Stolarsky Interspersions

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1. Introduction.

In his paper [4], David R. Morrison discusses two very interesting Stolarsky arrays and writes that "It would be interesting to have a classification of the Stolarsky arrays." Here, we introduce a class of Stolarsky arrays which we call *Stolarsky interspersions*. These arrays are in one-to-one correspondence with the zero-one sequences that begin with 1, and they include two arrays discussed by Morrison [4], namely, Stolarsky's original array (Table 1) of 1977 and the Wythoff array (Table 2), introduced by Morrison.

Throughout, let $\alpha = (1+\sqrt{5})/2$. The indexing of all sequences will start with 1. The notation s_j will be accompanied by an identifier such as "the sequence" or "the number". Rows and columns of arrays will frequently be regarded as sequences; thus, the notation $a(i,j)$ can have any of four possible meanings: an array (both i and j vary), a row (j varies), a column (i varies), or a single number; nearby words will specify the intended meaning.

A sequence s_j is a *positive Fibonacci sequence* if

(F1) the recurrence $s_j = s_{j-1} + s_{j-2}$ holds for all $j \geq 3$;
(F2) there exists J for which $s_j > 0$ for all $j > J$.

A *Stolarsky array* is an array $A = a(i,j)$ of positive integers such that

(S1) for every i, the ith row, $a(i,j)$, is a positive Fibonacci sequence;
(S2) every positive integer occurs exactly once in A.

An array A is a *Stolarsky interspersion* if A satisfies (S1)-(S2) and also properties (S3) and (S4):

(S3) every column of A is an increasing sequence;
(S4) if u_j and v_j are distinct rows of A, and p and q are any indices for which $u_p < v_q < u_{p+1}$, then $u_{p+1} < v_{q+1} < u_{p+2}$.

The *interspersion property*, (S4), may be described as follows: as soon as a term of one row fits between successive terms of another row, all succeeding terms of each of these rows fit individually between individual terms of the other row. See Table 1 and Table 2. Arrays that satisfy (S1)-(S4) are called *interspersions*, and properties of interspersion proved in [4] and [3] hold, in particular, for Stolarsky interspersions.

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Table 1. The original Stolarsky array

1	2	3	5	8	13	21	34	55	89	144	⋯
4	6	10	16	26	42	68	110	178	288	466	
7	11	18	29	47	76	123	199	322	521	843	
9	15	24	39	63	102	165	267	432	699	1131	
12	19	31	50	81	131	212	343	555	898	1453	
14	23	37	60	97	157	254	411	665	1076	1741	
17	28	45	73	118	191	309	500	809	1309	2118	
20	32	52	84	136	220	356	576	932	1508	2440	
22	36	58	94	152	246	398	644	1042	1686	2728	
⋮											

Table 2. The Wythoff array

1	2	3	5	8	13	21	34	55	89	144	⋯
4	7	11	18	29	47	76	123	199	322	521	
6	10	16	26	42	68	110	178	288	466	754	
9	15	24	39	63	102	165	267	432	699	1131	
12	20	32	52	84	136	220	356	576	932	1508	
14	23	37	60	97	157	254	411	665	1076	1741	
17	28	45	73	118	191	309	500	809	1309	2118	
19	31	50	81	131	212	343	555	898	1453	2351	
22	36	58	94	152	246	398	644	1042	1686	2728	
⋮											

In (R3) just below, $[x]$ denotes the greatest integer $\leq x$. Later we use $((x))$ to denote the fractional part, $x - [x]$, of x.

Call an array $A = a(i,j)$ a *regular array* if

(R1) the first row of A is given by $a(1,1) = 1$, $a(1,2) = 2$, and $a(1,j) = a(1,j-1) + a(1,j-2)$ for all $j \geq 3$;

(R2) for all $q \geq 1$, $a(q+1,1) =$ least positive integer not in the first q rows of A;

(R3) for all $q \geq 1$, $a(q+1,2) \in \{[\alpha a(q,1)], [\alpha a(q,1)+1]\}$;

(R4) for all $q \geq 1$, $a(q+1,j) = a(q+1,j-1) + a(q+1,j-2)$ for all $j \geq 3$.

2.{ Stolarsky interspersions } = { regular arrays } .

The heading of this section indicates the main result in the article, which takes the form of Theorem 1 and Theorem 2. We begin with five lemmas.

Lemma 1.1. *Suppose u is a positive integer. Then*

$$((\alpha((\alpha u)))) = \begin{cases} \alpha((\alpha u)) & \text{if } ((\alpha u)) < ((\alpha((\alpha u)))) \\ \alpha((\alpha u)) - 1 & \text{if } ((\alpha u)) > ((\alpha((\alpha u)))) \end{cases}.$$

Proof: First, suppose $((\alpha u)) < \left((\alpha((\alpha u)))\right)$. That is, $((\alpha u)) < \alpha\,((\alpha u)) - [\alpha((\alpha u))]$, so that $[\alpha((\alpha u))] < (\alpha - 1)\,((\alpha u)) < 1$, whence $[\alpha((\alpha u))] = 0$, which implies $\alpha((\alpha u)) = \left((\alpha((\alpha u)))\right)$. On the other hand, if $((\alpha u)) > \left((\alpha((\alpha u)))\right)$, we similarly find that $[\alpha((\alpha u))] > (\alpha - 1)\,((\alpha u)) > 0$, so that $[\alpha((\alpha u))] = 1$, which implies $\left((\alpha((\alpha u)))\right) = \alpha((\alpha u)) - 1$. ∎

Lemma 1.2. *Suppose u is a positive integer. Then*

$$\left((\alpha u - \alpha((\alpha u)))\right) = ((\alpha u)) - \alpha((\alpha u)) + 1.$$

Proof:

$$\begin{aligned}\left((\alpha u - \alpha((\alpha u)))\right) &= \begin{cases} 1 + ((\alpha u)) - \alpha((\alpha u)) & \text{if } ((\alpha u)) < \left((\alpha((\alpha u)))\right) \\ ((\alpha u)) - \alpha((\alpha u)) & \text{if } ((\alpha u)) > \left((\alpha((\alpha u)))\right) \end{cases} \\ &= ((\alpha u)) - \alpha((\alpha u)) + 1 \text{ by Lemma 1.1.}\end{aligned}$$

∎

Lemma 1.3. *Suppose u is a positive integer. Then* $\left[\alpha[\alpha u]\right] + 1 = u + [\alpha u]$.

Proof:

$$\begin{aligned}\left[\alpha[\alpha u]\right] + 1 &= \alpha[\alpha u] + 1 - ((\alpha[\alpha u])) \\ &= \alpha\left\{\alpha u - ((\alpha u))\right\} + 1 - ((\alpha[\alpha u])) \\ &= \alpha^2 u + 1 - \alpha((\alpha u)) - ((\alpha[\alpha u])) \\ &= u + \alpha u + 1 - \alpha((\alpha u)) - \left((\alpha^2 u - \alpha((\alpha u)))\right) \\ &= u + [\alpha u] + ((\alpha u)) + 1 - \alpha((\alpha u)) - \left((\alpha u - \alpha((\alpha u)))\right) \\ &= u + [\alpha u] \text{ by Lemma 1.2.}\end{aligned}$$

∎

Lemma 1.4. *Suppose v is a positive integer of the form* $[\alpha u]$ *for some positive integer u. Then* $[\alpha v + \alpha] = [\alpha v + 2]$. *If v is of the form* $[\alpha u + 1]$, *then* $[\alpha v + \alpha] = [\alpha v + 1]$.

Proof: It is known that v is of the form $[\alpha u]$ if and only if $((\alpha v)) > 2 - \alpha$. (See, for example, Section 2 of Fraenkel, Porta, and Stolarsky [2].) This inequality is equivalent to each of the following: $((\alpha v)) + ((\alpha)) > 1$, $((\alpha v)) > ((\alpha v + \alpha))$, $[\alpha v] - [\alpha v + \alpha] > \alpha$, $[\alpha v] - [\alpha v + \alpha] = 2$, and finally, $[\alpha v + \alpha] = [\alpha v + 2]$.

If v is not of the form $[\alpha u]$ then it must be of the form $[\alpha u + 1]$, and since then $[\alpha v + \alpha] \neq [\alpha v + 2]$, we have $[\alpha v + \alpha] = [\alpha v + 1]$. ∎

Lemma 1.5. *Suppose u_j is a row of a regular array. Then either*

$$u_{2k} = [\alpha u_{2k-1}] \text{ and } u_{2k+1} = [\alpha u_{2k} + 1] \text{ for all } k \geq 1,$$

or else

$$u_{2k} = [\alpha u_{2k-1} + 1] \text{ and } u_{2k+1} = [\alpha u_{2k}] \text{ for all } k \geq 1.$$

Proof: By (R3), $u_2 = [\alpha u_1]$ or else $u_2 = [\alpha u_1 + 1]$, so we have two cases.
Case 1. $u_2 = [\alpha u_1]$. Suppose for arbitrary $m \geq 1$ that $u_{2m} = [\alpha u_{2m-1}]$. Then

$$\begin{aligned} u_{2m+1} &= [\alpha u_{2m-1}] + u_{2m-1} \text{ by (R4)} \\ &= [\alpha[\alpha u_{2m-1}]] + 1 \text{ by Lemma 1.3} \\ &= [\alpha u_{2m}] + 1. \end{aligned}$$

Now suppose for arbitrary $m \geq 1$ that $u_{2m+1} = [\alpha u_{2m} + 1]$. Then

$$\begin{aligned} u_{2m+2} &= [\alpha u_{2m}] + u_{2m} \text{ by (R4)} \\ &= [\alpha[\alpha u_{2m}]] + 2 \text{ by Lemma 1.3} \\ &= [\alpha[\alpha u_{2m}] + \alpha] \text{ by Lemma 1.4} \\ &= [\alpha u_{2m+1}] \text{ by induction hypothesis.} \end{aligned}$$

We conclude that $u_{2k} = [\alpha u_{2k-1}]$ and $u_{2k+1} = [\alpha u_{2k} + 1]$ for all $k \geq 1$.
Case 2. $u_2 = [\alpha u_1 + 1]$. The proof in this case is quite similar to that of Case 1 and is omitted. ∎

Theorem 1. *Every regular array is a Stolarsky interspersion.*

Proof: Let $A = a(i,j)$ be a regular array. Each positive integer occurs at least once because of (R2). We shall show next that no entry of A can appear more than once. Suppose $\hat{u} = a(i,j)$ and $\hat{v} = a(m,n)$ are terms of a regular array $A = a(i,j)$ and that $\hat{v} = \hat{u}$; we assert that $(m,n) = (i,j)$. If not, assume $\hat{u}$ is the least positive integer for which there is a number $\hat{v}$ as described. Now, neither i nor m is 1 because of the manner in which terms in Column 1 are chosen. Therefore, $\hat{u} = [\alpha u]$ or $\hat{u} = [\alpha u + 1]$ for some u, by Lemma 1.5; similarly, $\hat{v} = [\alpha v]$ or $\hat{v} = [\alpha v + 1]$ for some v.

Case 1. $\hat{u} = [\alpha u]$ and $\hat{v} = [\alpha v]$, or $\hat{u} = [\alpha u + 1]$ and $\hat{v} = [\alpha v + 1]$. Then $[\alpha u] = [\alpha v]$, from which easily follows $u = v$, so that $a(i, j-1) = a(m, n-1)$. But $u < \hat{u}$, contrary to our choice of $\hat{u}$ as the least repeated term of the array. Thus, Case 1 cannot occur.

Case 2. $\hat{u} = [\alpha u]$ and $\hat{v} = [\alpha v + 1]$, or $\hat{u} = [\alpha u + 1]$ and $\hat{v} = [\alpha v]$. It suffices to consider only the former. From $[\alpha u] = [\alpha v + 1]$ follows $\alpha u - ((\alpha u)) = \alpha v - ((\alpha v)) + 1$, so that

$$v = u + \frac{1}{\alpha}\{((\alpha v)) - ((\alpha u)) - 1\}.$$

Writing this as $v = u + Q$, we note that Q, as an integer strictly between $-2/\alpha$ and 0, must be -1, so that $u = v + 1$. The equation $[\alpha u] = [\alpha v + 1]$ therefore implies

$$[\alpha v + \alpha] = [\alpha v + 1]. \tag{1}$$

Now, since v immediately precedes $\hat{v} = [\alpha v + 1]$ in a row of a regular array, $v = [\alpha w]$ for some positive integer w, by Lemma 1.5. Therefore, by Lemma 1.4, Equation (1) cannot occur, and we conclude that no positive integer occurs more than once in A. We show next that A has the interspersion property, (S4).

Suppose u_j and v_j are distinct rows of A, and for convenience, suppose $u_1 < v_1$. (The argument if $u_1 > v_1$ is analogous to what follows.) Let p be the index for which $u_p < v_1 < u_{p+1}$. Lemma 1.5 shows that there are four cases:

Case 1. $v_2 = [\alpha v_1]$, $u_{p+1} = [\alpha u_p]$, and $u_{p+2} = [\alpha u_{p+1} + 1]$. As a first induction step, clearly

$$u_{p+1} < v_2 < u_{p+2}. \tag{2}$$

Assume for arbitrary $m \geq 2$ that $u_{p+k-1} < v_k < u_{p+k}$. Then adding the inequalities

$$\begin{aligned} &u_{p+m-1} < v_m < u_{p+m} \text{ and } u_{p+m-2} < v_{m-1} < u_{p+m-1} \\ &\text{gives } u_{p+m} = u_{p+m-1} + u_{p+m-2} < v_m + v_{m-1} = v_{m+1} \\ &\qquad < u_{p+m} + u_{p+m-1} = u_{p+m+1}. \end{aligned}$$

By induction, (S4) holds.

Case 2. $v_2 = [\alpha v_1]$, $u_{p+1} = [\alpha u_p + 1]$, and $u_{p+2} = [\alpha u_{p+1}]$. Since $[\alpha u_p + 1]$ lies in Row u_j and $[\alpha v_1]$ lies in Row v_j, we have $[\alpha u_p + 1] \neq [\alpha v_1]$. Assume that $[\alpha u_p + 1] > [\alpha v_1]$. Then $\alpha u_p + 1 - ((\alpha u_p)) > \alpha v_1 - ((\alpha v_1))$, so that $v_1 - u_p < 2/\alpha$, whence $v_1 = u_p + 1$. But then $[\alpha v_1] = [\alpha u_p + \alpha] \geq [\alpha u_p + 1]$, a contradiction. Therefore, $[\alpha u_p + 1] < [\alpha v_1]$, so that $u_{p+1} < v_2$. The condition $v_1 < u_{p+1}$ implies $v_2 < u_{p+2}$. Thus, the first step (2) holds, and just as in the proof for Case 1, (S4) holds.

Case 3. $v_2 = [\alpha v_1 + 1]$, $u_{p+1} = [\alpha u_p]$, and $u_{p+2} = [\alpha u_{p+1} + 1]$. Here, (S2) holds, and (S4) follows as in the proof for Case 1.

Case 4. $v_2 = [\alpha v_1 + 1]$, $u_{p+1} = [\alpha u_p + 1]$, and $u_{p+2} = [\alpha u_{p+1} + 1]$. Again, (S2) holds, and (S4) follows as in the proof for Case 1.

Theorem 2. *Every Stolarsky interspersion is a regular array.*

Proof: First, it is easy to show that (S1)-(S3) imply (R1), (R2), (R4). We show next that (R4) holds also. Suppose $A = a(i,j)$ is a Stolarsky interspersion. For any $i \geq 2$, let

$$u = a(i,1), \quad a(i_1,j_1) = u-1, \quad \text{and} \quad a(i_2,j_2) = u+1.$$

We shall show first that $a(i_2,j_2+1) - a(i_1,j_1+1) = 3$, then that $a(i,2)$ is one of the numbers $a(i_1,j_1+1)+1$ or $a(i_1,j_1+1)+2$, and finally that $a(i_1,j_1+1)+1 = [\alpha u]$. We shall then conclude that $a(i,2)$ is one of the numbers $[\alpha a(i,1)]$ or $[\alpha a(i,1)]+1$.

Let C denote the sequence $a(i,1)$; that is, Column 1 of the array A. Let v_1 be the number of terms in C that are $\leq a(i_1,j_1)$, and let v_2 be the number of terms in C that are $\leq a(i_2,j_2)$. Now $v_2 - v_1 = 1$, since otherwise $u-1$ or $u+1$, as well as u, belongs to C; but this is impossible, for if two consecutive positive integers w and $w+1$ belong to C, then let $a(k,1)$ be the greatest number in C that is less than w, so that $a(k,1) < w < w+1 < a(k,2)$, contrary to (S4).

By Lemma 3 in [3], we have

$$v_1 = a(i_1,j_1+1) - a(i_1,j_1) \text{ and } v_2 = a(i_2,j_2+1) - a(i_2,j_2), \text{ so that}$$
$$a(i_2,j_2+1) - a(i_1,j_1+1) = v_2 - v_1 + a(i_2 j_2) - a(i_1,j_1)$$
$$= 1 + u + 1 - (u-1) = 3.$$

Since $a(i_1,j_1) < a(i,1) < a(i_1,j_1+1)$, we have $a(i,2) > a(i_1,j_1+1)$, by (S4). Similarly, since $a(i,1) < a(i_2,j_2) < a(i,2)$, we have $a(i,2) < a(i_2,j_2+1)$. Thus, $a(i,2)$, lying strictly between $a(i_1,j_1+1)$ and $a(i_2,j_2+1)$, must be $a(i_1,j_1+1)+1$ or $a(i_1,j_1+1)+2$. There are two cases:

Case 1. $u-1 = [\alpha m]$ for some positive integer m. By Lemma 1.5, $a(i_1,j_1+1) = [\alpha u - \alpha + 1]$. Let $v = u-1$ in Lemma 1.4 to obtain $[\alpha v + \alpha] = [\alpha v + 2]$. That is, $[\alpha u] = [\alpha u - \alpha + 2]$, so that $a(i_1,j_1+1)+1 = [\alpha u]$.

Case 2. $u-1 = [\alpha m + 1]$ for some positive integer m. By Lemma 1.5, $a(i_1,j_1+1) = [\alpha(u-1)]$. Let $v = u-1$ in Lemma 1.4 to obtain $[\alpha v] = +[\alpha v + \alpha - 1]$. That is, $[\alpha u - \alpha] = [\alpha u - 1]$, so that $a(i_1,j_1)+1 = [\alpha u]$. ∎

3. Classification of Stolarsky interspersions.

Theorem 3. *The set of Stolarsky interspersions are in one-to-one correspondence with the set of all zero-one sequences that begin with 1.*

Proof: Let A be a Stolarsky interspersion. For $i \geq 2$, each entry $a(i,1)$ in Column 1 is uniquely determined by the entries of the preceeding rows, and all column entries in columns numbered ≥ 3 are determined by the recurrence $a(i,j) =$

$a(i,j-1)+a(i,j-2)$ for $j \geq 3$. Therefore, the sequence $a(i,2)$, which is Column 2, determines A.

Since the Stolarsky interspersions are in one-to-one correspondence with the regular arrays, we have $a(i,2) \in \{[\alpha a(i,1)], [\alpha a(i,1)+1]\}$ for all $i \geq 2$. Define the *classification sequence of A* by

$$\phi_1(A) = 1 \text{ and } \phi_i(A) = \begin{cases} 0 & \text{if } a(i,2) = [\alpha a(i,1)] \\ 1 & \text{if } a(i,2) = [\alpha a(i,1)+1] \end{cases} \quad \text{for } i = 2,3,\ldots.$$

It is clear that the mapping $A \to \phi_i(A)$ is a one-to-one function from the set of all Stolarsky interspersions onto the set of all zero-one sequences that begin with 1. Conversely, every zero-one sequence beginning with 1 yields a Stolarsky interspersion, by (R3) and Theorem 1. ∎

As suggested by the last sentence of the proof of Theorem 3, it is helpful to think of (B3) as a means of constructing all possible Stolarsky interspersions: for each row beyond the first, after the first term u (which is determined by (R2)), we can, by (R3), choose either $[\alpha u]$ or $[\alpha u + 1]$ to be the second term. The availability of both of these numbers means that neither has occurred previously in the construction. Thus, we have the following corollary.

Corollary 3.1. *Suppose* $A = a(i,j)$ *is a Stolarsky interspersion. If* $a(i,2) = [\alpha u]$ *then* $[\alpha u + 1]$ *occurs in a row of A numbered higher than* i. *If* $a(i,2) = [\alpha u + 1]$ *then* $[\alpha u]$ *occurs in a row of A numbered higher than* i.

Theorem 4. *The classification sequence of the Wythoff array W is given by* $\phi_i(W) = 1$ *for all* $n \geq 1$.

Proof: Column 2 of W is given (for example, Morrison [4]) by $w(i,2) = [\alpha^2[\alpha i]]$. By Lemma 1.3, then, $w(i,2) = [\alpha[\alpha[\alpha i]]] + 1$, which is $[\alpha w(i,1)] + 1$. ∎

Theorem 4 shows the Wythoff array to be maximal among the Stolarsky interspersions in the sense that its classification sequence dominates all others: $\phi_i(W) \geq \phi_i(A)$ for all $i \geq 1$, for all Stolarsky interspersions A. It is natural to inquire about the minimal array, W', given by $\phi_i(W') \leq \phi_i(A)$ for all $i \geq 1$, for all A. We call W' the *Wythoff dual*. (Generally, every A has a *dual* A' defined by

$$\phi_1(A) = 1 \text{ and } \phi_i(A') = \begin{cases} 1 + \phi_i(A) \text{ if } \phi_i(A) = 0 \\ 1 - \phi_i(A) \text{ if } \phi_i(A) = 1 \end{cases} \quad \text{for } i = 2,3,\ldots.$$

It is not difficult to prove that the first column of the Wythoff dual is given by

$$w'(1,1) = 1 \text{ and } w'(i,1) = [\alpha[\alpha i - \alpha + 1] + 1] \text{ for } i = 2,3,\ldots.$$

Column 2 is given by $w'(i,2) = [\alpha w'(i,1)]$, Column 3 by $w'(i,3) = [\alpha w'(i,2) + 1]$, and so on, in accord with Lemma 1.5.

Consider the array $B = b(i,j)$ defined as follows:

(B1) the first row of B is given by $b(1,1) = 1$, $b(1,2) = 2$, and $b(1,j) = b(1,j-1) + b(1,j-2)$ for all $j \geq 3$;

(B2) for all $q \geq 1$, $b(q+1,1) =$ least positive integer not in the first q rows of A;

(B3) for all $q \geq 1$, $b(q+1,2) =$ least positive integer b such that no term of the Fibonacci sequence $b(q+1,2), b(q+1,2)+b, \ldots$ is equal to any $b(i,j)$ for $1 \leq i \leq q$ and $j \geq 1$;

(B4) for all $q \geq 1$, $b(q+1,j) = b(q+1,j-1) + b(q+1,j-2)$ for all $j \geq 3$.

Note that (B1)-(B4) are like (R1)-(R4) except for (B3), which with (B2) suggests the name *least-least array* for B.

Conjecture. *The least-least array is the Wythoff dual.*

The classification scheme presented in the proof of Theorem 3 depends on which of two possible numbers occupies the second column of a Stolarsky interspersion, for each row after the first. It is natural to ask where the other number — the one not appearing in Column 2 — lies. According to Theorem 5 below, this other number always lies in Column 1.

Lemma 5.1. *Suppose $A = a(i,j)$ is a Stolarsky interspersion. If $v = a(i,1)$ for some $i \geq 2$, then $v+1 = a(k,j)$ for some $k \leq i-1$ and $j \geq 1$.*

Proof: If not, then $v+1 = a(i+1,1)$, by (R2). Since $v-1 = a(h,j)$ for some $h \leq i-1$ and $j \geq 1$, we have $a(h,j) < v < v+1 < a(h,j+1)$, contrary to (S4). ∎

Theorem 5. *Suppose $A = a(i,j)$ is a Stolarsky interspersion and $i \geq 2$. Let $v = a(i,1)$. If $a(i,2) = [\alpha v]$, then $[\alpha v + 1] = a(k,1)$ for some $k \geq i+1$. If $a(i,2) = [\alpha v + 1]$, then $[\alpha v] = a(k,1)$ for some $k \geq i+1$.*

Proof: *Case 1.* $a(i,2) = [\alpha v]$. We have $[\alpha v+1] = a(k,j)$ for some $k \geq i+1$, by Corollary 3.1, for some $j \geq 1$. If $j \geq 2$, let $u = a(k,j-1)$. By Lemma 1.5, $[\alpha v+1] \in \{[\alpha u],[\alpha u+1]\}$. Since $u \neq v$, we have $[\alpha v+1] = [\alpha u]$. But then, as argued in Case 2 of the proof of Theorem 1, $u = v+1$. This means, by Lemma 5.1, that u lies in a row of A numbered $\leq i-1$, contrary to $u = a(k,j-1)$. Therefore, $j = 1$.

Case 2. $a(i,2) = [\alpha v+1]$. The proof is similar to that of Case 1 and is omitted. ∎

4. Row-swapping in Stolarsky interspersions.

For any array $A = a(i,j)$, let R_i denote Row i of A and let R'_k denote all of Row k except the first term. The (i,k) *row-swap* of A, written $A\langle i,k\rangle$, is the array that results from A by interchanging R_i and R'_k. We use the term *row-swap* to mean also the operation that carries A onto $A\langle i,k\rangle$. For example, the succession

$\langle 2,3\rangle, \langle 5,8\rangle, \langle 10,16\rangle$ of row-swaps, applied to the Stolarsky array S (see Table 1) results in an array (which we write as $S\langle\langle 2,3; 5,8; 10,16\rangle\rangle$ whose first twelve rows agree with those of the Wythoff array (Table 2).

Lemma 6.1. $(([\alpha m]/\alpha)) + 1/\alpha > 1$ *for every positive integer* m.

Proof: $(([\alpha m]/\alpha)) + 1/\alpha > 1 = \big(((\alpha-1)[\alpha m])\big) + \alpha - 1 = ((\alpha[\alpha m])) + \alpha - 1$, which exceeds $(2-\alpha) + \alpha - 1$, by the Fraenkel-Porta-Stolarsky inequality used to prove Lemma 1.5. ∎

Lemma 6.2. *For every positive integer* m, $((\alpha m))/\alpha + (([\alpha[\alpha m])) = 1$, $[[\alpha m + 1]/\alpha] = m$, *and* $[[\alpha m]/\alpha] = m - 1$.

Proof: Let $m \geq 1$. Below, we use the inequality in Lemma 6.1 to evaluate $(([\alpha m + 1]/\alpha))$ as $(([\alpha m]/\alpha)) + 1/\alpha - 1$:

$$\begin{aligned}
[[\alpha m + 1]/\alpha] &= [\alpha m + 1]/\alpha - (([\alpha m + 1]/\alpha)) \\
&= \{\alpha m + 1 - ((\alpha m))\}/\alpha - (([\alpha m + 1]/\alpha)) \\
&= m + 1/\alpha - ((\alpha m))/\alpha - (([\alpha m]/\alpha)) - 1/\alpha + 1 \\
&= m + 1 - \{((\alpha m))\}/\alpha + (([\alpha m]/\alpha))\},
\end{aligned}$$

so that $((\alpha m))/\alpha + (([\alpha m]/\alpha))$ is an integer. It is clearly positive and less than 2, so it must be 1, so that the first two assertions are proved. For the third, we have $1/\alpha + (([\alpha m]/\alpha)) > 1$ by Lemma 6.1. Then $1 = [1/\alpha + (([\alpha m]/\alpha))]$ $= [[\alpha m + 1]/\alpha - [[\alpha m]/\alpha]] = [[\alpha m + 1]/\alpha] - [[\alpha m]/\alpha] = m - [[\alpha m]/\alpha]$. ∎

Theorem 6. *Suppose* $A = a(i,j)$ *is a Stolarsky interspersion. Let* i *and* k *be distinct positive integers. Then the row-swap* $A\langle i,k\rangle$ *is a Stolarsky interspersion if and only if*

$$a(k,2) - a(k,1) = a(i,1). \tag{3}$$

Proof: Write $A\langle i,k\rangle$ as $A' = a'(i,j)$, and suppose A' is a Stolarsky interspersion. Then in Row i of A', we have $a'(i,3) = a'(i,2) + a'(i,1)$, and (3) follows.

For the converse, suppose A is a Stolarsky interspersion and i and k are indices for which (3) holds. Let $u = a(i,1)$. By Lemma 1.5, $a(i,2) \in \{[\alpha u], [\alpha u + 1]\}$, and $a(k,1) \in \{[\alpha m], [\alpha m + 1]\}$ for some $m \geq 1$.

Case 1. $a(i,2) = [\alpha u]$ and $a(k,1) = [\alpha m + 1]$. We shall show that $m = u$:

$$\begin{aligned}
u &= a(k,2) - a(k,1) \\
&= [\alpha[\alpha m + 1]] - [\alpha m + 1] \text{ by Lemma 1.5} \\
&= [(\alpha - 1)[\alpha m + 1]] \\
&= [[\alpha m + 1]/\alpha] \\
&= m \text{ by Lemma 6.2.}
\end{aligned}$$

Case 2. $a(i,2) = [\alpha u]$ and $a(k,1) = [\alpha m]$. We shall show that this cannot occur:

$$\begin{aligned} u &= a(k,2) - a(k,1) \\ &= [\alpha[\alpha m] + 1] - [\alpha m] \text{ by Lemma 1.5} \\ &= 1 + [[\alpha m]/\alpha] \\ &= m \text{ by Lemma 6.2.} \end{aligned}$$

But then $a(k,1) = a(i,2)$, contrary to (S2).

Case 3. $a(i,2) = [\alpha u + 1]$ and $a(k,1) = [\alpha m + 1]$. The method of proof for Case 2 readily shows Case 3 to be impossible.

Case 4. $a(i,2) = [\alpha u + 1]$ and $a(k,1) = [\alpha m]$. The method of proof for Case 1 yields $u = m$.

The array $A\langle i,k\rangle$ has the same rows as A except for Row i and Row k. It follows that (R1)-(R4) hold for $A\langle i,k\rangle$; in particular, (R4) holds as a result of Equation (3), and (R3) holds as shown in Cases 1-4 above. By Theorem 1, therefore, $A\langle i,k\rangle$ is a Stolarsky interspersion. ∎

Theorem 6 shows that infinitely many Stolarsky interspersions B can be obtained via a sequence of row-swaps from any given Stolarsky interspersion A. One must not ask for too much, however, for it is not true that *every* B can be obtained from every A. For example, one can start with the original Stolarsky array (Table 1) and find a sequence of row-swaps that place all the terms of row 2 (that is, $4, 6, 10, 16, \ldots$) into column 1. It is clear that the resulting array cannot be returned by row-swapping to the original array. One wonders just which Stolarsky arrays can be obtained from others by row-swapping.

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