On $2-(v,4,\lambda)$-Designs without Pair Intersections

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1. Introduction

A $2\text{-}(v,4,\lambda)$*-design* is a collection of (not necessarily distinct) 4-element subsets (called blocks) of a v-element set V, such that every 2-element subset of V belongs to exactly λ blocks. A design is called *simple* if it contains no repeated blocks. For example, it is well-known that a $2\text{-}(v,4,1)$-design (being simple) exists if and only if $v \equiv 1$ or $4 \mod 12$, see Hanani [3]. In Gronau and Mullin [2] *super-simple* $2\text{-}(v,4,2)$-designs, i.e. designs in which any two blocks have at most two elements in common, were introduced and studied. It was shown that a super-simple $2\text{-}(v,4,2)$-design exists if and only if $v \equiv 1 \mod 3$, $v \neq 4$. In this paper we will study *WPI-designs* (designs without pair intersections), i.e. designs in which any two blocks do not intersect in exactly 2 elements. In some sense one may feel that this condition seems to be an anticoncept to the balancy of the pairs. In fact it turned out that the WPI-condition is very strong, so the well-known necessary conditions are not all sufficient for large enough v. Nevertheless we have made the following general observation:

Theorem 1. *For all $k \geq 4$ there are simple $2\text{-}(v,k,\lambda)$-WPI-designs.*

Proof: Fix an integer t with $0 \leq t \leq k-3$. By Wilson's theorem, see [8], there are $2\text{-}(v,k+t,1)$-designs for all sufficiently large v satisfying the well-known necessary conditions. Now replace every $(k+t)$-element block by their k-element subsets. It is simple to see that this yields a $2\text{-}(v,k,\lambda)$-WPI-design with $\lambda = \binom{k+t-2}{t}$. ∎

In the present paper we will describe completely the existence and structure of $2\text{-}(v,4,\lambda)$-WPI-designs.

2. On $2\text{-}(v,4,\lambda)$-WPI-designs

Let a $2\text{-}(v,4,\lambda)$-WPI-design be given. Consider an arbitrary block A. Assume that $A = \{1,2,3,4\}$. Let a denote the replication number of the block A. If $a = \lambda$, then all other blocks of the design intersect A in at most one point.If

$a < \lambda$, then there must be a block intersecting A in exactly 3 points by the WPI-property. We may assume that this block is just $B = \{1,2,3,5\}$. Let b denote the replication number of the block B. Observe that any two blocks which intersect A in at least 3 points, have at least 3 points in common. Since $a < \lambda$, there must be another block containing 1 and 4. It is simple to check that we may assume that this is $C = \{1,2,4,5\}$ using the WPI-property. Let c denote the replication number of the block C. By the same argument there must be another block containing 3 and 4. It is simple to check that we may assume that this is $D = \{1,3,4,5\}$ using the WPI-property. Let d denote the replication number of the block D. According to the pair $\{1,2\}$ we have $a + b + c \leq \lambda$, with $a, b, c > 0$. Now let us look at the pair $\{2,3\}$. So far its replication (in the blocks A and B) is $a + b$. So there must be another block containing $\{2,3\}$. This must be $E = \{2,3,4,5\}$. Let e denote the replication number of the block E. By the WPI-property any other block of the design intersects each of the blocks A, B, C, D, and E in at most one point.Counting the replication numbers of the 10 pairs of $\{1,2,3,4,5\}$, we get the following system of equations: Any 3 of the variables a, b, c, d, e sum up to λ. This system has the unique solution $a = b = c = d = e = \frac{\lambda}{3}$. This provides $\lambda \equiv 0 \mod 3$. Resuming we have proved:

If $\lambda \equiv 1$ or $2 \mod 3$, then the design consists of λ copies of a 2-$(v,4,1)$-design.

If $\lambda \equiv 0 \mod 3$, then the design consists either of λ copies of a 2-$(v,4,1)$-design or of $\frac{\lambda}{3}$ copies of a 2-$(v,4,3)$-WPI-design.

Moreover, the structure of the last design is clear too: The design consists of some 3-fold blocks and some groups of 5 blocks each of which are exactly the five 4-element subsets of a 5-element set. Deleting from the 3-fold blocks two and replacing the five 4-elements subsets of the 5-element sets by this 5-element set yield a pairwise balanced design, namely a 2-$(v, \{4,5\}, 1)$-design, i.e. a collection of 4- and 5-element subsets of V in which any 2-element subset of V occurs in exactly one block. The last construction can be used in the opposite direction too, i.e. a 2-$(v,4,3)$-WPI-design exists if and only if a 2-$(v, \{4,5\}, 1)$-design exists. So far we can state the following

Theorem 2. *A 2-$(v,4,\lambda)$-WPI-design exists if and only if*

- $v \equiv 1$ *or* $4 \mod 12$ *if* $\lambda \equiv 1$ *or* $2 \mod 3$ *and*
- $v \equiv 0$ *or* $1 \mod 4, v \neq 8, 9,$ *or* $12,$ *if* $\lambda \equiv 0 \mod 3$.

Theorem 3. *A simple 2-$(v,4,\lambda)$-WPI-design exists if and only if*

- $v \equiv 1$ *or* $4 \mod 12,$ *if* $\lambda = 1$ *and*
- $v \equiv 1$ *or* $5 \mod 20,$ *if* $\lambda = 3$.

Proof: Using the above structure assertion we have to mention only that a 2-$(v,4,1)$-design exists if and only if $v \equiv 1$ or $4 \mod 12$, see Hanani [3] and to refer to the next section. ∎

3. Pairwise balanced designs with block sizes 4 and 5

The existence of pairwise balanced designs were studied for many block sizes, but we did not find this set. For the general techniques, which we will use, see also Beth, Jungnickel, and Lenz [1]. We will prove the

Theorem 4. *A* 2-$(v, \{4,5\}, 1)$*-design exists if and only if* $v \equiv 0$ *or* $1 \mod 4$, $v \neq 8, 9,$ *or* 12.

Proof: We will use the following constructions:

1. Take a 2-$(v,4,1)$-design. It is known that such design exists if and only if $v \equiv 1$ or $4 \mod 12$, see Hanani [3].
2. Take a 2-$(v,5,1)$-design. It is known that such design exists if and only if $v \equiv 1$ or $5 \mod 20$, see Hanani [3].
3. Take a 2-$(v,5,1)$-design. Delete just one point. This yields a 2-$(v,\{4,5\},$ $1)$-design with $v \equiv 0$ or $4 \mod 20$, see Hanani [3].
4. Take a 2-$(v,5,1)$-design. Delete just 4 points on a block and delete the resulting 1-element block. This yields a 2-$(v,\{4,5\},1)$-design $v \equiv 1$ or $17 \mod 20$, see Hanani [3].
5. Take a 2-$(v,5,1)$-design. Delete the 5 points of a block. This yields a 2-$(v,\{4,5\},1)$-design with $v \equiv 0$ or $16 \mod 20$, see Hanani [3].

So far we have solutions for the residue classes $\mod 60$, which are given in Table 1.

v	1	4	5	8	9	12	13	16	17	20
Constr.	1	1	2	·	·	·	1	1	4	3

v	21	24	25	28	29	32	33	36	37	40
Constr.	2	3	1	1	·	·	·	5	1	1

v	41	44	45	48	49	52	53	56	57	60
Constr.	2	3	2	·	1	1	·	5	4	3

Table 1: The residue classes for which we have solutions

Next we will use resolvable designs. It is well-known that a resolvable 2-$(v,4,1)$-designs exists if and only if $v \equiv 4 \mod 12$, see Hanani, Ray-Chaudhuri, and Wilson [4]. Associate to every of the $a = \frac{v-1}{3}$ parallel-classes a new point and enlarge every block of a parallel-class by this new associated point. Finally, add the set of the associated new points as a block. This yields a 2-$(v+a,\{5,a^*\},1)$-design with exactly one block of size a. Now we have a few more constructions:

6. Take $v = 28$, i.e. $a = 9$. Delete exactly 8 points of a block of size 9 and delete the resulting 1-element block. This yields a 2-$(29,\{4,5\},1)$-design.
7. Take $v = 28$, i.e. $a = 9$. Delete exactly 5 points of a block of size 9. This yields a 2-$(32,\{4,5\},1)$-design.

8. Take $v = 28$, i.e. $a = 9$. Delete exactly 4 points of a block of size 9. This yields a 2-$(33, \{4,5\}, 1)$-design.
9. Take $v = 40$, i.e. $a = 13$. This yields a 2-$(53, \{5, 13^*\}, 1)$-design. Replace the 13-element block by a 2-$(13, 4, 1)$-design. So we get a 2-$(53, \{4,5\}, 1)$-design.
10. In Lamken, Mills, and Wilson [5] a 2-$(57, \{5, 9^*\}, 1)$-design was constructed. Delete the 9-element block. This yields a 2-$(48, \{4,5\}, 1)$-design.

For $v = 8, 9$ or 12 there is no solution. Assume the contrary. Then not all points can lie on one block only. Take a block B (this has size at least 4) and a point x not belonging to B. Then the pairs consisting of point of B and x define mutually different blocks, each of size at least 4. On these (at least 4 blocks) lie at least $4(4-1)+1 = 13$ different points, contradicting our assumption. This proof idea is due to Mullin [6]. So far we have proved the Theorem 4 for all $v \leq 60$. Next we take transversal designs with block size 5. These ones exist for all group sizes $m \geq 4$, $m \neq 6$, and $m \neq 10$. Let $0 \leq n \leq m$. Delete $m - n$ points of the last group getting a group divisible design with block sizes 4 and 5 and four groups of size m and one group of size n. Now we have two recursive constructions:

11. Replace each the groups by a 2-$(m, \{4,5\}, 1)$-design or by a 2-$(n, \{4,5\}, 1)$-design, respectively. This yields a 2-$(v, \{4,5\}, 1)$-design with $v = 4m + n$.
12. Adjoin a new point ∞ to the point set and enlarge every group by this point ∞. Apply the construction 11. So we get a 2-$(v, \{4,5\}, 1)$-design from a 2-$(m+1, \{4,5\}, 1)$ and a 2-$(n+1, \{4,5\}, 1)$-design with $v = 4m+n+1$.

We apply these constructions 11 and 12 to the open cases of orders v with $61 \leq v \leq 252$, the corresponding parameters are given in Table 2.

v	68	69	72	89	92	93	108	113	128
Constr.	11	11	11	11	12	11	12	11	11
m	17	17	17	21	19	20	23	28	32
n	0	1	4	5	15	13	15	1	0

v	129	132	149	152	153	168	173	188	189
Constr.	11	11	11	11	11	11	11	11	11
m	32	33	37	37	37	41	40	41	41
n	1	0	1	4	5	4	13	24	25

v	192	209	212	213	228	233	248	249	252
Constr.	11	11	11	11	11	11	11	11	11
m	44	52	52	52	56	57	61	61	57
n	16	1	4	5	4	5	4	5	24

Table 2: Constructions for the remaining cases

Finally we prove the theorem for all $v \geq 253$. We know that

- a 2-$(m+1, \{4,5\}, 1)$-design exists for all $m = 12s$, $s \geq 5$, see Construction 1,
- a 2-$(n+1, \{4,5\}, 1)$-design exists for all $13 \leq n+1 \leq 60$, $n+1 \equiv 0$ or $1 \mod 4$.

By Construction 12 we get the existence of a 2-$(v, \{4,5\}, 1)$-design with $v = 48s + n + 1$, i.e. for all

$$v \in \mathcal{G}(s) = \{x : 48s + 13 \leq x \leq 48(s+1) + 12, x \equiv 0 \text{ or } 1 \mod 4\}.$$

Since

$$\bigcup_{s=5}^{\infty} \mathcal{G}(s) = \{x : x \geq 61, x \equiv 0 \text{ or } 1 \mod 4\},$$

the proof is complete. ∎

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