A Characterization of Well-Covered Claw-Free Graphs Containing no 4-Cycles

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Abstract. We show how a claw-free well-covered graph containing no 4-cycle, with any given independence number m, can be constructed by linking together m subgraphs, each isomorphic to either K_2 or K_3. We show further that the only well-covered connected claw-free graphs containing no 4-cycle that cannot be constructed in this way are K_1 and the cycle graphs on 5 and 7 vertices respectively.

1. Introduction

In this paper, G will denote a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$. We shall further assume that G is claw-free (that is, G contains no induced subgraph isomorphic to $K_{1,3}$) and that G contains no 4-cycle as a (not necessarily induced) subgraph.

A set $J \subseteq V(G)$ is said to be *independent* if no pair of vertices of J is adjacent. The size of the largest maximal independent set is called the *independence number* of G and denoted by $\beta(G)$. In 1970, Plummer [5] introduced the concept of a *well-covered* graph, as a graph in which every maximal independent set has the same size, $\beta(G)$. These graphs are of interest because whereas the problem of determining the independence number of an arbitrary graph is NP-complete, in the case of a well-covered graph, it can be found by determining the size of any one maximal independent set.

Various approaches to the problem of characterizing families of well-covered graphs have been tried and the reader is referred to [9] for an excellent survey of progress. One approach has been to restrict the cycle lengths contained in the graph. Well-covered graphs of girth 8 or more have been characterized by Finbow and Hartnell [1]; their result was extended to include well-covered graphs of girth 5 or more by Finbow, Hartnell and Nowakowski [2]. The same three authors [3] have also characterized well-covered graphs containing no cycles of length 4 or 5. However, when the cycle restriction is relaxed so that only cycles of length 4 are debarred, the problem of characterization appears much more difficult. Gasquoine, Hartnell, Nowakowski and Whitehead [5] have described techniques for building a family of connected well-covered graphs containing no 4-cycles. Twenty-seven graphs in this family, that are also edge-critical with respect to the property of being well-covered, are illustrated in an appendix to [5]. The authors also give seven other examples of such edge-critical graphs known to them: K_1, K_2, K_3, C_5 and three graphs on respectively 13, 14 and 16 vertices. Of these 34 graphs, 20 can be used as building blocks, in a manner described in [5], to produce well-covered graphs of larger order. However, no complete characterization of well-covered graphs containing no 4-cycles is given. In this paper, we add the extra restriction that the graph is claw-free.

It follows from a result of van Rooij and Wilf [10], that a claw-free graph containing no 4-cycle is necessarily a line graph. The graphs characterized in this paper are therefore a subclass of well-covered line graphs. The edge analogue of the well-covered property for graphs is the property that every maximal matching in a graph G is also a *maximum*. Such graphs are called *equimatchable*. Thus a graph G is equimatchable if and only if its line graph $L(G)$ is well-covered. The problem of characterizing equimatchable graphs was first suggested by Grünbaum [4] in 1974. A characterization has been given by Lewin in [7] and a good characterization, that leads to a polynomial recognition algorithm, by Lesk, Plummer and Pulleybank in [6]. However, neither of these characterizations is descriptive.

2. Notation and Preliminary Results

We shall use the following notation. Let $v \in V(G)$. Then we denote the neighbourhood of v by $N(v)$ and the closed neighbourhood, $N(v) \cup \{v\}$, by $N[v]$. For any subset $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by S. With a slight abuse of notation, we shall denote the independence number of $\langle S \rangle$ by $\beta\langle S \rangle$. A cycle graph on n vertices will be denoted by C_n. The minimum vertex degree in G is denoted by $\delta(G)$.

Lemma 1. *When $\beta(G) = 1$, then G is isomorphic to one of K_1, K_2 or K_3.*

Proof: Since $\beta(G) = 1$, $V(G)$ is a clique. The result follows by observing that G contains no 4-cycle. ∎

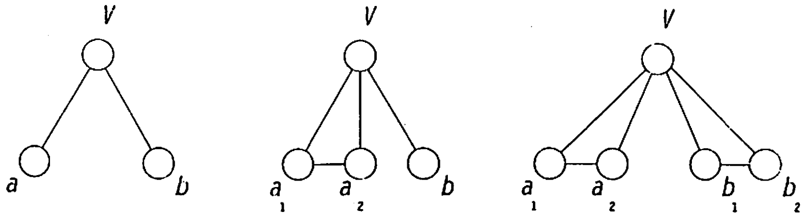
Lemma 2. *Let $v \in V(G)$. Then $\beta\langle N(v) \rangle \leq 2$ and, when $\beta\langle N(v) \rangle = 2$, then $\langle N(v) \rangle$ is disconnected and each of its two components is isomorphic to either K_1 or K_2.*

Proof: Since G contains no induced subgraph isomorphic to $K_{1,3}$, it follows that $\beta\langle N(v) \rangle \leq 2$, for all $v \in V(G)$. Further, since G contains no 4-cycle, $N(v)$ contains no path of length 2. Thus, when $\beta\langle N(v) \rangle = 2$, then $N(v)$ is disconnected and the result follows. ∎

It follows from Lemma 2 that, for any $v \in V(G)$, the induced subgraph $v \in V(G)$ is one of the three possibilities shown in Figure 1.

The following result is well known and has appeared in several other papers on well-covered graphs (see e.g. [2], where a more general version is proved as Lemma 1).

Lemma 3. *Suppose that $\beta(G) \geq 2$. Let $v \in V(G)$ and denote the subgraph $G - N[v]$ by H. Then H is well-covered and $\beta(H) = \beta(G) - 1$.*



(i) (ii) (iii)

Figure 1

Proof: Since $\beta(G) \geq 2$, $V(H) \neq \emptyset$. Let J be any maximal independent set in H. Then $J \cup \{v\}$ is a maximal independent set in G. Thus $|J| = \beta(G) - 1$, for all choices of J, and hence H is well-covered. ∎

Corollary 3.1. *Suppose $\beta(G) \geq 2$. Let $v \in V(G)$ be such that $\beta\langle N(v)\rangle = 2$ and let a, b be independent vertices in $N(v)$. Let J be a maximal independent set in $H = G - N[v]$. Then J contains a neighbour of a or a neighbour of b.*

Proof: Suppose there is a maximal independent set J in H containing no neighbour of either a or b. Then $J \cup \{a, b\}$ is a maximal independent set in G. But by Lemma 3, $\beta(G) = |J| + 1$, so this is impossible. ∎

We make the following definitions. A *basic* subgraph is an induced subgraph B, isomorphic to K_2 or K_3, and such that B contains a vertex u for which $N[u] = V(B)$; the vertex u is called a *hidden* vertex of B. Thus, when $B \cong K_2$, a hidden vertex of B has degree 1 in G; and when $B \cong K_3$, a hidden vertex of B has degree 2 in G.

Taking a hidden vertex of a basic subgraph as the vertex v of Lemma 3, we have the following corollary.

Corollary 3.2. *Suppose $\beta(G) \geq 2$ and G contains a basic subgraph B. Then $G - V(B)$ is well-covered with independence number $\beta(G) - 1$.*

3. A Construction

We show in this section how a claw-free well-covered graph containing no 4-cycle, with any given independence number m, can be constructed by linking together m basic subgraphs.

Theorem 4. *Suppose G is a graph such that $V(G)$ can be partitioned into subsets $V_1, V_2, \ldots, V_m$, such that $\langle V_i\rangle$ is a basic subgraph, $i = 1, 2, \ldots, m$. Then G is well-covered. Further, G is claw-free and contains no 4-cycle if and only if the following conditions are also satisfied:*

(a) *there is at most one edge between the sets V_i and V_j, $i \neq j$;*
(b) *if $x_i \in V_i$ is adjacent to both $x_j \in V_j$ and to $x_k \in V_k$, where i, j, k are distinct, then x_j and x_k are also adjacent;*

(c) *a vertex of V_i is adjacent to a vertex of at most two other subsets in the partition, distinct from V_i.*

Proof: We first show that G is well-covered. Since the vertices of each subset $\langle V_i \rangle, i = 1, 2, \ldots, m$, form a clique, at most one vertex from each subset can be included in any maximal independent set in G. However, since each subset V_i contains a vertex v_i such that $N[v_i] \subseteq V_i$, every maximal independent set must include a vertex of V_i. Hence G is well-covered, with $\beta(G) = m$.

The proof is concluded by noting that each of the three conditions (a), (b), (c) is necessary for G to contain no 4-cycle, and that taken together they are also sufficient. The conditions (b) and (c) are each necessary to ensure that G is claw-free and, taken together, they are also sufficient. ∎

If G satisfies the conditions of Theorem 4 and is also connected, we shall call G a *basic chain*.

We now describe a recursive method for constructing a basic chain, with independence number m, from a given sequence of m basic subgraphs, $B_1, B_2, \ldots, B_m$, where for $2 \leq i \leq m-2$, not both B_i and B_{i+1} are isomorphic to K_2. We adopt the notation that $V(B_i) = \{v_i, a_i\}$, when $B_i \cong K_2$, and $V(B_i) = \{v_i, a_i, b_i\}$, when $B_i \cong K_3$, where in either case, v_i denotes a hidden vertex of B_i. Then, the graph G_i is defined recursively as follows:

$$G_1 = B_1;$$
$$V(G_i) = V(G_{i-1}) \cup V(B_i), 2 \leq i \leq m;$$

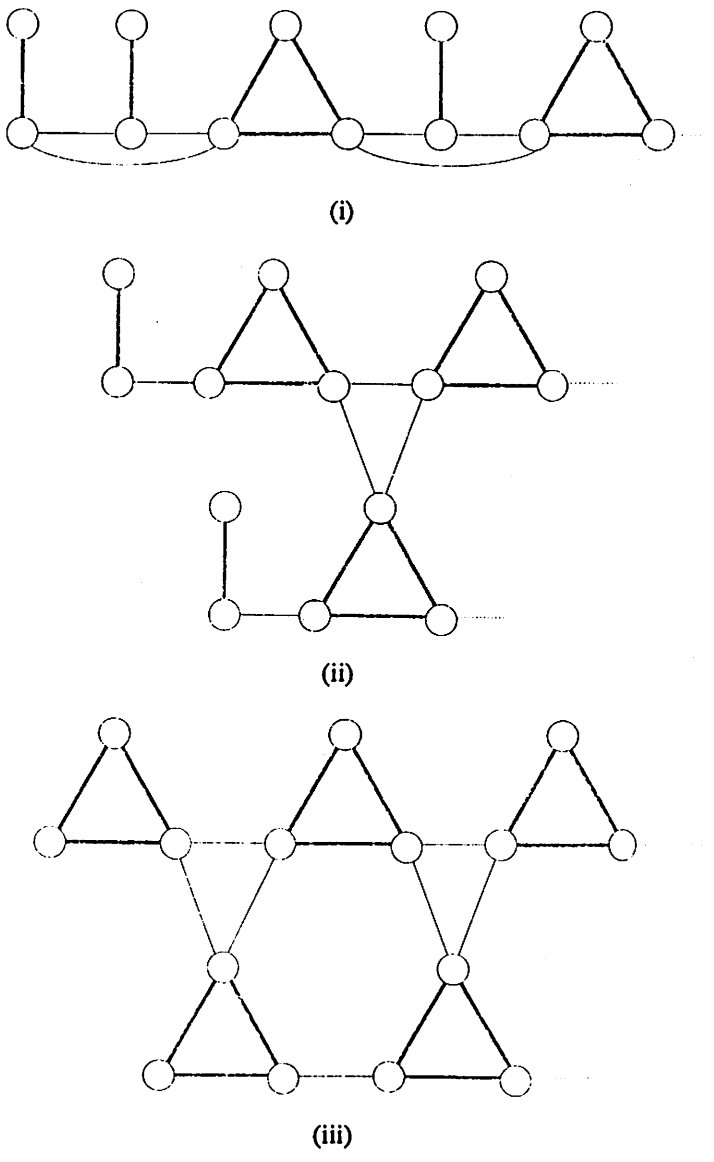
When $B_1 \cong K_3$, then

$$E(G_i) = \begin{cases} E(G_{i-1}) \cup E(B_i) \cup \{a_i b_{i-1}\} & \text{when } B_{i-1} \cong K_3, 2 \leq i \leq m; \\ E(G_{i-1}) \cup E(B_i) \cup \{a_i a_{i-1}, a_i b_{i-2}\} & \text{when } B_{i-1} \cong K_2, 2 \leq i \leq m. \end{cases}$$

When $B_1 \cong K_2$, then

$$E(G_2) = E(G_1) \cup E(B_2) \cup \{a_2 a_1\};$$
$$E(G_3) = E(G_2) \cup E(B_3) \cup \{a_3 a_2, a_3 a_1\}; \quad \text{when } B_2 \cong K_2$$
$$E(G_i) = \begin{cases} E(G_{i-1}) \cup E(B_i) \cup \{a_i b_{i-1}\} & \text{when } B_{i-1} \cong K_3, 3 \leq i \leq m; \\ E(G_{i-1}) \cup E(B_i) \cup \{a_i a_{i-1}, a_i b_{i-2}\} & \text{when } B_{i-1} \cong K_2, 4 \leq i \leq m. \end{cases}$$

An example of a basic chain constructed by this algorithm, with $B_1 \cong B_2 \cong K_2$, is illustrated in Figure 2(i), where strong lines represent the edges of basic subgraphs and weak lines represent the edges linking them. It will be seen that this gives an essentially linear method of linking the basic subgraphs. Other ways of linking them to form a basic chain are possible, if a sufficient proportion of them are isomorphic to K_3. Examples are shown in Figure 2(ii) and 2(iii).



(i)

(ii)

(iii)

Figure 2

4. The Characterization

We shall assume throughout this section that G is well-covered.

We first investigate the structure of G when G contains no basic subgraph. Thus, in particular, $\beta\langle N(v)\rangle = 2$, for all $v \in V(G)$. Our strategy is to consider the possible structure of the subgraph $G-N[v]$, and we start, in the following lemma, by showing that $G-N[v]$ is connected, for all $v \in V(G)$.

Lemma 5. *Suppose $\beta(G) \geq 2$, and G contains no basic subgraph. Then for all $v \in V(G)$, the subgraph $H = G - N[v]$ is connected.*

Proof: Suppose for some $v \in V(G)$, the subgraph H is disconnected. By hypothesis, v is not a hidden vertex of a basic subgraph and hence, by Lemma 2, $\langle N(v)\rangle$ has two components, A and B, say. Then we can find $a \in V(A)$, $b \in V(B)$ such that the neighbour(s) of a in H and the neighbour(s) of b in H are in different components of H. Let x_1, y_1 (where possibly $x_1 = y_1$) be the neighbours of a in H and let x_2, y_2 (where possibly $x_2 = y_2$) be the neighbours of b in H. Denote the (distinct) components of H containing x_i and y_i by H_i, $i = 1, 2$. Note that when $x_i \neq y_i$ then x_i and y_i are adjacent, since G is claw-free.

Since G contains no basic subgraph, each of x_1, y_1, x_2, y_2 has a neighbour in H, which is not a vertex of either of the induced subgraphs $\langle a, x_1, y_1\rangle$ or $\langle b, x_2, y_2\rangle$. Let x_i' (y_i') be such a neighbour of x_i (y_i), in H_i, where $x_i' = y_i'$ if $x_i = y_i$, $i = 1, 2$. Then, when $x_i \neq y_i$, we note that x_i' and y_i' are not adjacent, since G contains no 4-cycle. The restriction on 4-cycles also implies that when $x_i \neq y_i$, then $x_i' \neq y_i'$. Denote by S the subset of *distinct* vertices from among x_1', y_1', x_2', y_2'. Then S is independent and can be extended to a maximal independent set J of H. We note that J contains no vertex adjacent to either a or b. But this contradicts Corollary 3.1, proving that $G - N[v]$ is connected. ∎

Lemma 6. *Suppose that $\beta(G) \geq 2$ and G contains no basic subgraph. Suppose further that for some $v \in V(G)$, $H = G - N[v]$ contains a basic subgraph. Then*

(a) *each hidden vertex in H is adjacent to exactly one vertex of $N(v)$; further, no vertex in $N(v)$ is adjacent to more than one hidden vertex in H;*

(b) *$\delta(G) = 2$, when H contains a basic K_2, and $\delta(G) \leq 3$, when H contains a basic K_3.*

Proof: (a) Since G contains no basic subgraph, every hidden vertex in H is adjacent to at least one vertex of $N(v)$. However, no vertex of H is adjacent to more that one vertex of $N(v)$, since then G would contain a 4-cycle through v.

Now let $a \in N(v)$ and suppose a is adjacent to vertices $x, y \in V(H)$ such that x, y are hidden vertices of basic subgraphs of H. Then since G is claw-free, x, y are adjacent and hence, since they are both hidden vertices, they are vertices of the same basic subgraph B, say, of H. Clearly $B \ncong K_3$, since otherwise G would contain a 4-cycle through a and the vertices of B. So the only possibility is that $B \cong K_2$ and hence $B = \langle x, y\rangle$. However, this is only possible if $B \cong H$. But then $\langle a, x, y\rangle$ is a basic subgraph of G, contrary to hypothesis. Thus each vertex of $N(v)$ is adjacent to at most one hidden vertex of H.

(b) This follows immediately, from the first assertion in (a). ∎

Lemma 7. *Suppose $\beta(G) \geq 2$ and G contains no basic subgraph. Let v be a vertex of minimum degree $\delta(G)$ and suppose that $H = G - N[v]$ is isomorphic to a basic chain. Then either $\beta(G) = 2$, $H \cong K_2$ and $G \cong C_5$; or $\beta(G) = 3$, H is a chain of two basic K_2's and $G \cong C_7$.*

Proof: It follows from Lemma 6, that two cases can arise, according to whether $\delta(G) = 2$ or $\delta(G) = 3$.

Case (a) Suppose $\delta(G) = 2$ (so that $\langle N[v] \rangle$ is given by Figure 1(i)); then by Lemma 6(a), H contains exactly two hidden vertices, each adjacent to a distinct vertex of $N(v)$. Since each basic subgraph in the chain contains a hidden vertex, H contains at most two basic subgraphs. Now, if H contains a basic K_3, then two vertices of this K_3 must be hidden, by the 4-cycle restriction. But this would imply that H contains at least three hidden vertices in all, so this possibility does not arise. Thus either $H \cong K_2$ and $G \cong C_5$ or H is a chain of two basic K_2's and $G \cong C_7$. It is easily seen that both C_5 and C_7 are well-covered, so both these possibilities arise. We note that $\beta(C_5) = 2$ and $\beta(C_7) = 3$.

Case (b) Suppose $\delta(G) = 3$ (so that $\langle N[v] \rangle$ is given by Figure 1(ii)). By Lemma 6(b), every basic subgraph in the chain H is a basic K_3. Further, since $|N(v)| = 3$, it follows from Lemma 6(a) that H contains exactly three hidden vertices, each adjacent to a distinct vertex of $N(v)$. Denote the vertex of degree 1 in $N[v]$ by b. Then since $\delta(G) = 3$, b has two neighbours, say u, w, in H, where we may take u to be the (unique) hidden vertex adjacent to b. Then since G is claw-free, the vertices u, w are adjacent. But this implies that w is in the same basic subgraph as u (since u is a hidden vertex). However, this is impossible, since otherwise G would contain a 4-cycle through b and the three vertices of this basic subgraph. Hence this case does not arise. ∎

Lemma 8. *Suppose $\beta(G) \geq 2$ and G contains no basic subgraph. Let v be a vertex of minimum degree $\delta(G)$. Then $H = G - N[v]$ is not isomorphic to any of K_1, C_5 or C_7.*

Proof: Suppose first that $H \cong K_1$ and let $V(H) = \{w\}$. Then since $\deg_G w \geq \deg_G v \geq 2$, w is adjacent to at least two vertices of $N(v)$. But this is impossible, since then G would contain a 4-cycle through v and w. Thus $H \ncong K_1$.

Next suppose that H is isomorphic to one of C_5 or C_7 and let a, x be two adjacent vertices with $a \in N(v)$ and $x \in V(H)$. Now x has two neighbours in H and hence one of these, y say, must also be adjacent to a, since G is claw-free. Thus the vertices of H adjacent to a vertex of $N(v)$ occur in pairs; hence (since in either case, $V(H)$ has odd size), there is at least one vertex of H of degree 2 in G. Thus $\delta(G) = 2$ and we may assume that $\langle N[v] \rangle$ is given by Figure 1(i), with $N(v) = \{a, b\}$, where a, b are independent vertices.

Now suppose that H is the 5-cycle $xyzws$, where $x, y \in N(a)$, $z, w \in N(b)$. But then $\{v, x, w\}$ and $\{a, w\}$ are both maximal independent sets in G, which is impossible, since G is well-covered. Thus $H \ncong C_5$.

Lastly, suppose that H is the 7-cycle $xyzwstu$, where $x, y \in N(a)$. Just two possibilities arise according to whether $N(b) \cap H = \{z, w\}$ or $\{w, s\}$. But in either case, G contains the maximal independent sets $\{v, x, w, t\}$ and $\{a, w, t\}$, which is again impossible, since G is well-covered. Thus $H \not\cong C_7$, completing the proof of the lemma. ∎

In the next two lemmas, we consider the structure of G in the case when G contains a basic subgraph B.

Lemma 9. *Suppose $\beta(G) \geq 2$ and G contains a basic subgraph B. Let H be a connected component of $G - V(B)$. Then H is not isomorphic to any of K_1, C_5 or C_7.*

Proof: Let v denote a hidden vertex of B, so that $V(B) = N[v]$. We shall adopt the notation that $V(B) = \{v, a\}$ when $B \cong K_2$ and $V(B) = \{v, a, b\}$ when $B \cong K_3$. Let x denote a vertex of H adjacent to the vertex $a \in B$. We note that a is the only vertex of B adjacent to x, by the 4-cycle restriction.

Suppose first $H \cong K_1$ so that $V(H) = \{x\}$. We extend $\{a\}$ to a maximal independent set J in G and note that J contains no neighbour of either x or v. Then $(J - \{a\}) \cup \{v, x\}$ is also a maximal independent set in G of size $|J| + 1$. But this is impossible, since G is well-covered. Hence $H \not\cong K_1$.

Now suppose that H is isomorphic to C_5 or C_7. In either case, x has two independent neighbours in H and hence one of these, say y, must also be adjacent to a, since G is claw-free; further, x, y are the only vertices of H adjacent to a, by Lemma 2. When $H \cong C_5$, then $H - N[a]$ is a path zws, say. Then both $\{a, w\}$ and $\{a, z, s\}$ are maximal independent sets in the induced subgraph $G' = \langle V(B) \cup V(H) \rangle$. Let $J \cup \{a, w\}$ be a maximal independent set in G. Then $J \cup \{a, z, s\}$ is also a maximal independent set in G, but of different cardinality. However, this is impossible, since G is well-covered. Thus $H \not\cong C_5$. A similar argument shows that $H \not\cong C_7$. ∎

Lemma 10. *Suppose $\beta(G) \geq 2$ and G contains a basic subgraph B such that each component of $G - V(B)$ is a basic chain. Then G is also a basic chain.*

Proof: Let v denote a hidden vertex of B, so that $V(B) = N[v]$. We shall adopt the notation that $V(B) = \{v, a\}$ when $B \cong K_2$ and $V(B) = \{v, a, b\}$ when $B \cong K_3$. Let $H = G - V(B)$. Then by Corollary 3.2, $\beta(H) = B(G) - 1 \geq 1$. Since H is a basic chain, there exists a partition $\{V_1, V_2, \ldots, V_m\}$ of $V(H)$ such that $\langle V_i \rangle$ is a basic subgraph of H, $i = 1, 2, \ldots, m$. Then $\beta(H) = m$ and hence $\beta(G) = m + 1$. We show first that $\{V_1, V_2, \ldots, V_m, V(B)\}$ is a partition of $V(G)$, such that $\langle V_1 \rangle, \langle V_2 \rangle, \ldots, \langle V_m \rangle, B$ are basic subgraphs of G. Thus, we need to show that $\langle V_i \rangle$ has a hidden vertex in G, $i = 1, 2, \ldots, m$.

Suppose that $\langle V_1 \rangle$ has no hidden vertex. Then since $\langle V_1 \rangle$ is a basic subgraph of H, we may assume that there is an edge between V_1 and $V(B)$, say $x \in V_1$ is adjacent to $a \in V(B)$.

Suppose first that $\langle V_1 \rangle \cong K_2$ and $V_1 = \{x, y\}$. Then the vertex y is not also adjacent to a vertex of B. For, by the 4-cycle restriction, this would be possible only if y were adjacent to a. In this case, we extend $\{a\}$ to a maximal independent set J in G. Then a is the only vertex of $V(B) \cup V_1$ in J. Observing that any maximal independent set in G contains at most one vertex from each of $V_2, V_3, \ldots, V_m$, we have $|J| \leq m$, contradicting the supposition that G is well-covered. Thus y is adjacent to a vertex of $V_2 \cup V_3 \cup \cdots \cup V_m$. Now suppose $\langle V_1 \rangle \cong K_3$ and $V_1 = \{x, y, z\}$. Then, by the 4-cycle restriction, xa is again the only edge between V_1 and $V(B)$. Thus, in either case, y has a neighbour $y' \in V_2$, say. Also, when $\langle V_1 \rangle \cong K_3$, z has a neighbour $z' \in V_3$, say, where V_1, V_2, V_3 are distinct subsets in the partition of $V(H)$, by condition (a). Further, the vertices y' and z' are independent, since otherwise $yy'z'z$ would be a 4-cycle in G. Then the set $\{a, y'\}$ when $\langle V_1 \rangle \cong K_2$, or $\{a, y', z'\}$ when $\langle V_1 \rangle \cong K_3$, can be extended to give a maximal independent set of size at most m in G, which is impossible, since G is well-covered. Thus $\langle V_1 \rangle, \langle V_2 \rangle, \ldots, \langle V_m \rangle, B$ are basic subgraphs of G.

Finally, we note that as G is claw-free and contains no 4-cycle, then conditions (a), (b), (c) of Theorem 4 hold. Then, since by assumption G is connected, we conclude that G is a basic chain. ∎

Theorem 11. *Let G be a connected well-covered claw-free graph containing no 4-cycle. Then either G is a basic chain or G is isomorphic to one of the graphs K_1, C_5 or C_7.*

Proof: We have established in Theorem 4 and Lemmas 1 and 7 that all these cases can arise. It remains to show that these are the only possibilities. We shall prove this by induction on $\beta(G)$, making repeated use of Lemma 3 and Corollary 3.2.

We introduce some notation. In the case where G contains a basic subgraph B, let $V(B) = \{v, a\}$ when $B \cong K_2$ and $V(B) = \{v, a, b\}$ when $B \cong K_3$, where v denotes a hidden vertex of B. We denote the subgraph $G - V(B) = G - N[v]$ by H. In the case where G contains no basic subgraph, we let v denote a vertex of G of minimum degree $\delta(G)$ and, as above, we denote the subgraph $G - N[v]$ by H.

We first establish the theorem in the cases when $\beta(G) \leq 3$.

Case (a) $\beta(G) = 1$. The result was established in Lemma 1.

Case (b) $\beta(G) = 2$. By Lemma 3, $\beta(H) = 1$ and H satisfies the conditions of the theorem. Hence, by Lemma 1, H is isomorphic to one of K_1, K_2 or K_3. Suppose first that G contains no basic subgraph. Then by Lemma 8, $H \ncong K_1$ and so by Lemma 7, $H \cong K_2$ and $G \cong C_5$. Now suppose that G contains a basic subgraph B. Then by Lemma 9, $H \ncong K_1$. Hence by Lemma 10, G is a basic chain.

Case (c) $\beta(G) = 3$. Then by Lemma 3, $\beta(H) = 2$ and H is well-covered. Clearly, H is claw-free and contains no 4-cycle. Suppose first that G contains no basic subgraph. Then by Lemma 5, H is connected and hence, by case (b) above, H is a basic chain or $H \cong C_5$. The latter alternative is impossible, by Lemma 8.

Hence $G \cong C_7$, by Lemma 7. Now suppose that G contains a basic subgraph B. Then by Corollary 3.2, H is well-covered with $\beta(H) = 2$. By Lemma 9, no component of H is isomorphic to K_1 or C_5. Thus each component of H is a basic chain and hence, by Lemma 10, G is also a basic chain.

This establishes the truth of the theorem when $\beta(G) \leq 3$. Now suppose that the theorem is true for all graphs G such that $1 \leq \beta(G) < m$; let G be a graph satisfying the conditions of the theorem, for which $\beta(G) = m, m \geq 4$.

Assume first that G contains no basic subgraph. Then by Lemmas 3 and 5, H also satisfies the conditions of the theorem with $\beta(H) = m - 1$. Hence either H is a basic chain or $m = 4$ and $H \cong C_7$, by case (c) above. The first possibility is ruled out by Lemma 7 and the second by Lemma 8. Thus we may assume that G contains a basic subgraph B. Then by Corollary 3.2, H is well-covered with $\beta(H) \geq 3$. By Lemma 9, no component of H is isomorphic to one of K_1, C_5 or C_7. Hence, by cases (a) to (c) above and the induction hypothesis, each component of H is a basic chain. But then G is also a basic chain, by Lemma 10, and this is therefore the only possibility for G when $m \geq 4$. This completes the proof of the theorem. ∎

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