Toward Computing $m(4)$

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1 Introduction

A hypergraph is called bipartite [6] if its vertices can be colored in two colors so that no hyperedge is monochromatic. Erdös [7] defines a function $m(n)$ to be the minimal number of hyperedges in an n-uniform non-bipartite hypergraph. It is easy to see that $m(2) = 3$ and the corresponding hypergraph is a triangle. The Fano plane, the projective plane of order 2, shows that $m(3) \leq 7$; furthermore, it is not too difficult to prove that $m(3) = 7$.

Let G be an n-uniform hypergraph on v vertices. Since there are 2^{v-1} different 2-colorings of $V(G)$ and every hyperedge is monochromatic under 2^{v-n} of them, a non-bipartite hypergraph G must have at least $2^v/2^{v-n} = 2^{n-1}$ hyperedges. Thus, (see [9]) $m(n) \geq 2^{n-1}$. It turns out (see [9]) and ([15]) that a better lower bound can be obtained by using *balanced 2-colorings*, for which the color classes represent a partition of $V(G)$ into sets of equal size (within one element). If v and m are, respectively, the number of vertices and the number of hyperedges in a non-bipartite n-uniform hypergraph, then

$$m \geq \begin{cases} \frac{v!((v/2)-n)!}{2(v/2)!(v-n)!}, & \text{if } v \text{ even;} \\ \frac{v!(\lceil v/2 \rceil - n)!}{\lfloor v/2 \rfloor !(v+1-n)!}, & \text{else.} \end{cases} \qquad (*)$$

Although much stronger lower bounds are known for the asymptotics of $m(n)$ (see [4],[12]), they do not provide an improvement for small n. In

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this paper, we are concerned with the case of $n = 4$. Abbott and Hanson [2] constructed a hypergraph showing that $m(4) \leq 24$. Their construction was independently improved by Seymour[11] and Toft[14]. The new hypergraph[1] contains 23 quadruples of an 11-element set; thus, $m(4) \leq 23$.

If a non-bipartite n-uniform hypergraph contains two vertices not in any hyperedge, then contracting them yields a new non-bipartite n-uniform hypergraph with a fewer vertices. Thus, the problem of computing $m(4)$ can be considered on those hypergraph for which every pair of vertices belongs to a hyperedge. Selfridge ([13]) defines $m_v(n)$ to be the minimal number of hyperedges in a non-bipartite n-uniform hypergraph with v vertices satisfying the condition on vertex pairs. Everywhere in this paper, the hypergraphs are assumed to satisfy this condition. It is easy to see that in such a hypergraph, every vertex belongs to at least $(v-1)/(n-1)$ hyperedges, implying

$$m_v(n) \geq \left\lceil \frac{v}{n} \left\lceil \frac{v-1}{n-1} \right\rceil \right\rceil . \qquad (**)$$

For the case of $n = 4$, more is known. Abbot and Liu ([3]) proved $24 \leq m_9(4) \leq 26$; Exoo shows in [10], that $m_{10}(4) \leq 25$. The anonymous referee to this paper presents the following constructions showing that $m_k(4) \leq 26$ for $k = 12, 13, 14$. For $k = 14$, the quadruples are

$$\begin{array}{llllll}
(1,2,3,8) & (1,4,5,8) & (1,6,7,8) & (2,4,6,8) & (2,5,7,8) & (3,4,7,8) \\
(3,5,6,8) & (1,2,3,9) & (1,4,5,9) & (1,6,7,9) & (2,4,6,9) & (2,5,7,9) \\
(3,4,7,9) & (3,5,6,9) & (1,2,3,a) & (1,4,5,a) & (1,6,7,a) & (2,4,6,a) \\
(2,5,7,a) & (3,4,7,a) & (3,5,6,a) & (8,9,a,b) & (8,9,1,c) & (8,9,a,d) \\
(8,9,a,e) & (b,c,d,e). & & & &
\end{array}$$

For $k = 13$, replace the last two quadruples in the above family by $(1,8,9,a)$ and $(1, b, c, d)$; for $k = 12$, replace the last three quadruples by $(1, 8, 9, a)$, $(2, 8, 9, a)$ and $(1, 2, b, c)$.

In 1977, P. Aizley and J.L. Selfridge have stated in [1], that they can prove that $m(4) \geq 19$, but they never published their argument.

A summary of the results related to computing $m_v(4)$ is given in the table below. The values inside the square brackets are lower and upper bounds for the corresponding value of v.

v	8	9	10	11	12	13	14	15	16
$m_v(4)$	35	[24,26]	[21,25]	[17,23]	[17,26]	[18,26]	[18,26]	[19, ?]	[20, ?]

[1] Both constructions turns out to be isomorphic, although it was not recognized at the time. To the authors' knowledge, the isomorphism of the hypergraphs was first established with the use of the software system *SetPlayer*[5].

Our main result improves the lower bounds of the entries for $v = 10$, 11, 12, 13.

Theorem.

$$m_k(4) \geq \begin{cases} 23 & \text{if } k = 10; \\ 20 & \text{if } k = 11, 12; \\ 17 & \text{if } k = 13. \end{cases}$$

Consequently, $m(4) \geq 17$. □

The hypergraphs considered in the paper are always 4-uniform, that is, they are collections of quadruples (sets of size 4). The union of these sets is called the vertex set of the hypergraph and the sets are called hyperedges. Every coloring of a hypergraph is a 2-coloring of its vertex set. A coloring is called *balanced* (resp. *unbalanced*) *on a given set T of vertices* if the number of vertices in T colored in one of the colors differs from that for the other color by at most (resp. more than by) one. A coloring which is balanced (resp. unbalanced) on the vertex set of the hypergraph is simply called *balanced* (resp. *unbalanced*). If a coloring ϕ is monochromatic on a hyperedge Q, then we say that Q blocks ϕ. The set of colorings blocked by a given hyperedge Q is denoted $C(Q)$; the *blocking degree* of a hyperedge Q with respect to a given set $\mathcal{S}$ of colorings is the number $bdeg(Q;\mathcal{S}) = |\mathcal{S} \cap C(Q)|$.

Our approach involves an analysis of pairwise intersections of the quadruples. We repeatedly branch the problem into subproblems according to the sizes of the hyperedge pairwise intersections. In one case, the hyperedge intersections imply a sizable overlap of the sets of colorings blocked by the hyperedges, and this can be used to increase the lower bound. In the other case, we define a certain subset $\mathcal{S} \subset \mathcal{B}$ and prove that the reduction in the size of the set of colorings is still smaller than that in the blocking degrees. This leads to the same lower bound as in the first case.

We say that two hyperedges Q' and Q'' are *distant* if $|Q' \cap Q''| = 1$ or 2; otherwise, the hyperedges are called *close*. Given a hypergraph $\mathcal{H}$, we define an auxiliary graph $G(\mathcal{H})$ whose vertices are hyperedges of $\mathcal{H}$, and two vertices are adjacent iff the hyperedges are close. A hyperedge which is an isolated vertex in $G(\mathcal{H})$ is also called *isolated*. Given an order of the hyperedges in a hypergraph, we define the *actual* degree of a hyperedge Q to be the number of colorings blocked by Q and not blocked by any preceding hyperedge. We assume that the vertices of $G(\mathcal{H})$ (equivalently, the hyperedges in $\mathcal{H}$) are always ordered in such a way that for every component of $G(\mathcal{H})$, every vertex, except for the first, is adjacent to at least one preceding vertex of the component. We call such an ordering *standard*. A standard ordering implies that for every hyperedge which is not the first in its component, its actual degree is smaller than its blocking degree.

2 The $v = 10$ case

Let $\mathcal{B}$ be the set of all balanced coloring of the set $\{1, 2, \ldots, 10\}$. Then it is straightforward to compute that $|\mathcal{B}| = 126$ and the blocking degree of every quadruple is 6. Given two quadruples Q_1 and Q_2, the number $|C(Q_1) \cap C(Q_2)|$ of colorings in $\mathcal{B}$ blocked by both hyperedges is determined by the value of $|Q_1 \cap Q_2|$, as the following blocking table shows.

$\lvert Q_i \cap Q_j \rvert$	0	1	2	3
$\lvert C(Q_i) \cap C(Q_j) \rvert$	2	0	0	1

Let $\mathcal{H} = \{Q_1, Q_2, \ldots, Q_m\}$ be a non-bipartite hypergraph on 10 vertices. We consider two cases.

Case 1: There exist at most 7 isolated quadruples.

Assume that the hyperedges are ordered in the standard way. Then, every hyperedge which is not the first in its component of $G(\mathcal{H})$ blocks at most 5 colorings not blocked by some preceding hyperedge. Since there are at most 7 isolated hyperedges, at least half of the remaining $m - 7$ hyperedges block ≤ 5 colorings not blocked by some preceding hyperedge. Thus, we have $7 \times 6 + (m-7) \times \frac{11}{2} \geq 126$ implying $m \geq 23$.

Case 2: The number of isolated quadruples of the hypergraph is at least 8.

Any collection of eight quadruples of a set with 10 elements, contains at least two with two or more elements in common. It can be seen by observing that the quadruples cover $8\binom{4}{2} = 48$ pairs, while the number of pairs is $\binom{10}{2} = 45$. Thus, at least one pair is covered by at least two quadruples. Since the quadruples are isolated hyperedges, it follows that there are two isolated hyperedges with exactly two elements in common. Let these be $Q_1 = (1, 2, 3, 4)$ and $Q_2 = (1, 2, 5, 6)$ and let $\mathcal{S}$ be the subset of $\mathcal{B}$ comprised of all colorings that are unbalanced on both Q_1 and Q_2, and are not blocked by either one. Furthermore, let $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ be the partitioning of $\mathcal{S}$ defined by:

$\mathcal{S}_1$: the set of colorings unbalanced on $Q_1 \cap Q_2$;

$\mathcal{S}_2$: the set of colorings balanced on $Q_1 \cap Q_2$ and on $V - (Q_1 \cup Q_2)$;

$\mathcal{S}_3$: the set of colorings balanced on $Q_1 \cap Q_2$ and unbalanced on $V - (Q_1 \cup Q_2)$.

Notice that if a coloring is unbalanced on a set with at most three elements, then it is monochromatic on this set. It is straightforward to verify that $|\mathcal{S}| = 30$, $|\mathcal{S}_1| = 16$, $|\mathcal{S}_2| = 12$, $|\mathcal{S}_3| = 2$. Define the type of a quadruple Q to be a triple (a, b, c) where $a = |Q \cap \{1,2\}|$; $b = |Q \cap \{3,4\}|$, $c = |Q \cap \{5,6\}|$. Since Q_1 and Q_2 are isolated, $1 \le a + b \le 2$ and $1 \le a + c \le 2$. It is easy to check, that for any quadruple, its blocking degrees with respect to sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ depend on the type of the quadruple only; in addition, any two quadruples whose types are (a, b, c) and (a, c, b) have the same degrees. The table below shows the dependence of the blocking degrees on the type of a quadruple; the empty entries in the table correspond to 0's.

TYPE	(2,0,0)	(1,1,1)	(0,2,2)	(1,1,0)	(0,2,1)	(1,0,0)	(0,1,1)
SAMPLE	1 2 7 8	1 3 5 7	3 4 5 6	1 3 7 8	3 4 5 7	1 7 8 9	3 5 7 8
$\mathcal{S}_1$		1					2
$\mathcal{S}_2$				1			
$\mathcal{S}_3$			2			1	

Thus, from the table we see that the colorings from sets $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ are blocked by quadruples of different types. To block $\mathcal{S}_1$ we need at least 8 quadruples (the types are either $(1,1,1)$ or $(0,1,1)$); to block colorings from $\mathcal{S}_2$ we need 12 quadruples of type $(1,1,0)$, and to block colorings from $\mathcal{S}_3$, we need at least one quadruple. It implies that at least 23 quadruples[2] are needed.

3 The $v = 11, 12$ case

The blocking table for this case is as follows.

$\lvert Q_i \cap Q_j\rvert$	0	1	2	3
$\lvert C(Q_i) \cap C(Q_j)\rvert$	6	0	1	7

As before, $\mathcal{H} = \{Q_1, Q_2, \ldots, Q_m\}$ is a non-bipartite hypergraph and $A_1, \ldots, A_p$ are the components of $G(\mathcal{H})$.

Case 1: $G(\mathcal{H})$ contains at least one isolated vertex.

Let $Q_1 = \{1,2,3,4\}$ be an isolated hyperedge, and let $\mathcal{S}_1$ be the set of all balanced colorings that are unbalanced on Q_1, but not blocked by this hyperedge. We easily check that for every $i = 2, \ldots m$, if $|Q_i \cap Q_1| = 1$ (resp. $|Q_i \cap Q_1| = 2$) then $bdeg(Q_i; \mathcal{S}_1) = 13$ (resp. $bdeg(Q_i; \mathcal{S}_1) = 12$). The bound $m \ge 19$ follows immediately. To prove a stronger bound, we branch the case into two subcases.

Case 1.1: There is at least one more isolated hyperedge in $\mathcal{H}$.

[2]A more careful analysis shows that to block the colorings from $\mathcal{S}_1$, we need 12 quadruples; this gives the lower bound of 27 for the case.

Let Q_2 be an isolated hyperedge different from Q_1 and let $\mathcal{S}_2 \subset \mathcal{S}_1$ be the set of colorings in $\mathcal{S}_1$ unbalanced on both Q_1 and Q_2, but not blocked by Q_2. Then, if $|Q_1 \cap Q_2| = 1$, say $Q_2 = (1,5,6,7)$, then $|\mathcal{S}_2| = 106$ and $\forall Q \in \mathcal{H}, Q \neq Q_1, Q_2$, $bdeg(Q'; \mathcal{S}_1) \leq 6$. This implies $6 \times (m-2) \geq 106$, yielding $m \geq 20$.

Let now $|Q_1 \cap Q_2| = 2$, say $Q_2 = (3,4,5,6)$. Then $||d"k"9"bekKW$, $[6Q \in \mathcal{H}, Q \neq Q_1, Q_2, (1,2,5,6)$, $bdeg(Q; \mathcal{S}_2) \leq 6$. Consequently, $12 + 6 \times (m-3) \geq 112$, and $m \geq 20$.

Case 1.2: Q_1 is the only isolated hyperedge.

Lemma 1 *For every component A of $G(\mathcal{H})$, $\sum_{Q \in A} adeg(Q) \leq 12|A|$.*

Proof: If A contains a hyperedge Q intersecting Q_1 in two elements, then we place Q first in the component. Obviously, $adeg(Q') \leq 12$ for every hyperedge in the component.

On the other hand, if every hyperedge in A intersects Q_1 in one element, then we can verify that any of hyperedges except for the first blocks at least two colorings in $\mathcal{S}_1$ in common with a preceding hyperedge. The lemma follows. □

Completing case 1.2, as well as the whole case 1, is now trivial:

$$224 \leq \sum_{i=2}^{p} 12|A_i| = 12(m-1) \text{ and } m \geq 1 + \lceil 224/12 \rceil = 20.$$

Case 2: Every component of $G(\mathcal{H})$ contains at least two vertices.

First, we prove that $m \geq 19$. From the blocking table, it follows that for every hyperedge Q which is not numbered first in its component, $adeg(Q) \leq 22$. Therefore,

$$462 \leq \sum_{i=1}^{m} adeg(Q_i) \leq= \sum_{i=1}^{p} (28 + 22(|A_i| - 1)) = 22m + 6p.$$

But, for the case under consideration, $p \leq m/2$, implying $462 \leq 25m$, and $m \geq 19$.

To improve the bound, we need two more auxiliary statements, first of which can be checked directly.

Lemma 2 *Let P and Q be two disjoint quadruples and let $\mathcal{S}$ be the set of balanced colorings that are unbalanced on both P and Q. If R and S are close quadruples both distant from P and Q, then $bdeg(R; \mathcal{S}) \leq 6$, and at least one coloring in $\mathcal{S}$ is blocked by R and by S.* □

Lemma 3 *Let $(R_1, S_1), (R_2, S_2), \ldots, (R_8, S_8)$ be 8 pairs of distinct quadruples of a set with ≤ 12 elements. Then, there are 8 pairs of elements each covered by two quadruples from different pairs.*

Proof. For every $i = 1, \ldots 8$, the quadruples of the pair (R_i, S_i) cover at least 9 pairs of elements, making the total number of covered pairs of elements 72. On the other hand, since the cardinality c of the ground set ≤ 12, the number of sets with two elements is at most 66. This proves the lemma. □

Now we proceed to proving $m \geq 20$ in case 2. Let us assume that the bound is wrong and $m = 19$. Then, from

$$p \leq m/2 \text{ and } 22m + 6p \leq 462$$

it follows that $8 \leq p \leq 9$. It is easy to see then that for the sizes $\{|A_i|\}$ $(i = 1, \ldots, p)$ the only possibilities, up to reordering, are

$$(5,2,2,2,2,2,2,2), (4,3,2,2,2,2,2,2), \text{ and}$$
$$(3,2,2,2,2,2,2,2,2).$$

Let there be a component consisting of two disjoint hyperedges. Then we consider the set $\mathcal{S} \subset \mathcal{B}$ of colorings that are unbalanced on both hyperedges of the component. Using Lemma 2 we have

$$|\mathcal{S}| \leq \sum_{i=1}^{p} (6 + 5(m_i - 1)) = 5m + p \leq 5 \times 19 + 9 = 89.$$

But it is easy to compute that $|\mathcal{S}| = 112$, which proves that every component of size two consists of hyperedges intersecting in three elements. Now we can easily reject $(5,2,2,2,2,2,2,2)$ and $(4,3,2,2,2,2,2,2)$ as possible lists of components' sizes. For example, for the latter one (similar for the former), it must have been that

$$462 \leq (28 + 3 \times 22) + (28 + 2 \times 22) + (28 + 21) \times 6 = 460.$$

Thus, if $m = 19$, then $G(\mathcal{H})$ has 9 components, one is of size 3, and the rest are of size 2. We apply Lemma 3 to find 8 pairs of elements that are covered by hyperedges from different components. From the blocking table, we see that if two quadruples intersect in two elements, there is a coloring in B that is blocked by each. Therefore, for at least 4 components $\{A_i\}$ of size two, the value $\sum_{Q \in A_i} adeg(Q)$ is smaller than $\sum_{Q \in A_i} bdeg(Q)$, implying

$$\sum_{i=1}^{p} \sum_{Q \in A_i} adeg(Q_i) \leq \sum_{i=1}^{m} bdeg(Q_i) - 4$$
$$\leq (28 + 2 \times 22) + (28 + 21) \times 6) - 4 = 460,$$

which is impossible since $|S| = 460$. This completes the analysis for $v = 11, 12$.

4 The $v = 13$ case

There are $\binom{13}{6} = 1716$ balanced 2-colorings of $\{1, 2, \ldots, 13\}$; each quadruple blocks 120 of them. The blocking table is as follows.

$\vert Q_i \cap Q_j\vert$	0	1	2	3
$\vert C(Q_i) \cap C(Q_j)\vert$	20	1	8	36

Let $\mathcal{H} = \{Q_1, Q_2, \ldots, Q_m\}$ is a non-bipartite hypergraph and $A_1, \ldots, A_p$ are the components of $G(\mathcal{H})$.

Case 1: There is at least one isolated quadruple, say $Q_1 = (1, 2, 3, 4)$.

Let $\mathcal{S}$ be the set of all balanced colorings that are not blocked by Q_1 and are unbalanced on it. It is easy to check that $|S| = 840$ and for any quadruple Q with $|Q \cap Q_1| = 0$ or 1, the blocking degree $bdeg(Q) = 56$. This gives $m \geq 1 + 840/56 = 16$. If $m = 16$, then all $Q_2, \ldots, Q_{16}$ are distant from Q_1 and no two of them block the same coloring in $\mathcal{S}$.

To eliminate the latter possibility, we describe the conditions under which two quadruples Q' and Q'' that are distant from Q_1 can block disjoint sets of colorings in $\mathcal{S}$.

It is easy to see that the quadruples themselves must not be disjoint; moreover, their intersection must not contain more than one element. Thus, $|Q'' \cap Q''| = 1$. Furthermore, the previous condition can be used to prove that each of the quadruples must meet Q_1 in one element.

In summary, if $m = 16$, then the intersection of any two quadruples contains one element only. The rest is simple: there are $16 \times \binom{4}{2} = 96$ pairs covered by the hyperedges, and there are just $\binom{13}{2} = 78$ pairs to cover, so several of the hyperedges must intersect in two or more elements. The contradiction completes this case.

Case 2: Every component of $\mathcal{H}$ contains at least two hyperedges.

Call a component type 1 if it is comprised of two disjoint hyperedges; any other component is called type 2. If every component of $G(\mathcal{H})$ is type 2, then

$$1716 \leq \sum_i \sum_{Q \in A_i} bdeg(Q) < 107m \text{ and } m \geq 17.$$

If $G(\mathcal{H})$ has a type 1 component, let $Q_1 = (1,2,3,4)$ and $Q_2 = (5,6,7,8)$ be the hyperedges of that component and let $\mathcal{S}$ be the subset of $\mathcal{B}$ consisting of all colorings that are unbalanced on both Q_1 and Q_2. Then $|S| = 416$ and every hyperedge blocks at most 28 colorings[3] in $\mathcal{S}$. We have $28(m-2) \geq 416$, or $m \geq 17$. □

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[3]To check the latter, remember that every hyperedge distinct from Q_1 and Q_2 meets both these hyperedges in one or two elements.

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