ and $y$ is the length (i.e., the number of edges) of a shortest path joining $x$ and $y$. In [1], G. Bacso and Z. Tuza generalized the classical concepts of centre and dominating set to the notions of $i$-center and $k$-dominating set. The **$i$-center** $C_i(G)$ of $G$ is the set of vertices such that $x \in C_i(G)$ if for all $v \in V(G)$, $d(x,v) \leq i$. A set $D \subseteq V$ **$k$-dominates** a set of vertices $U$ of $G$ if for every $x \in U$ there is a $y \in D$ such that $d(x,y) < k$. In particular, a set of vertices $D$ of $V(G)$ is called a **$k$-dominating set** of $G$ if $D$ $k$-dominates $V(G)$.

P. Erdös, M. Saks and V. T. Sós [3] proved that $C_k(G)$ is nonempty for any $P_{2k+1}$-free connected graph $G$. In the light of this result, Bacsó and Tuza [1] used the $i$-center and $k$-dominating set to give a characterization for $P_t$-free graphs. Furthermore, they studied the relation between the $i$-center and $k$-dominating set and proved that if $G$ is a connected graph and $C_i(G)$ is a $d$-dominating set, then $C_{i+1}(G)$ is a $(d-1)$-dominating set. For $P_t$-free graphs, they proposed the following conjecture.

**Conjecture 1.** *(Bacsó and Tuza [1]) If $G$ is a $P_{2k+1}$-free connected graph, then $C_i$-dominates $C_{i+1}$ for $k \leq i \leq 2k-2$.*

In general, this conjecture is not true. Tuza [5], O. Favaron and J. L. Fouquet [4] as well as the authors, have independently constructed counterexamples. Although Conjecture 1 does not hold in general, the following result is proved in [4].

**Theorem 1.1.** *Let $G$ be a $P_t$-free graph. Then for every $i \geq \lfloor \frac{t}{2} \rfloor$, $C_i$ 2-dominates $C_{i+2}$.*

Based on this result, Favaron and Fouquet [4] posed the following problem.

**Conjecture 2.** *(Favaron and Fouquet [4]) For a $P_t$-free graph $G$, if $i \geq \lfloor \frac{t}{2} \rfloor$, then $C_i(q+1)$-dominates $C_{i+q} (q \geq 1)$.*

In section 3 of this paper, we prove this conjecture and hence generalize Theorem 1.1.

## 2. The Generalized Neighborhood Lemma

A result which plays an important role in dealing with $P_t$-free graph is "Neighborhood Lemma" [1]. The special case $t = 2k+1$ and $d = k+1$ was first proved by F. R. K. Chung (see [1]).

A vertex $v$ in $G$ is an **$i$-neighbor** of $x$ if and only if $d_G(x, v) = i$. We denote all the $i$-neighbors of $x$ by $V_i(x)$. So $\bigcup_{i=1}^{k} V_i(x)$ is the set of vertices other than $x$, whose distances to $x$ are at most $k$. This set is also denoted by $N_k(x)$. A path from $u$ to $v$ will be called an $u$–$v$ path. Let $P$ be a path and $x, y$ be two vertices on $P$. The subpath of $P$ from $x$ to $y$ is called an $x$–$y$ segment of $P$.

**Lemma 2.1.** *(Neighborhood Lemma, Bacsó and Tza [1]) Let $u$ and $v$ be two nonadjacent vertices in a $P_t$-free graph $(t \geq 4)$. If $s$ is the second vertex of an $u$–$v$ path with $d = d(u, v)$, then $N_{t-d}(u) \subseteq N_{t-d}(s)$.*

We will require a generalization of this lemma in the proof of the main theorem.

**Lemma 2.2.** *(Generalized Neighborhood Lemma) Let $u$ and $v$ be two nonadjacent vertices in a $P_t$-free graph $(t \geq 4)$. If $s$ is is the $j$-th $(j \geq 2)$ vertex of an $u$–$v$ path (i.e., $d(s, u) = j$) with length $d = d(u, v)$, then $N_{t-d}(u) \subseteq N_{t-d+j-2}(s)$.*

**Proof:** Let $x \in N_{t-d}(u)$ be an arbitrary vertex, and choose an $x$–$u$ path $R$ of length $d(x, u)$. Denote the $u$–$v$ path by $D$. Since $D$ and $R$ share the vertex $u$, their union connects $x$ and $v$.

We can choose two (not necessarily distinct) vertices $y \in R$ and $z \in D$, such that the $x$–$y$ segment of $R$ together with the $z$–$v$ segment of $D$ induces an $x$–$v$ path $P$. This can be done if we choose $y \in R$ such that $y$ is adjacent to a vertex of $D$ and $d(x, y)$ is the smallest. The vertex in $D$ which is adjacent to $y$ and closest to $v$ is $z$. Since the graph $G$ is $P_t$-free, the length of the path $P$ is at most $t-2$.

If $z$ belongs to the $u$–$s$ segment of $D$, then

$$d(x,s) \leq d(x,v) - d(v,s) \leq t-2 - (d-j) = t-d+j-2.$$

If $z$ belongs to the $s$–$v$ segment, then

$$\begin{aligned} d(x,s) &\leq d(x,y) + d(y,z) + d(z,s) \\ &\leq d(x,y) + d(y,z) + d(u,y) + d(y,z) - j \\ &= d(u,x) + 2d(y,z) - j \\ &\leq t-d+2-j \leq t-d+j-2. \end{aligned}$$

Therefore, $x \in N_{t-d+j-2}(s)$. ∎

## 3. The Main Theorem

In this section we prove Conjecture 2, which is our main result and generalizes Theorem 1.1.

**Theorem 3.1.** *Let $G$ be a $P_t$-free connected graph. If $i \geq \lfloor \frac{t}{2} \rfloor$, then $C_i(q+1)$-dominates $C_{i+q}$ for $q \geq 1$.*

**Proof:** If $t < 4$, then $G$ is a complete graph. $C_i(G) = V(G)$, the theorem is obviously true. Suppose $t \geq 4$. Since $G$ is a $P_t$-free graph, for any $u, v \in V$, we have $d(u,v) \leq t-2$ or $C_{t-2}(G) = V$. To prove the theorem, we need only consider $i+q \leq t-2$, i.e. $q \leq t-i-2 \leq 2i+1-i-2 \leq i-1$.

We have that $C_{i+q} = (C_{i+q} - C_{i+q-1}) \cup (C_{i+q-1} - C_{i+q-2}) \cup \cdots \cup (C_{i+1} - C_i) \cup C_i$. For any $x \in C_{i+q} - C_i$, there is a $p$, $1 \leq p \leq q$, such that $x \in C_{i+p} - C_{i+p-1}$. If we can show that $x$ is $(p+1)$-dominated by $C_i$, then of course $x$ is $(q+1)$-dominated by $C_i$. This will prove the theorem. Also, notice that $p \leq i-1$.

Let $x \in C_{i+p} - C_{i+p-1}$. Let $B = \{b \in V_{p+1}(x) : \exists w \in V_{i+p}(x)$ such that $d(b,w) = i-1\}$. Then $B \neq \emptyset$. If $B \cap C_i \neq \emptyset$, then $C_i(p+1)$-dominates $x$ and we are done. Suppose then that $B \cap C_i = \emptyset$. We will obtain a contradiction.

**Claim.** *For any $s \in B$, $V_{i+r}(s) \subseteq \bigcup_{k=r-1}^{p} V_{i+k}(x)$ for $1 \leq r \leq p$.*

Let $s \in B$, and $w$ be an $(i-1)$-neighbor of $s$ in $V_{i+p}(x)$. There is an $x$–$w$ path $xx_1x_2 \ldots x_p x_{p+1} \ldots x_{i+p-1} w$ with length $i+p$, and $s$ is the $(p+1)$-th vertex $x_{p+1}$ along this path. Note that $d(x_{p-1}, w) = i+p-(p-1) = i+1$, and also $i \geq \frac{t-1}{2}$, that is $t-i-1 \leq i$. There is an integer $m \geq 0$ such that $t-i-1+m = i$, or $t-(i+1-m) = i$. Let $d \leq i+1-m$ be an integer. Then $d(x_{p-1}, w) \geq d$.

When $p$ is odd, applying the Neighborhood Lemma to the paths $x$-$x_d$, $x_2$-$x_{d+2}, \ldots, x_{p-1}$-$x_{p-1+d}$, respectively, we have

$$N_{t-d}(x) \subseteq N_{t-d}(x_2) \subseteq \cdots \subseteq N_{t-d}(x_{p-1}) \subseteq N_{t-d}(s).$$

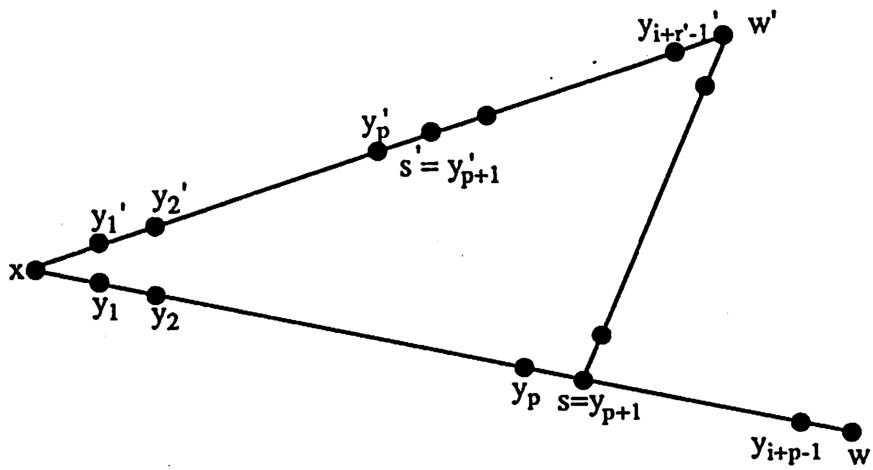


Figure 1

If $p$ is even, applying the Neighborhood Lemma to the paths $x$–$x_d$, $x_2$–$x_{d+2}$, $\ldots$, $x_{p-4}$–$x_{p-4+d}$, and applying the Generalized Neighborhood Lemma (with $j = 3$) to the path $x_{p-2}$–$x_{p-2+d}$, we obtain

$$N_{t-d}(x) \subseteq N_{t-d}(x_2) \subseteq \cdots \subseteq N_{t-d}(x_{p-2}) \subseteq N_{t-d+1}(s).$$

In any case, we have $N_{t-d}(x) \subseteq N_{t-d+1}(s)$. Thus for any $j/gei$, letting $j = t-d$, we have $N_j(x) \subseteq N_{j+1}(s)$ for $j \geq i$.

Let $y \in V_{i+r}(s)$. Then $d(s,y) = i + r$. If $d(x,y) < i + r - 1$, then $y \in N_{i+r-2}(x) \subseteq N_{i+r-1}(s)$ which is impossible. But $d(x,y) \leq i + p$ as $x \in C_{i+p}$, therefore, $y \in \bigcup_{k=r-1}^{p} V_{i+k}(x)$. The claim is proved.

Let $w$ be any $(i-1)$-neighbor of $s$ in $V_{i+p}$ and $w'$ be a $(i+r)$-neighbor of $s$ for some $r \geq 1$. By the above claim, the shortest $x$–$w'$ path $P_{xw'}$ has length $i + r'$, where $r - 1 \leq r' \leq p$ and $i + y'/gep + 1$.

Let $y_j$ (resp. $y_j'$) be a vertex of $V_j(x) \cap P_{xw}$ (resp. $V_j(x) \cap P_{xw'}$) for $j \geq 0$. In particular, $x = y_0 = y_0'$ and $s = y_{p+1}$. We use $s'$ to denote $y_{p+1}'$ if $i + r' \geq p + 1$. The paths $x$–$w$ and $x$–$w'$ are labeled as $P_{xw} = xy_1y_2\ldots y_py_{p+1}\ldots y_{p+i-1}w$ and $P_{xw'} = xy_1'y_2'\ldots y_p'y_{p+1}'\ldots y_{i+r'-1}'w'$ (see Figure 1).

Let $h = \left\lceil \frac{p+r'-r}{2} \right\rceil$. We show the following claims.

(1) $y_l = y_k'$ *is only possible for* $l = k \leq h$.

Suppose that $y_l = y_k'$. If $l \neq k$, say $l < k$, then the path $x$–$y_l(= y_k')$–$w'$ has length $l + i + r' - k < i + r'$ (for the case $l > k$, consider the path $x$–$y_k(= y_l)$–$w$). This is a contradiction.

If $l = k$, and $l > p + 1$, then the path $s$–$y_l( = y_k')$–$w'$ has length

$$l - (p+1) + i + r' - l = i + r' - p - 1 \geq i + r$$

and hence $r' \geq p + r + 1$. This is impossible as $r \geq 1$ by the hypothesis and $r' \leq p$ by the claim.

So we can assume $l \leq p + 1$. Then the path $s$–$y_l(= y'_k)$–$w'$ has length

$$p + 1 - l + i + r' - l = i + r' + p + 1 - 2l \geq i + r$$

or $l \leq \frac{p+r'-r+1}{2}$. Since $l$ is an integer, we have $l \leq h$.

(2) *Assume $y_l y'_l \in E$, then $l \leq h + 1$, and $y_{h+1} y'_{h+1} \in E$ only if $\frac{p+r'-r}{2}$ is an integer.*

If $l > p + 1$, the path $s$–$y_l$–$y'_l$–$w'$ has length

$$l - (p + 1) + 1 + i + r' - l = -p + i + r' \geq i + r,$$

this implies $r' \geq p + r$. Again, this is a contradiction as $r' \leq p$ and $r \geq 1$.

So we assume $l \leq p + 1$. The path $s$–$y_l$–$y'_l$–$w'$ has length

$$p + 1 - l + 1 + i + r' - l = p + 2 + i + r' - 2l \geq i + r.$$

Thus $l \leq \frac{p+r'-r}{2} + 1 \leq h + 1$. So, if $l = h + 1$, then $\frac{p+r'-r}{2}$ is an integer.

(3) *Assume $y_l y'_k \in E$, then $|l - k| \leq 1$.*

Suppose to the contrary that $y_l y'_k \in E$ and $k - l > 1$, say. Then the path $x$–$y_l$–$y'_k$–$w'$ has length $l + 1 + i + r' - k < i + r'$ which is a contradiction.

(4) *Assume $y_l y'_{l+1} \in E$, then $l \leq h + 1$.*

If $l \leq p + 1$, the path $s$–$y_l$–$y'_{l+1}$–$w'$ has length

$$p + 1 - l + 1 + i + r' - (l + 1) = p + i + r' - 2l + 1 \geq i + r,$$

which implies $l \leq h + 1$.

If $l > p + 1$, the path $s$–$y_l$–$y'_{l+1}$–$w'$ has length

$$l - (p + 1) + 1 + i + r' - (l + 1) = -p + i + r' - 1 \geq i + r,$$

hence $r' \geq p + r + 1$. As before, this is a contradiction.

(5) *Assume $y_l y'_{l-1} \in E$, then $l \leq \frac{p+r-r'+3}{2}$, or $r = 1, r' = p$.*

If $l \leq p + 1$, the path $s$–$y_l$–$y'_{l-1}$–$w'$ has length

$$p + 1 - l + 1 + i + r' - (l - 1) = p + 3 + i + r' - 2l \geq i + r.$$

Hence $l \leq \frac{p+r-r'+3}{2}$.

If $l > p + 1$, the path $s$–$y_l$–$y'_{l-1}$–$w'$ has length

$$l - (p + 1) + 1 + i + r' - (l - 1) = -p + 1 + i + r' \geq i + r.$$

Together with $r' \leq p$, we have that $r = 1$ and $r' = p$.

The claims are proved.

Unless $k = l - 1$ and $(r, r') = (l, p)$, we can now construct an induced path from $w$ to $w'$ that has length at least $t - 1$. There are two cases.

Case 1. There exist vertices $y_l$ and $y'_k$ such that $y_l = y'_k$ and the union of paths $y_l$–$w$ and $y'_k$–$w'$ form an induced path from $w$ to $w'$.

In this case, we have $l = k \leq h$ by (1), and the $w$–$w'$ path has length

$$\begin{aligned} i + p - l + i + r' - l &\geq 2i + p + r' - 2\frac{p + r' - r + 1}{2} \\ &\geq 2i + r - 1 \geq 2i \geq t - 1. \end{aligned}$$

Case 2. There exist vertices $y_l$ and $y'_k$ such that $y_l y'_k \in E$ and paths $y_l$–$w$, $y'_k$–$w'$ together with the edge $y_l y'_k$ form an induced path between $w$ and $w'$.

In this case, $y_l y'_k \in E$, which implies that $|l - k| \leq 1$ by (3). If $k = l + 1$ or $l = k < h + 1$, the path $w$–$w'$ has length

$$\begin{aligned} i + p - l + 1 + i + r' - k &\geq 2i + p + r' - 2l \\ &\geq 2i + r - 1 \geq 2i \geq t - 1. \end{aligned}$$

If $l = k = h + 1$, then $\frac{p+r'-r}{2}$ is an integer by (2) again. In this case, the $w$–$w'$ path has length

$$\begin{aligned} i + p - l + 1 + i + r' - l &= 2i + p + r' + 1 - (p + r' - r + 2) \\ &= 2i + r - 1 \geq 2i \geq t - 1. \end{aligned}$$

If $k = l - 1$, and $(r, r') \neq (1, p)$, then $l \leq \frac{p+r-r'+3}{2}$, by (5) the $w$–$w'$ path has length

$$\begin{aligned} i + p - l + 1 + i + r' - (l - 1) &= 2i + p + r' - 2l + 2 \\ &\geq 2i + r - 1 \geq 2i \geq t - 1. \end{aligned}$$

Hence, in any case above we have an induced path $w$–$w'$ with length at least $t - 1$. This contradicts that $G$ is $P_t$-free graph.

If $k = l - 1$ and $(r, r') = (1, p)$, we cannot obtain contradiction as above. In this case, suppose that we have chosen $s \in B$, such that $|V_{i-1}(s)|$ is maximum. But we have $w \in V_{i-1}(s')$ for each $w \in V_{i-1}(s)$ (see Figure 1), hence $|V_{i-1}(s')| > |V_{i-1}(s)|$, which contradicts the choice of $s$. This shows that $s \in C_i$. ∎

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