When is a complex matrix a character table? A reduction to vertex independence

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ABSTRACT. In [3] R. Brauer asked the question: When is an $n \times n$ complex matrix X the ordinary character table of some finite group? It is shown that the problem can be reduced in polynomial time to that of VERTEX INDEPENDENCE. We also pose and solve some (much) simpler problems of a related combinatorial nature.

1 Multiplicity structures

Definition 1. In [18, 19] a *multiplicity structure (MS) of order* n is defined as a pair $\mu = (\mathbf{h}, \gamma)$, where $\mathbf{h}$ is a sequence h_i of n positive integers and γ is an array γ_{ij}^k $(i, j, k = 1, 2, \ldots, n)$ of n^3 non-negative integers satisfying

$$h_i h_j = \sum_{k=1}^{n} h_k \gamma_{ij}^k \tag{1}$$

In the sequel we let $g = \sum_1^n h_i$, choose a fixed set G of order g and a fixed partition of G into disjoint sets C_i with $|C_i| = h_i$ $(i = 1, 2, \ldots, n)$. For $a \in C_k$ put $\gamma_{ij}^a = \gamma_{ij}^k$ By (1) we can draw up a $g \times g$ multiplication table in such a way that any block $C_i \times C_j$ has a repeated exactly γ_{ij}^a times for $a \in G$, and we say that the corresponding binary operation '$\circ$' and the multiplicity structure μ are *compatible.*

If $(G, \circ)$ is a group and the C_i are its conjugate classes, we call μ the *conjugate class* (CC) structure of $(G, \circ)$. Then by convention, $C_1 = \{e\}$, where e is the neutral element. Besides this CC structure there are many ways in which an MS can arise in a group; see for example [12, 1, 11]. Related concepts can be found in [6, 7, 8, 13, 14, 4].

2 The conjugate class structure

Brauer's problem is equivalent to asking for a characterization of those MS's which are CC structures [5, 9]. Thus, for a given $n \times n$ complex matrix $X = [\chi_j^i]$ to be the ordinary character table of some finite group, necessarily the columns X_j must be mutually orthogonal, closed under complex conjugation and have lengths given by $|X_j|^2 = h_j'$, where h_j' is a positive integer. Putting $g = h_1'$, the numbers $h_j = g/h_j'$ are integers, row $X^1 = [1, \ldots, 1]$, column $X_1 > 0$ and the equations

$$\gamma_{ij}^k = \frac{h_i h_j}{g} \sum_{r=1}^{n} \frac{\chi_i^r \chi_j^r \overline{\chi_k^r}}{\chi_1^r} \tag{2}$$

have to define an MS $\mu = (\mathbf{h}, \gamma)$. Then, provided X has these properties, μ is the CC structure of a group $(G, \circ)$ iff X is the character table of $(G, \circ)$.

3 Other notation

Given a finite simple graph $\mathcal{H} = (V, E)$, with vertex set V and edge set E, a set $S \subseteq V$ is *independent* if no two vertices of S form an edge and the *vertex independence number* $\alpha(\mathcal{H})$ is the largest size of an independent set [2]. The problem VERTEX INDEPENDENCE ie the determination of $\alpha(\mathcal{H})$ is equivalent to many other combinatorial problems; for example, through a construction of E. Lawler, to THREE DIMENSIONAL MATCHING [15, 10].

In what follows, if $m \geq 0$ is an integer, we let $[m] = \{1, \ldots, m\}$, so that $[0] = \emptyset$. Disjoint unions are written as sums.

4 Main result

Theorem 1. [19]. *Let $\mu = (\mathbf{h}, \gamma)$ be an MS such that the integers h_i divide $g = \sum_1^n h_i$ with $h_1 = 1$ and $\gamma_{11}^1 = 1$. Then there is a graph $\mathcal{H}$ constructible from μ in polynomial time such that μ is a conjugate class structure iff $\mathcal{H}$ has independence number $\alpha(\mathcal{H}) = g^3$.*

Proof: Partition a set G of order $g \geq 2$ into n classes C_i with $|C_i| = h_i$ as above with $C_1 = \{e\}$. The partition defines an equivalence relation '$\equiv$' on G. The vertex set of our graph X is the union

$$V = \sum_{i,j,k} C_i \times C_j \times G \times C_k \times G \times G \times [\gamma_{ij}^k] \times [g/h_i],$$

and two distinct vertices (written as strings) $\xi = abcxyzfr$ and $\xi' = a'b'c'x'y'z'f'r'$ form an edge if at least one of the following conditions holds:

(i) $abc = a'b'c'$.

(ii) $ab = a'b', \quad x \neq x'$.

(iii) $bc = a'b', \quad y \neq x'$.

(iv) $xc = a'b', \quad z \neq x'$.

(v) $ay = a'b', \quad z \neq x'$.

(vi) $a \equiv a', b \equiv b', c = c', x = x', f = f'$.

(vii) $bx = b'x', \quad a \neq a'$.

(viii) $ax = a'x', \quad b \neq b'$.

(ix) $\xi' = acbey'bf'r', \quad z \neq b$.

(x) $\xi = abaxxzfr, \xi' = ab'ax'z'f'r, \quad b \neq b'$.

Sufficiency. Let $S \subseteq V$ be independent with $|S| = g^3$. By rule (i) for the formation of edges, there is a bijection $abc \leftrightarrow \xi = \xi(abc) = abcxyzfr$ between G^3 and S, so that the triple abc uniquely determines the other entries of ξ. Rule (ii) defines a binary operation on G given by $x = a \circ b$ where x is independent of c. Rule (iii) ensures that $b \circ c = y$ and (iv) that $x \circ c = z$ while $a \circ y = z$ by (v), so that the associative law holds. To show compatibility, fix c and for $x \in G$ let J_{ij}^x be the collection of those $abcxyzfr \in S$ with $ab \in C_i \times C_j$. By rule (vi), $|J_{ij}^x| \leq \gamma_{ij}^x$ and using equation (1),

$$h_i h_j = \sum_{x \in G} |J_{ij}^x| \leq \sum_{x \in G} \gamma_{ij}^x = h_i h_j,$$

from which compatibility follows.

Rules (vii) and (viii) ensure that $(G, \circ)$ is a group, and since $\gamma_{11}^1 = 1$, the neutral element must be e. Write the inverse of a as a^{-1}, and let $Z(a)$ be the centralizer of a. To see that C_i consists of complete conjugate classes, let $b \in C_i$ and $a \in G$. Find $\xi = \xi(aba^{-1})$ and $\xi' = \xi(aa^{-1}b)$. If $\xi = \xi'$ then $a \circ b \circ a^{-1} = b$ while if $\xi \neq \xi'$, rule (ix) says that $a \circ b \circ a^{-1} \equiv b$. Finally, suppose that $b \neq b'$ are in $Z(a)$ and $a \in C_i$. By (x) the last entries in $\xi(aba)$ and $\xi(ab'a)$ must be distinct so that $|Z(a)| \leq g/h_i$, with equality holding. Hence C_i is the conjugate class containing a.

Necessity. Suppose that μ is the CC structure of the group $(G, \circ)$. To each $ab \in G^2$ associate two integers $(ab)' \geq 1$ and $(ab)^* \geq 1$ so that the following hold: Given $1 \leq i, j \leq n$ and $x \in G$, the numbers $(ab)'$ run once through $[\gamma_{ij}^x]$ as $ab \in C_i \times C_j$ with $a \circ b = x$. For fixed $a \in C_i$ we set $(ab)^* = 1$ if $b \notin Z(a)$ and let $(ab)^*$ run through $[g/h_i]$ as $b \in Z(a)$. Then the elements

$abc(a \circ b)(b \circ c)(a \circ b \circ c)(ab)'(ab)^*$ for $abc \in G^3$ form an independent set in $\mathcal{H}$, and $\alpha(\mathcal{H}) = g^3$ by rule (i).

Since $\mathcal{H}$ has clearly been constructed from μ in polynomial time [10], the statement follows. □

Remark 1. Taking the observations of section (2) into consideration, we have clearly reduced (in polynomial time) the original form of Brauer's problem to VERTEX INDEPENDENCE.

Remark 2. The assumed properties of μ are extremely meagre. It is just possible that by adding sufficiently many known properties, enough structure may be introduced to make some equivalent of the combinatorial problem more tractable. Among the more elementary of these are the following [5, 9]:

(i) $\gamma^i_{1i} = 1$

(ii) $\sum_s \gamma^s_{ij}\gamma^r_{sk} = \sum_s \gamma^r_{is}\gamma^s_{jk}$

(iii) There is an involution $r \leftrightarrow \bar{r}$ of $[n]$ which leaves 1, h_k and γ^k_{ij} invariant and allows $h_k\gamma^{\bar{k}}_{ij}$ to be symmetric in all its indices.

Like these, further necessary properties of μ can be inferred from those of the χ^i_j and equation (2).

5 Some related problems and their solutions

[18, 19]

Theorem 2. *A given MS $\mu = (\mathbf{h}, \gamma)$ is compatible with a quasi-group (ie a binary operation on G for which both cancellation laws hold) iff for each i and k*

$$\sum_{j=1}^{n} \gamma^k_{ij} = \sum_{j=1}^{n} \gamma^k_{ji} = h_i.$$

Theorem 3. *μ is compatible with a commutative binary operation iff $\gamma^k_{ij} = \gamma^k_{ji}$ and for each i*

$$\sum \{h_k : \gamma^k_{ii} \text{ is odd }\} \leq h_i.$$

Theorem 4. *The MS μ is compatible with an operation '$\circ$' such that $a \circ b \neq b \circ a$ for $a \neq b$ iff $h_k = 1$ implies that*

$$\gamma^k_{ii} \leq \frac{1}{2}h_i(h_i + 1) \text{ and } \gamma^k_{ij} + \gamma^k_{ji} \leq h_i h_j \quad (i \neq j).$$

We omit the easy proofs of theorems 2 and 3.

Proof of Theorem 4: For $Y \subseteq G \times G$, let $Y' = \{(b,a) : (a,b) \in Y\}$ and $diagY = \{(a,b) \in Y : a = b\}$. Calling Y *independent* if $Y \cap Y' \subseteq diagY$, these sets form the independent sets of a matroid [17] on G^2 with rank function given by

$$\rho(X) = |X| - \frac{1}{2}(|X \cap X'| - |diagX|) \quad (X \subseteq G^2).$$

Although it is not immediately obvious, it is not hard to see that μ is compatible with a binary operation of the required kind iff the following two conditions are valid:

(i) For each $1 \leq i \leq n$ there are disjoint independent sets X_i^a of orders γ_{ii}^a with
$$\sum_{a \in G} X_i^a = C_i \times C_i.$$

(ii) For each pair $1 \leq i < j \leq n$ there are disjoint independent sets Y_{ij}^a of orders $\gamma_{ij}^a + \gamma_{ji}^a$ such that
$$\sum_{a \in G} Y_{ij}^a = C_i \times C_j + C_j \times C_i.$$

Now let M_i^a be the matroid restricted to $C_i \times C_i$ and truncated at γ_{ii}^a, and for $i < j$ let N_{ij}^a be its restriction to $C_i \times C_j + C_j \times C_i$ and truncated at $\gamma_{ij}^a + \gamma_{ji}^a$. The Nash-Williams formula [16, 17] applied to the union of the matroids M_i^a over $a \in G$ then gives necessary and sufficient conditions for (i) to hold, and similarly the union of the N_{ij}^a determines the conditions for (ii). These are the conditions of theorem 4. □

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