Bandwidth of the Composition of Certain Graph Powers

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ABSTRACT. The composition of two graphs G and H, written $G[H]$, is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent to (u_2, v_2) if u_1 is adjacent to u_2 in G or if $u_1 = u_2$ and v_1 is adjacent to v_2 in H. The rth *power* of graph G, denoted G^r, is the graph with vertex set $V(G)$ and edge set $\{(u,v): d(u,v) \leq r \text{ in } G\}$. The bandwidth of graph G is min max $|f(u) - f(v)|$, where the max is taken over each edge $uv \in E(G)$, and the min is over all proper numberings, f. This paper establishes tight upper and lower bounds for the bandwidth of an arbitrary graph composition $G[H]$, with the upper bound based only on $|V(H)|$ and the bandwidth of G. In addition, the exact bandwidth of the composition of $G[H]$ is established for G the power of a path or a cycle.

1 Introduction and terminology

All graphs are assumed to be undirected, simple, and finite. For $G = (V, E)$, either V or $V(G)$ will be used to denote the set of vertices of G, and either E or $E(G)$ will denote the set of edges of G. For $u, v \in V$, the *distance* $d(u, v)$ *between* u and v is the minimum of the lengths of all paths between u and v in G. $\delta(G)$ denotes the minimum degree of any vertex in G. For any $S = \{u_1, u_2, \ldots, u_n\} \subseteq V(G)$, the *neighborhood* $N(S)$ is the set of all vertices v in $V(G) - S$ such that v is adjacent to at least one vertex in S.

Bandwidth on graphs, and the analogous problem of bandwidth on matrices, has been studied since the early 1950s (see [1].) Following the notation of [1] and [10], we may define bandwidth as follows. Let $G = (V, E)$ be a graph on n vertices. A 1-1 mapping $f: V \to \{1, 2, \ldots, n\}$ will be called a *proper numbering* of G. The *bandwidth of a proper numbering* f of G, denoted $B_f(G)$, is the number

$$\max\{|f(u) - f(v)|: \text{ edge } uv \in E(G)\},$$

and the *bandwidth* of G, denoted $B(G)$, is the number

$$\min\{B_f(G): f \text{ is a proper numbering of } G\}.$$

A proper numbering f of G is a *bandwidth numbering* of G if $B_f(G) = B(G)$.

The decision problem corresponding to finding the bandwidth of an arbitrary graph was shown to be NP-complete in [12]. In [7] it was shown that the problem is NP-complete even for trees of maximum degree 3. In general, many upper and lower bounds are known which relate bandwidth to various graph invariants; however, the exact value of the bandwidth is only known for a few special classes of graphs. It is easy to find the bandwidth for graphs such as K_n, P_n (the path on n vertices), C_n (the cycle on n vertices), $K_{1,n}$ (the star on $n+1$ vertices) and others. [8] has established the bandwidth for hypercubes (or n-cubes). [3] found the bandwidth for $P_n \times C_m$, and for a number of other special graphs. [2] and [1] contain a number of survey results pertaining to solved problems. [1] also surveys a number of bounds on bandwidth. [6] provides results relating to the relationship between bandwidth and bandsize, and [11] provides some insight on the relationship between bandwidth and VLSI layout width.

[10] gives the bandwidth $B(G_1 + G_2)$ for $|V(G_1)| = n_1 \geq n_2 = |V(G_2)|$ with $B(G_1) \leq \lceil n_1/2 \rceil$, and also provides bounds for $B(G_1 + G_2)$ in other cases. [9] extends these results to the sum of k graphs.

2 Bounding the bandwidth of graph compositions

Given a graph composition $G[H]$, we denote the vertices in $\{(u,v): v \in V(H)\}$ as H_u. We first develop an upper bound for the bandwidth of the composition of two graphs.

Theorem 1.

$$B(G[H]) \leq B(G) \cdot |V(H)| + |V(H)| - 1.$$

Proof: Let f be a bandwidth numbering of G. Let m denote $|V(H)|$. Define a proper numbering F on $G[H]$ by numbering the vertices within H_u (arbitrarily) with integers $(f(u) - 1) \cdot |V(H)| + 1$, $(f(u) - 1) \cdot |V(H)| + 2, \ldots, f(u) \cdot |V(H)|$. Vertices x and y in distinct subgraphs H_u and H_w, are adjacent in $G[H]$ if and only if u is adjacent to w in G which implies that $|f(u) - f(w)| \leq B(G)$. For $f(u) > f(w)$, we must have $|F(x) - F(y)| \leq f(u) \cdot |V(H)| - [(f(w) - 1) \cdot |V(H)| + 1] = (f(u) - f(w)) \cdot |V(H)| + |V(H)| - 1 <$

$B(G) \cdot |V(H)| + |V(H)| - 1$. If x is adjacent to y and both are in the same subgraph H_u, then $|F(x) - F(y)| \le |V(H)| - 1$. Thus, in any case, we have $B(G[H]) \le B_F(G[H]) \le |F(x) - F(y)| \le B(G) \cdot |V(H)| + |V(H)| - 1$. □

It is easy to verify the following bandwidth computations which are used in the remainder of the paper: $B(P_m) = 1$, $B(C_m) = 2$, $B(K_m) = m - 1$, $B((P_m)^r) = r$ for $r < m$, $B((C_m)^r) = \min\{2r, m - 1\}$, and (for $\overline{K}_m =$ the complement of K_m) $B(\overline{K}_m) = 0$. To see that the upper bound given in Theorem 1 is tight, we note that for $G = K_n$ and $H = K_m$, $G[H] = K_{nm}$ and $B(G[H]) \le (n-1)m + m - 1 = nm - 1 = B(K_{nm})$.

The following two corollaries are direct consequences of Theorem 1.

Corollary 1. *For $|V(H)| = n$ and $m > r$, $B((P_m)^r[H]) \le (r+1)n - 1$.*

Corollary 2. *For $|V(H)| = n$, $B((C_m)^r[H]) \le (2r+1)n - 1$.*

We next develop a lower bound for the bandwidth of the composition of two graphs. For graph G of order n let $\eta(G)$ denote max min $|N(A)|$ where the maximum is over all k with $1 \le k \le n$ and the minimum is over all $A \subseteq V(G)$ with $|A| = k$. Bound $\eta(G)$ was first defined in [8].

Theorem 2. $B(G[H]) \ge \eta(G) \cdot |V(H)| + \delta(H)$.

Proof: We first define functions $p\colon 2^{V(G[H])} \to 2^{V(G)}$ and $\Phi\colon 2^{V(G[H])} \to \{1, 2, \ldots, |G[H]|\}$. For all $S \subseteq V(G[H])$, $p(S) = \{u\colon S \cap V(H_u) \ne \phi\}$ and $\Phi(S) = |p(S)|$ (as defined in [8]). Let $T_k = \{S\colon S \subseteq V(G[H]),\ \Phi(S) = k,$ and there is an $x \in S$ such that $\Phi(S - \{x\}) < k\}$. From [8] we must then have $B(G[H]) \ge \max\min |N(S)|$, where the maximum is over all k such that $1 \le k \le n$ and the minimum is over all $S \in T_k$. Since for all $S \in T_k$, $|N(S)| \ge |N_G(p(S))| \cdot |V(H)| + \delta(H)$ and $\{p(S)\colon S \in T_k\} = \{A\colon A \subseteq V(G), |A| = k\}$ it follows that $B(G[H]) \ge \max\min(|N_G(A)| \cdot |V(H)| + \delta(H)) = \eta(G) \cdot |V(H)| + \delta(H)$, where the maximum is over all k such that $1 \le k \le n$ and the minimum is over all $A \subseteq V(G)$ such that $|A| = k$. □

If $H = K_1$ it is clear that the bound established by Theorem 2 is tight. The following two corollaries follow immediately.

Corollary 3. $B(G[K_m]) \ge \eta(G) \cdot m + m - 1$.

In fact, note that $\eta(G) \cdot m + m - 1 \le B(G[K_m]) \le B(G) \cdot m + m - 1$.

Corollary 4. *If $B(G) = \eta(G)$, then $B(G[Km]) = B(G) \cdot m + m - 1$.*

3 Bandwidth involving powers of graphs

Lemmas 1 and 2, which are used in this section, are previously known results.

Lemma 1 (From [1]). *If H is a subgraph of G, then $B(H) \le B(G)$.*

Given proper numbering f of G, let $u_i = f^{-1}(i)$.

Lemma 2 (From [8]). *Let f be a proper numbering on G, a graph on p vertices. Then, for every $x \in [1,p]$, $B_f(G) \geq |N(\{u_1,u_2,\ldots,u_x\})|$.*

We now consider compositions on $G[H]$ for G a power of a graph path.

Lemma 3. *Let $G = (P_{r+2})^r[\overline{K}_n]$. Then $B(G) \geq (r+1)n-1$.*

Proof: Suppose f is a proper numbering of G. Let $u_j = f^{-1}(j)$, $V((P_{r+2})^r) = \{v_1,v_2,\ldots,v_{r+2}\}$, and $P_{r+2} = v_1v_2\ldots v_{r+2}$. Let $\Gamma_j = V((\overline{K}_n)_j)$ for each $1 \leq j \leq r+2$ where $(\overline{K}_n)_j$ is the jth copy of $\overline{K}_n$. We consider two cases for the location of u_1.

Case I. $u_1 \notin \Gamma_1 \cup \Gamma_{r+2}$. Then $N(\{u_1\})$ contains all but $n-1$ vertices of G so that u_1 is adjacent to u_j for some $j \geq (r+2)n-(n-1) = (r+1)n+1 \Rightarrow f(u_j) - f(u_1) \geq (r+1)n \Rightarrow B_f(G) > (r+1)n-1$.

Case II.. $u_1 \in \Gamma_1 \cup \Gamma_{r+2}$. Let $y = \min\{i\colon u_i \notin \Gamma_1 \cup \Gamma_{r+2}\}$. Then $u_y \in N(\{u_1\})$, and $\max\{i\colon u_i \in N(\{u_1\})\} \geq nr+y-1$ since $|N(\{u_1\})| = nr$. Thus **A**: $B_f(G) \geq nr+y-2$. From Lemma 2, we have **B**: $B_f(G) \geq |N(\{u_1,\ldots,u_y\})| = |V(G)|-|\{u_1,\ldots,u_y\}| = (r+2)n-y$. Adding inequalities **A** and **B** and dividing by 2 gives $B_f(G) \geq (r+1)n-1$.

Thus, whether Case I or Case II holds, $B_f(G) \geq (r+1)n-1$, therefore $B(G) \geq (r+1)n-1$. □

Theorem 3. *For $G = (P_m)^r[H]$, $|V(H)| = n$, and $m \geq r+2$, $B(G) = (r+1)n-1$.*

Proof: First note that $(P_{r+2})^r[\overline{K}_n] \subseteq (P_m)^r[\overline{K}_n] \subseteq (P_m)r^{|}H]$. Then by Lemma 3, Lemma 1 and Corollary 1 we have

$$(r+1)n-1 \leq B(P_{r+2})^r[\overline{K}_n]) \leq B((P_m)^r[H]) \leq (r+1)n-1$$

Thus $B(G) = (r+1)n-1$. □

Corollary 5 follows directly from Theorem 3.

Corollary 5. *For H any graph with $|V(H)| = n$ and $m \geq 3$, $B(P_m[H]) = 2n-1$.*

The following corollaries may be derived from Theorem 3 and from results providing the bandwidth of the sum of two graphs in [9].

Corollary 6.

$$B(P_m[P_n]) = \begin{cases} 3 & \text{if } m=n=2 \\ 2n - \lceil (n+1)/2 \rceil & \text{if } m=2 \text{ and } n \neq 2 \\ 2n-1 & \text{if } m \geq 3 \end{cases}$$

Corollary 7.

$$B(P_m[C_n]) = \begin{cases} 5 & \text{if } m=2 \text{ and } n=3 \\ 6 & \text{if } m=2 \text{ and } n=4 \\ 2n-\lceil (n+1)/2 \rceil & \text{if } m=2 \text{ and } n\geq 5 \\ 2n-1 & \text{if } m\geq 3 \end{cases}$$

Corollary 8.

$$B(P_m[\overline{K}_n]) = \begin{cases} n+\lceil n/2 \rceil -1 & \text{if } m=2 \\ 2n-1 & \text{if } m\geq 3 \end{cases}$$

We next consider compositions $G[H]$ for G a power of a graph cycle.

Lemma 4. *For $G=(C_m)^r[\overline{K}_n]$ and $m\geq 2r+2$, $B(G)\geq (2r+1)n-1$.*

Proof: Suppose f is a proper numbering of G. Let $u_j=f^{-1}(j)$, $V((C_m)^r) = \{v_1, v_2, \ldots, v_m\}$, $C_m = v_1v_2\ldots v_mv_1$, and Γ_j be the copy of $\overline{K}_n$ corresponding to v_j. Without loss of generality we may suppose that $u_1 \in \Gamma_1$. Let $y=\min\{i\colon u_i \in N(\{u_1\})\}$. Then $\max\{i\colon u_i\in N(\{u_1\})\}\geq 2nr+y-1$ which implies **A**: $B_f(G)\geq 2nr+y-2$. From Lemma 2 we have **B**: $B_f(G)\geq |N(\{u_1,\ldots,u_y\})|\geq (2r+2)n-y$. Adding **A** and **B** and dividing by 2 we obtain $B_f(G)\geq (2r+1)n-1$ which implies $B(G)\geq (2r+1)n-1$. □

Theorem 4. *For $G=(C_m)^r[H]$, $|V(H)|=n$, and $m\geq 2r+2$, $B(G)=(2r+1)n-1$.*

Proof: From Lemma 4, Lemma 1, Corollary 2, and the fact that $(C_m)^r[\overline{K}_n] \subseteq (C_m)^r[H]$, we obtain

$$(2r+1)n-1\leq B((C_m)^r[\overline{K}_n])\leq B((C_m)^r[H])\leq (2r+1)n-1.$$

Thus $B(G)=(2r+1)n-1$. □

Corollary 9 follows directly from Theorem 4.

Corollary 9. *For H any graph with $|V(H)|=n$ and $m\geq 4$, $B(C_m[H])=3n-1$.*

The following corollaries may also be derived from Theorem 4.

Corollary 10.

$$B(C_m[P_n]) = \begin{cases} 2 & \text{if } m=3 \text{ and } n=1 \\ 5 & \text{if } m=3 \text{ and } n=2 \\ 2n+\lfloor (n-1)/2 \rfloor & \text{if } m=3 \text{ and } n\geq 3 \\ 3n-1 & \text{if } m\geq 4 \end{cases}$$

Corollary 11.

$$B(C_m[C_n]) = \begin{cases} 2n+2 & \textit{if } m=3 \textit{ and } 3 \le n \le 5 \\ 2n+\lfloor (n-1)/2 \rfloor & \textit{if } m=3 \text{ and } n \ge 6 \\ 3n-1 & \textit{if } m \ge 4 \end{cases}$$

Corollary 12.

$$B(C_m[\overline{K}_n]) = \begin{cases} 2n+\lfloor (n-1)/2 \rfloor & \textit{if } m=3 \\ 3n-1 & \textit{if } m \ge 4 \end{cases}$$

4 Conclusions

The bandwidth for the composition of two graphs has been bounded above and below and all bounds have been shown to be tight. In addition, exact values for bandwidth have been established for a number of graph compositions involving graph powers on paths and cycles.

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References

[1] P.Z. Chinn, J. Chvatalova, A.K. Dewdney, and N.E. Gibbs, "The bandwidth problem for graphs and matrices - a survey", *Journal of Graph Theory*, vol. 6, pp. 223–254,1982.

[2] F.R.K. Chung, "Labelings of Graphs", *Selected Topics in Graph Theory*, vol. 3, pp. 151–168, 1988.

[3] J. Chvatalova, "*On the bandwidth problem for graphs*", Ph.D. Thesis, Department of Combinatorics and Optimization, University of Waterloo, 1980.

[4] J. Chvatalova, A.K. Dewdney, N.E. Gibbs, and R.R. Korfhage, "The bandwidth problem for graphs: a collection of recent results", Research Report No. 24, Department of Computer Science, University of Waterloo, 1975.

[5] P.G. Eitner, "The bandwidth of the complete multipartite graph", Presented at the Toledo Symposium on Applications of Graph Theory, 1979.

[6] P. Erdos, P. Hell, and P. Winkler, "Bandwidth versus bandsize", *Annals of Discrete Mathematics*, vol. 41, pp. 117–130, 1989.

[7] M.R. Garey, R.L. Graham, D.S. Johnson, and D.E. Knuth, "Complexity results for bandwidth minimization", *SIAM J. Appl. Math.*, pp. 477–495, MR57#18220, 1978.

[8] L.H. Harper, "Optimal numberings and isoperimetric problems on graphs", *J. Combinatorial Theory*, vol. 1, pp. 385–393, MR34#91, 1966.

[9] Y.L. Lai, J. Liu and K. Williams, "Bandwidth for the Sum of k Graphs", *Ars Combinatoria*, vol. 37, pp. 149–155, 1992.

[10] J. Liu, J. Wang and K. Williams, "Bandwidth for the Sum of Two Graphs", Proceedings of the Twenty-second Southeastern International Conference on Combinatorics, Graph Theory and Computing, *Congressus Numerantium*, Vol.. 82, pp. 79–85, 1991.

[11] F. Makedon, and I.H. Sudborough, "On minimizing width in linear layouts", *Discrete Applied Mathematics*, vol. 23, pp. 243–265, 1989.

[12] C.H. Papadimitriou, "The NP-completeness of the bandwidth minimization problem", *Computing*, vol. 16, pp. 263–270, MR53#14981, 1976.