Ramsey Numbers for Induced Regular Subgraphs

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Abstract. We consider two variations of the classical Ramsey number. In particular we seek the number of vertices necessary to force the existence of an induced regular subgraph on a prescribed number of vertices.

1. Introduction

If k is a positive integer, then the Ramsey number $r(k)$ is the smallest positive integer r such that every graph on r vertices contain k vertices which induce either an independent or complete subgraph. Rephrasing, any graph with $r(k)$ vertices contains k vertices which induce a regular subgraph of degree either zero or $k-1$. These numbers have proven to be notoriously difficult to compute. We consider some natural relaxations of the classical Ramsey problem in this paper. These relaxations yield two new classes of Ramsey type numbers. These two classes have been considered by Erdös [2, p.8] and Fajtlowicz [3, p.13].

The graph terminology used here mostly follows Chartrand and Lesniak [1]. All induced subgraphs in this paper are vertex induced subgraphs. Graphs on j vertices which are complete, cycles, independent, and paths will be denoted by K_j, C_j, I_j, and P_j, respectively.

Definition 1. *Let k be a positive integer. Then $N(k)$ is the least positive integer N such that every graph with N vertices contains an induced regular subgraph with exactly k vertices* [3, p.13].

Definition 2. *Let k be a positive integer. Then $n(k)$ is the least positive integer n such that every graph with n vertices contains an induced regular subgraph with at least k vertices* [2, p.8].

The existence of the numbers $N(k)$ and $n(k)$ follows from the existence of the Ramsey numbers as $n(k) \leq N(k) \leq r(k) \leq \binom{2k-1}{k-1}$.

*This research was partially supported by NSF grant 00365085-arp
†This research was partially supported by NSA grant MSPPS-054-91

We note the following numbers for small values of k:

(a) $n(1) = N(1) = 1$
(b) $n(2) = N(2) = 2$
(c) $N(3) = r(3) = 6$
(d) $n(3) = 5$.

Equation (d) follows from the fact that the cycle C_5 is the only graph on five vertices which is free of K_3 and I_3, while the path P_4 contains no regular induced subgraph on three or more vertices.

2. Results

We give our results after first defining some terminology. Let A and B be vertex subsets of a graph G. Then $\langle A \rangle$ denotes the subgraph of G induced by A. An $A - B$ edge is an edge with end vertices in A and B. The graph on four vertices with two disjoint edges is denoted by M_4. The j-wheel graph is denoted by W_j. For example, the graph W in Figure 1 is obtained by adding an isolated vertex to a five-wheel graph.

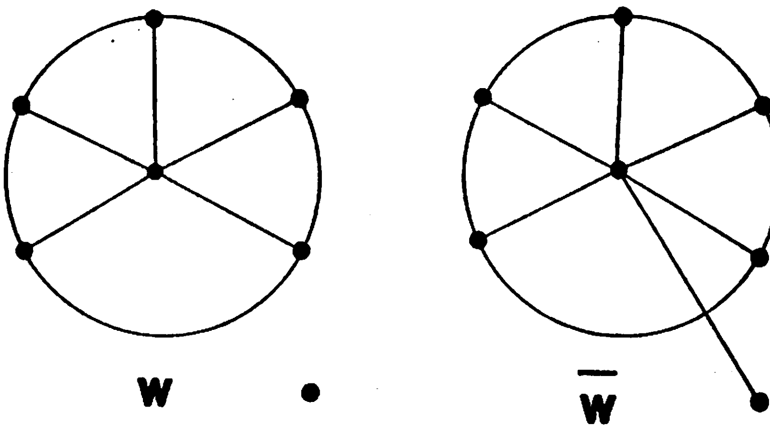

Figure 1

Our main result is the following structure theorem. This theorem will be used to compute $n(4)$ and $N(4)$.

Theorem 1. *Let G be a graph on seven vertices. Then G contains no induced regular subgraph on four vertices if and only if $G \cong W$ or $G \cong \overline{W}$.*

Proof. The graph W contains no induced regular subgraphs on four vertices and hence neither does its complement $\overline{W}$.

Suppose that G contains no induced regular subgraph on four vertices. Note that C_4, I_4, K_4, and M_4 are the regular graphs on four vertices. Since $K_4 \cong W_3$, G contains no three-wheel subgraphs. Assume that G contains a four-wheel subgraph H. The rim of H doesn't induce a C_4 in G. Thus two antipodal points of the rim are joined by an edge e in G. However, $\langle H \cup e\rangle$ contains K_4 as a subgraph; a contradiction. Thus

$$G \text{ contains no three- or four-wheel subgraphs.} \qquad (1.1)$$

The graph G has seven vertices and hence is not regular of degree 3. Thus either G or $\overline{G}$ contains a vertex v with degree at least 4. Assume the former without loss of generality. Let P be a longest path among the neighbors of v. Let $A = \{a_1, a_2, \ldots, a_j\}$ be the vertices of P, where adjacent vertices on P are consecutively listed. Let $B = \{b_1, b_2, \ldots, b_k\}$ be the neighbors of v not on P and $C = \{c_1, c_2, \ldots, c_\ell\}$ be the vertices of G not adjacent to v. Note that $|C| \leq 2$.

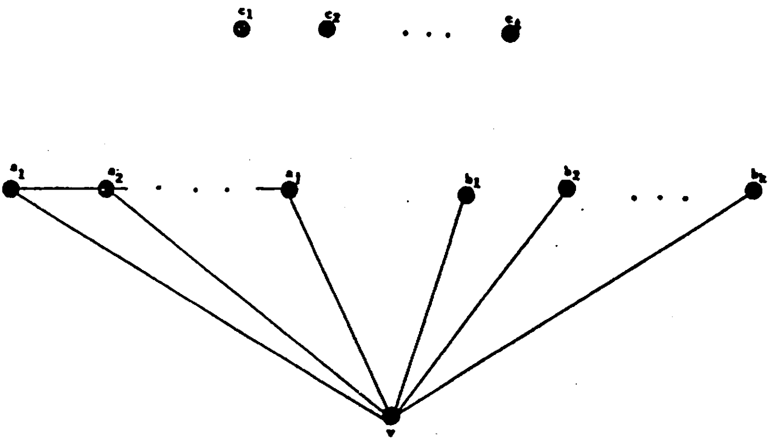

Figure 2. A subgraph of G.

There is an edge in the subgraph induced by the neighbors of v as $\deg(v) \geq 4$. Thus $|A| \geq 2$. Assume there is a B-B edge, say $b_1 b_2$. Then there is an $\{a_{j-1}, a_j\} - \{b_1, b_2\}$ edge as G doesn't contain an induced M_4. This yields a $j+1$ vertex path among the neighbors of v contradicting the choice of P. It follows that there are no B-B edges and likewise no $a_1 - B$ or $a_j - B$ edges. If $|B| \geq 3$, then $\langle a_j \cup B\rangle$ contains an edge e. Thus e is either an $a_j - B$ or B-B edge; a contradiction. Hence $|B| \leq 2$.

Suppose $|A| = 2$. Then $|B| = |C| = 2$ and $\langle B \cup C\rangle$ contains either a B-C or a C-C edge. Assume there is a C-C edge. Then $c_1 c_2$ is an edge and there exists a $\{b_1, v\} - \{c_1, c_2\}$ edge e. Evidently e is a B-C edge. Thus there is a B-C edge in either case. Suppose, without loss of generality $b_1 c_1$ is an edge. Then there is an $\{a_1, a_2\} - \{b_1, c_1\}$ edge. Assume that $a_1 c_1$ is an edge without loss of generality. Then $\langle a_1, b_1, c_1, v\rangle \cong C_4$; a contradiction. Thus $|A| \geq 3$.

Assume $|B| = 2$. Then there is an edge in $\langle a_1, a_j, b_1, b_2\rangle$. Hence $a_1 a_j \in E(G)$. It follows from (1.1) that $|A| \geq 5$. Thus $|V(G)| > |A \cup B| = 7$; a contradiction. Hence $|B| \leq 1$.

Assume $|A| \in \{3,4\}$. Then the edges of P are the only edges in $\langle A\rangle$ by (1.1). Let $S \subseteq A \cup B$ be an independent set of three vertices and $c \in C$. Then there exists a c-S edge as $c \cup S$ doesn't induce an I_4. Assume that c has two neighbors in S. Then these two neighbors together with c and v induce a C_4 in G; a contradiction. Thus

$$c \text{ has exactly one neighbor in } S. \tag{1.2}$$

Suppose $|A| = 3$. Then $|B| = 1$ and $|C| = 2$. Let $S = \{a_1, a_3, b_1\}$. Each of c_1 and c_2 has exactly one neighbor in S by (1.2). Suppose $c_1 c_2 \in E(G)$. Then, as G doesn't contain an induced M_4, there are $\{a_1, v\} - \{c_1, c_2\}$, $\{a_3, v\} - \{c_1, c_2\}$, and $\{b_1, v\} - \{c_1, c_2\}$ edges. It follows that either c_1 or c_2 has exactly two neighbors in S contradicting (1.2). Thus $c_1 c_2 \notin E(G)$. Suppose c_1 and c_2 share a common neighbor x in S. Then $(S \setminus \{x\}) \cup \{c_1, c_2\}$ induces an I_4 in G; a contradiction. Thus c_1 and c_2 have distinct neighbors, say x and y respectively, in S. Hence $\langle c_1, c_2, x, y\rangle \cong M_4$; a contradiction. It follows that $|A| \geq 4$.

Suppose $|A| = 4$. Let $c \in C$. It follows from the argument given in establishing (1.2) that c is not adjacent to two nonconsecutive vertices of the path P. Suppose $a_1 c \in E(G)$. Then $\langle a_1, a_3, a_4, c\rangle \cong M_4$; a contradiction. Thus $a_1 c \notin E(G)$ and likewise $a_4 c \notin E(G)$. Assume $|B| = 1$. Then $|C| = 1$. Let $S = \{a_1, a_4, b_1\}$. By (1.2), $b_1 c_1 \in E(G)$. Assume $a_2 b_1 \notin E(G)$. Then $\{a_2, a_4, b_1\}$ is an independent set. By (1.2), $a_2 c_1 \notin E(G)$. Thus $\langle a_1, a_2, b_1, c_1\rangle \cong M_4$; a contradiction. Hence $a_2 b_1 \in E(G)$. Assume $a_3 b_1 \notin E(G)$. Then $\{a_1, a_3, b_1\}$ is an independent set. By (1.2), $a_3 c_1 \notin E(G)$. Thus $\langle a_3, a_4, b_1, c_1\rangle \cong M_4$; a contradiction. It follows that $a_3 b_1 \in E(G)$ and $\langle a_2, a_3, b_1, v\rangle \cong K_4$; a contradiction. Hence $|B| = 0$ and $|C| = 2$. Since $\langle a_1, a_4, c_1, c_2\rangle$ contains an edge, $c_1 c_2 \in E(G)$. Hence $\langle a_1, c_1, c_2, v\rangle \cong M_4$; a contradiction. Thus $|A| \geq 5$.

There is an $\{a_1, a_2\} - \{a_4, a_5\}$ edge. It follows from (1.1) that $a_1 a_5 \in E(G)$. Moreover, all edges of $\langle a_1, a_2, a_3, a_4, a_5\rangle$ lie on the cycle $a_1, a_2, a_3, a_4, a_5, a_1$. Thus $\langle a_1, a_2, a_3, a_4, a_5, v\rangle \cong W_5$. Let u be the vertex $V(G) \setminus \{a_1, a_2, a_3, a_4, a_5, v\}$. We show that u may only be adjacent to v. Thus if $deg(u) = 0$, then $G \cong W$, while if $deg(u) = 1$, then $G \cong \overline{W}$. Suppose that u has a neighbor in A, say a_1 without loss of generality. Then there is an $\{a_1, u\} - \{a_3, a_4\}$ edge. By (1.1), we may assume that $a_3 u \in E(G)$ without loss of generality. As $\langle a_1, a_2, a_3, u\rangle \not\cong C_4$, $a_2 u \in E(G)$. Thus $\langle a_1, a_2, a_3, u, v\rangle \cong W_4$ contradicting (1.1). ∎

We next use our main theorem to compute the Ramsey type numbers $n(4)$ and $N(4)$.

Theorem 2. $(a)\, n(4) = 7$ $\quad (b)\, N(4) = 8$.

Proof. Let G be a graph on seven vertices. Both W and $\overline{W}$ contain induced C_5 subgraphs. It follows from Theorem 1.1 that G has an induced regular subgraph on

X

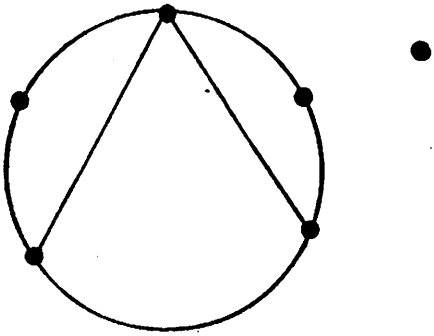

Figure 3

four or more vertices. On the other hand the graph X of Figure 3 has six vertices and no induced regular subgraph on four or more vertices. Thus $n(4) = 7$.

Let G be a graph on eight vertices. By Theorem 1, G contains W or $\overline{W}$ as a subgraph. By using the arguments given in the last paragraph of the proof of Theorem 1 it follows that any graph containing W or $\overline{W}$ as a proper subgraph has an induced regular subgraph on four vertices. On the other hand W and $\overline{W}$ have seven vertices and no induced regular subgraph on four vertices. Thus $N(4) = 8$. ∎

We next give some lower bounds for small values of our Ramsey type numbers.

Y

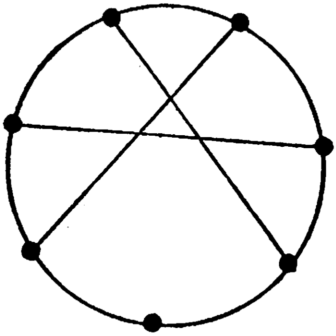

Figure 4

Theorem 3. $(a) n(5) \geq 12 \qquad (b) N(5) \geq 19 \qquad (c) N(6) \geq 18$.

Proof. It can be checked that the graph $P_4 + P_7$ has no induced regular subgraph on five or more vertices. Likewise, $C_9 + C_9$ has no induced regular subgraph on exactly five vertices. Parts (a) and (b) follow from these two facts. With more work one can check that the graph $(C_5 + W_5) \cup Y$ has no induced regular subgraph on exactly six vertices. Part (c) follows from this fact. ∎

We conclude with a general lower bound on $N(p)$ when p is prime.

Theorem 4. *If p is prime, then* $N(p) \geq (p-1)^2 + 1$.

Proof. Consider the graph G consisting of $p-1$ copies of K_{p-1}. If for some k with $0 \leq k \leq p-1$ there is a k-regular induced subgraph H with p vertices, then, since each component of H is complete, H must contain exactly $k+1$ vertices of each component which it intersects nontrivially. Hence p is a multiple of $k+1$. Since p is prime $k+1$ is p or 1. But this is impossible, since no component contains p vertices and there are not p components. ∎

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