A Note About the Closing S^3 Recognition Algorithm

Maria Rita Casali

Dipartimento di Matematica Pura ed Applicata
Università di Modena
Via Campi 213 B
I-41100 MODENA (Italy)

Abstract. In [V_2], Vince outlined three potential graph algorithms for S^3 recognition, namely *shelling*, *reducing*, and *closing*; on the other hand, he conjectured that the graph H_0 of Fig.1 - which proves the first two to fail - could be a counterexample for the third one, too. This note shows that the conjecture is false; so, the validity of the closing algorithm is still an open problem.

In recent years, the possibility of representing PL-manifolds of arbitrary dimension n by means of $(n+1)$-coloured graphs has been examined by different authors: see, for example, [P], [FGG], [LM], [BM], [V_1]. With this representation theory, it is possible to investigate, among other things, the 3-dimensional homeomorphism problem by a purely combinatorial approach. In particular, Vince [V_2] outlined three potential graph algorithms for S^3 recognition, namely *shelling*, *reducing*, and *closing*. Since each of these outputs NO for any closed 3-manifold M^3 not homeomorphic to S^3, it would be necessary - in order to confirm their validity - to prove that the output is surely YES for any manifold homeomorphic to S^3.

Unfortunately, Vince himself verifies that the shelling and reducing algorithms fail to recognize the 3-sphere represented by the 4-coloured graph H_0 of Fig.1 (which originally appeared in [Vo]). Moreover, H_0 is conjectured to be a counterexample to the closing algorithm validity.

In this paper, we shall prove the following

Proposition 1. *The closing algorithm outputs* YES *for the 4-coloured graph* H_0.

Thus, it remains an open problem to check whether the closing algorithm effectively recognizes S^3, or not.

In order to prove Proposition 1, let us recall how the algorithm works. Given a 4-coloured graph (Γ, γ) representing a closed 3-manifold M^3, let A be a subset of the edge set $E(\Gamma)$ of Γ. An element $e \in E(\Gamma) - A$ is said to be *dependent* on A if e belongs to a cycle C in $A \cup \{e\}$ not involving all four colours; in particular, if C involves the subset $\mathcal{B}$ of the colour set $\{0,1,2,3\}$, the edge e is said to be dependent on A *by a* $\mathcal{B}$*-residue*. A subset $A \subset E(\Gamma)$ is said to be *closed* if there

(*) Work performed under the auspicies of the G.N.S.A.G.A. of the C.N.R. (National Research Council of Italy) and financially supported by M.U.R.S.T. of Italy (project "Geometria Reale e Complessa").

is no edge in $E(\Gamma) - A$ which is dependent on A; the *closure* $cl\ (A)$ of A is defined to be the smallest closed subset of $E(\Gamma)$ containing A. Then, the closing algorithm outputs YES if there exists a spanning tree of Γ whose edge set A has $cl\ (A) = E(\Gamma)$, and outputs NO otherwise.

As far as the graph H_0 is concerned, it is easy to check that the spanning tree depicted in Fig.2 is such that the closure of its edge set coincides with the whole edge set of H_0. In details, the closing steps - illustrated in Fig.3 - are the following:
- adding the 1-coloured edges a_1, a_2, a_3, a_4, a_5 dependent by $\{0,1\}$- or $\{1,2\}$-residues, the 3-coloured edges b_1, b_2 dependent by $\{0,2\}$-residues, and the 3-coloured edges c_1, c_2, c_3, c_4, c_5 dependent by $\{2,3\}$- or $\{0,3\}$-residues (Fig. 3a);
- adding the 3-coloured edge d_1 dependent by a $\{1,3\}$-residue, and the 3-coloured edges e_1, e_2 dependent by $\{2,3\}$- or $\{0,3\}$-residues (Fig. 3b);
- adding the 1-coloured edge f_1 dependent by a $\{1,3\}$-residue, and the 1-coloured edges g_1, g_2 dependent by $\{0,1\}$- or $\{1,2\}$-residues (Fig. 3c);
- adding the 0-coloured edge h_1 dependent by a $\{0,1\}$-residue (Fig. 3d);
- adding the 2-coloured edge i_1 dependent by a $\{0,2\}$-residue (Fig. 3e);
- adding the 3-coloured edge l_1 dependent by a $\{2,3\}$-residue (Fig. 3f).

This proves Proposition 1. ∎

References

[BM]. J.Bracho - L.Montejano, *The combinatorics of colored triangulations of manifolds*, Geom. Dedicata 22 (1987), 303-328.

[FGG]. M.Ferri - C.Gagliardi - L.Grasselli, *A graph-theoretical representation of PL-manifold. A survey on crystallizations*, Aequationes Math. 31 (1986), 121-141.

[LM]. S.Lins - A.Mandel, *Graph-encoded 3-manifolds*, Discrete Math. 57 (1985), 261-284.

[P]. M.Pezzana, *Sulla struttura topologica delle varietà compatte*, Atti Sem. Mat. Fis. Univ. Modena 23 (1974), 269-277.

[V_1]. A.Vince, *n-graphs*, Discrete Math. 72 (1988), 367-380.

[V_2]. A.Vince, *Recognizing the 3-Sphere*, Graph theory and its applications, East and West (Jinon 1986), Annals New York Academy of Sciences 576 (1989), 571-583.

[Vo]. G.Volzone, *Alcune osservazioni sul riconoscimento di* $S^1 \times S^2$, Atti Sem. Mat. Fis. Univ. Modena 31 (1982), 228-247.