Extreme Values of the Edge-Neighbor-Connectivity

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Abstract. The edge-neighbor-connectivity of a graph G is the minimum size of all edge-cut-strategies of G, where an edge-cut-strategy consists of a set of common edges of double stars whose removal disconnects the graph G or leaves a single vertex or $\emptyset$. This paper discusses the extreme values of the edge-neighbor-connectivity of graphs relative to the connectivity, κ, and gives two classes of graphs — one class with minimum edge-neighbor-connectivity, and the other one with maximum edge-neighbor-connectivity.

I. Introduction

Gunther and Hartnell [1] [2] [3] introduced the idea of modeling a spy network by a graph whose vertices represent the stations and whose edges represent lines of communication. If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to network as a whole. Therefore, instead of considering the connectivity of a communication graph, in [5] we discussed the neighbor-connectivity‡ of a communication graph (removing some vertices and all of their adjacent vertices). Similarly, we can consider the edge analogue of (vertex) neighbor-connectivity: remove some edges, their incident nodes, and all of their incident edges.

Suppose that G=(V,E) is a graph. Let e be any edge in G. $N(e) = \{f \in E(G) \mid f \neq e, e \text{ and } f \text{ are adjacent}\}$ is the open edge-neighborhood of

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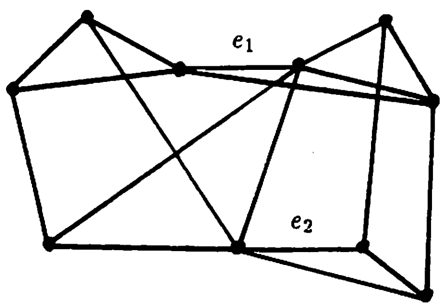
‡ A vertex set S is called a *vertex-cut-strategy* of a graph G if the removal of the closed neighborhood of S disconnects the graph G or leaves a clique or $\emptyset$. The *neighbor-connectivity* of G is the minimum size of all vertex-cut-strategies of G.

e, and $N[e] = N(e) \cup \{e\}$ is the closed edge-neighborhood of e. The **double star** with a common edge e is the closed neighborhood of e and the two vertices incident with e, denoted by DS(e). An edge e in G is said to be **subverted** when the DS(e) is deleted from G. In other words, if $e = [a, b]$, $G - DS(e) = G - \{a, b\}$. A set of edges $S=\{e_1, e_2, e_3, ..., e_m\}$ is called a **subversion strategy** if each of the edges in S has been subverted. Let G/S be the survival-subgraph left after each edge of S has been subverted from G. A subversion strategy S is called an **edge-cut-strategy** of G if the survival-subgraph G/S is disconnected, or is a single vertex, or is $\emptyset$. The **edge-neighbor-connectivity**, $\Lambda(G)$, of G is the minimum size of all edge-cut-strategies S of G.

Example 1:

G:



$\{e_1, e_2\}$ is a minimum edge-cut-strategy of G, hence $\Lambda(G) = 2$.

Figure 1

Note that the edge-neighbor-connectivity of a graph G is not always equal to the neighbor-connectivity of the line graph of G. Two examples are given here, and demonstrate that they are equal in some cases, but unequal in other cases.

Example 2: Let G be a star with n edges, $(n \geq 1)$, so $\Lambda(G) = 1$. Then the line graph of G, L(G), is the complete graph with n vertices, so the neighbor-connectivity of L(G) is 0. Therefore the edge-neighbor-connectivity of G $\neq$ the neighbor-connectivity of the line graph of G.

Example 3: Let G be a double star, as shown in Figure 2. Thus the line graph of G, L(G), is shown in Figure 3.

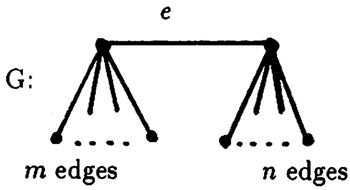


Figure 2

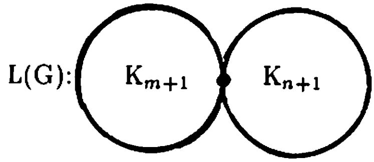


Figure 3

The edge-neighbor-connectivity of G = The neighbor-connectivity of L(G) = 1.

In this paper we give extreme values of the edge-neighbor-connectivity of graphs with the connectivity, κ. Then the relationship between the edge-neighbor-connectivity, Λ, and the edge-connectivity, λ, will be easily obtained, as shown later. Furthermore, for any fixed integers, m and n, we give two classes of graphs with order m, and connectivity n, where one class of the graphs has minimum edge-neighbor-connectivity, and the other has maximum edge-neighbor-connectivity.

II. The Upper and Lower Bounds of the Edge-Neighbor-Connectivity

Let G be a graph and $\kappa(G)$ be the connectivity of the graph G. $\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Lemma 2.1. If a graph G has a cut vertex then $\Lambda(G) = \kappa(G) = 1$.

Proof: Let v be a cut vertex of G. Then G$-v$ contains at least two components. If each of the components contains a single vertex, then G is a star. Hence any one edge forms an edge-cut-strategy of G. If at least one of the components, G_1, G_2, ..., G_r ($r \geq 2$), contains at least two vertices, without loss of generality, we may assume that G_1 contains at least two vertices. Now the subversion of any one edge connecting v and a vertex in G_1 disconnects the graph G. Therefore $\Lambda(G) = \kappa(G) = 1$. QED.

Lemma 2.2. Let G be a graph, and $T = \{e_1, e_2, ..., e_t\}$ be an edge set in G. Then $\Lambda(G) \leq \Lambda(G/T) + t$.

Proof: Let $T = \{e_1, e_2, ..., e_t\}$ be any edge set in G, and let $\Lambda(G/T) = n$.

Let edge set $T_1 = \{f_1, f_2, ..., f_n\}$ be a minimum edge-cut-strategy of G/T. Hence, $T \cup T_1$ is an edge-cut-strategy of G, and $\Lambda(G) \leq |T_1 \cup T| = |T_1|+|T|$ $= n + t = \Lambda\ (G/T) + t$. QED.

Lemma 2.3. Let G be a graph. If M is a maximum matching in G, then $\Lambda(G) \leq |M|$.

Proof: We show that M is an edge-cut-strategy of G. If it is not, then there exist at least two vertices v_1, v_2 in G/M, and there is a v_1 — v_2 path in G/M. Hence there is a matching M′ in G with $|M'| > |M|$, a contradiction. Therefore M is an edge-cut-strategy of G, and $\Lambda(G) \leq |M|$. QED.

Since the number of the edges in a maximum matching in a graph G is less than or equal to $\lfloor \frac{|V(G)|}{2} \rfloor$, it follows that:

Lemma 2.4. For any graph G = (V, E), $\Lambda(G) \leq \lfloor \frac{|V|}{2} \rfloor$.

The following result is central to this paper as it provides bounds for $\Lambda(G)$.

Theorem 2.5. $\lceil \frac{\kappa}{2} \rceil \leq \Lambda \leq \kappa$

Proof: First, we show that $\lceil \frac{\kappa}{2} \rceil \leq \Lambda$.

Let $\Lambda(G) = n$, and the edge set $T = \{[u_1, v_1], [u_2, v_2], ..., [u_n, v_n]\}$ be a minimum edge-cut-strategy of G. Thus if the survival-subgraph G/T is a single vertex, or is $\emptyset$, then $|V(G)| \leq 2n + 1$ or $2n$. Hence $\kappa(G) \leq 2n$ or $2n - 1$. If the survival-subgraph G/T is disconnected, then the vertex set $S = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ is a vertex cutset of G, and hence $\kappa(G) \leq |S| \leq 2n$.

Assume that $\Lambda(G) < \lceil \frac{\kappa(G)}{2} \rceil$, hence either

(1) if $\kappa(G) = 2r$ is even, then $\Lambda(G) < \frac{\kappa(G)}{2} = r$, and $\kappa(G) > 2\Lambda(G) = 2n$, a contradiction;

or

(2) if $\kappa(G) = 2r + 1$ is odd, then $\Lambda(G) < r + 1$, and $2r \geq 2\Lambda(G) = 2n$, hence $\kappa(G) = 2r + 1 > 2n$, a contradiction.

Therefore we have $\lceil \frac{\kappa}{2} \rceil \leq \Lambda$.

Next we prove that $\Lambda \leq \kappa$ by induction on κ. The result is true if $\kappa = 0$, since then G must be either trivial or disconnected. Suppose that it holds for all graphs with connectivity less than m, and let G be a graph with $\kappa(G) = m > 0$. If $m = 1$ then by Lemma 2.1, $\Lambda(G) = \kappa(G) = 1$.

Hence we merely need to show the result for the case of $m > 1$.

If $|V(G)| = m+1$ and $\kappa(G) = m$, then $G = K_{m+1}$, a complete graph. A matching with size $\lfloor \frac{m+1}{2} \rfloor$ in G is a minimum edge-cut-strategy of G, hence $\Lambda(G) = \lfloor \frac{m+1}{2} \rfloor \leq m = \kappa(G)$.

If $|V(G)| > m+1$ and $\kappa(G) = m$, then let $S = \{v_1, v_2, ..., v_m\}$ be a minimum vertex cutset of G and $G - S$ is disconnected. Now we consider the induced subgraph <S>:

Case (1) There is an edge in <S>, say $e = [v_i, v_j], i \neq j$. The subversion of e produces a subgraph $G'=G/\{e\}$ with $\kappa(G') \leq m-2$, since $S-\{v_i, v_j\}$ is a vertex cutset of G'. By induction on κ, $\Lambda(G') \leq \kappa(G') \leq m-2$. Hence, by Lemma 2.2, $\Lambda(G) \leq \Lambda(G') + 1 \leq (m-2) + 1 = m - 1 < \kappa(G) = m$.

Case (2) There is no edge in <S> (i.e. <S> is a subgraph of m isolated vertices). Assume that G−S contains t components, $G_1, G_2, ..., G_t$ $(t \geq 2)$. $|V(G_1)|+|V(G_2)|+...+|V(G_t)| \geq m$, otherwise $V(G_1) \cup V(G_2) \cup ... \cup V(G_t)$ is a vertex cutset of G with the size smaller than S. Each vertex in S is joined to each component of G−S, otherwise a proper subset of S is a vertex cutset of G. If the number of the components of G−S, t, is greater than or equal to 3, then the subversion of m edges, each of which has one different end in S and the other end in G_1 (these m edges may have the same ends in G_1), disconnects the graph G. Hence, the remainder of the proof considers the case where G−S contains only two components, G_1 and G_2. Again we emphasize that $|V(G_1)| + |V(G_2)| \geq m$ and each vertex in S is joined to some vertices in G_1 and some vertices in G_2.

Let $V(G_1) = \{u_1, u_2, ..., u_p\}$, $V(G_2) = \{w_1, w_2, ..., w_q\}$, and $S = \{v_1, v_2, ..., v_m\}$. If $p+q = m$ or $m+1$, then $|V(G)| = 2m$ or $2m+1$, and by Lemma 2.4, $\Lambda(G) \leq \lfloor \frac{|V(G)|}{2} \rfloor = m$. If $p+q \geq m+2$, then we consider two possibilities:

(i) $p < m+1$ (and therefore $q \geq 2$) —— the m edges of the subversion strategy are chosen in the following way:

(a) each of v_1, v_2, v_3, ..., v_{p-1}'s is an end of one of the chosen $p-1$ edges, and each of these $p-1$ edges has the other end in G_1,
and

(b) each of the remaining v_i's (i.e. v_p, v_{p+1}, ..., v_m's) is an end of one of the remaining $m-(p-1)$ edges, and each of these $m-(p-1)$ edges has the other end in G_2.

(ii) $p \geq m+1$ —— the m edges of the subversion strategy are chosen as follows:

Each of v_1, v_2, v_3, ..., v_m's is an end of one of the chosen m edges, and each of these m edges has the other end in G_1.

The subversion of the m edges in above two possibilities disconnects the graph G, since the subversion of these m edges removes all vertices in S, but leaves some vertices in G_1, and some vertices in G_2. Thus $\Lambda(G) \leq m = \kappa(G)$.

Therefore $\lceil \frac{\kappa}{2} \rceil \leq \Lambda \leq \kappa$. QED.

Corollary 2.6. $\Lambda \leq \lambda$.

Proof: Since $\kappa(G) \leq \lambda(G)$, for any graph G, and by Theorem 2.5, it is followed that $\Lambda(G) \leq \lambda(G)$, for any graph G. QED.

Let G be a graph, S be a minimum vertex cutset of G, and <S> be the induced subgraph of S in G, then we have the following corollaries:

Corollary 2.7. If $<S>$ contains the components $<S_1>, <S_2>, ..., <S_t>$, with $| V(<S_i>) | \geq 2$, for all $i = 1, 2, ..., t$, then $\Lambda\ (G) \leq \kappa(G) - t$.

Proof: $<S_1>, <S_2>, ..., <S_t>$ are components in $<S>$, and $|V(<S_i>)| \geq 2$, for all i, so there is at least one edge, $e_i = [u_i, v_i]$, in each component $<S_i>$, $i = 1, 2, ..., t$. The subversion of $\{e_1, e_2, ..., e_t\}$ produces a subgraph G', and $S' = S - \{u_1, u_2, ..., u_t, v_1, v_2, ..., v_t\}$ is a vertex cutset of G'. Hence $\kappa\ (G') \leq |S'| = \kappa\ (G) - 2t$. By Theorem 2.5,
$\Lambda\ (G') \leq \kappa\ (G') \leq \kappa\ (G) - 2t$. Therefore by Lemma 2.2,
$\Lambda\ (G) \leq \Lambda\ (G') + t \leq (\kappa\ (G) - 2t) + t = \kappa\ (G) - t$. QED.

Corollary 2.8. Let the edge set $M = \{e_1, e_2, ..., e_m\}$ be a maximum matching in $<S>$, then $\Lambda\ (G) \leq \kappa(G) - m$.

Proof: Let $G' = G/M$, $e_i = [u_i, v_i]$, where u_i, v_i are in S, $i = 1, 2, ..., m$. Then $S - \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_m\}$ is a vertex cutset of G'. Hence $\kappa\ (G') \leq \kappa\ (G) - 2m$. By Lemma 2.2 and Theorem 2.5,
$\Lambda\ (G) \leq \Lambda\ (G') + m \leq \kappa\ (G') + m \leq (\kappa\ (G) - 2m) + m = \kappa\ (G) - m$. QED.

Corollary 2.9. If $<S>$ has a maximum matching with the size $\lfloor \frac{|S|}{2} \rfloor$, then $\Lambda(G) = \lceil \frac{\kappa(G)}{2} \rceil$.

Proof: By Theorem 2.5, $\lceil \frac{\kappa(G)}{2} \rceil \leq \Lambda(G) \leq \kappa(G)$; by Corollary 2.8, $\Lambda(G) \leq \kappa(G) - \lfloor \frac{|S|}{2} \rfloor$. It follows that $\lceil \frac{\kappa(G)}{2} \rceil \leq \Lambda(G) \leq \lceil \frac{\kappa(G)}{2} \rceil$. Therefore $\Lambda(G) = \lceil \frac{\kappa(G)}{2} \rceil$. QED.

Corollary 2.10. If $|S|$ is even, and $<S>$ has a perfect matching, then

$\Lambda(G) = \frac{\kappa(G)}{2}$. Conversely, if $\Lambda(G) = \frac{\kappa(G)}{2}$ then either G= K_{2n+1}, a complete graph with order $2n+1$, or there is a minimum vertex cutset, S, of G, and the induced subgraph, <S>, has a perfect matching.

Proof: A perfect matching in <S> is a maximum matching in <S>, and the size of a perfect matching is $\frac{|S|}{2} = \frac{\kappa(G)}{2}$. Hence by Corollary 2.9 we obtain
$\Lambda(G) = \frac{\kappa(G)}{2}$.

Conversely, assume that $T = \{[u_1, v_1], [u_2, v_2], ..., [u_n, v_n]\}$ is a minimum edge-cut-strategy of G, then G/T is $\emptyset$, trivial, or disconnected. Since $\kappa(G) = 2\Lambda(G) = 2n$, the order of G must be greater than or equal to $2n+1$. Thus G/T is either trivial or disconnected. If G/T is trivial, then $G = K_{2n+1}$. If G/T is disconnected, then the vertex set $S = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ is a minimum vertex cutset, and <S> has a perfect matching T. QED.

The following result shows when the edge-neighbor-connectivity of the graph G reaches the maximum value.

Corollary 2.11. If $\Lambda(G) = \kappa(G) = m$, then for any minimum vertex cutset S of G, <S> must be a subgraph of m isolated vertices.

Proof: Let $S = \{v_1, v_2, ..., v_m\}$ be a minimum vertex cutset of G. If there is an edge $e = [v_i, v_j] (i \neq j)$ in <S>, then the subversion of e produces a subgraph G′ with $\kappa(G') \leq m-2$. Hence $\Lambda(G') \leq m-2$. But by Lemma 2.2, $\Lambda(G) \leq \Lambda(G') + 1 \leq (m-2) + 1 = m-1$, a contradiction. Therefore <S> must be a subgraph of m isolated vertices. QED.

The converse of the above theorem is not true, as shown by the following example:

Example 4:

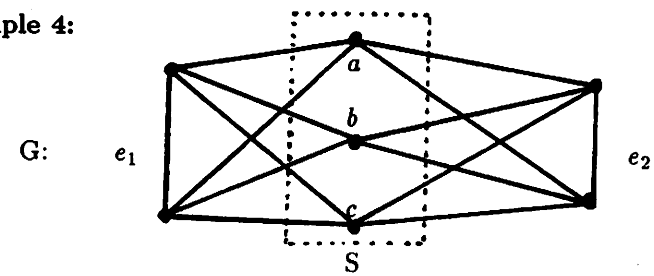


Figure 4

$\kappa(G) = 3$. <S> is a subgraph of 3 isolated vertices, $\{a, b, c\}$
$\{e_1, e_2\}$ is the minimum edge-cut-strategy, so $\Lambda(G) = 2$.

The following is a simple example for some results of this section.

Example 5:

G:

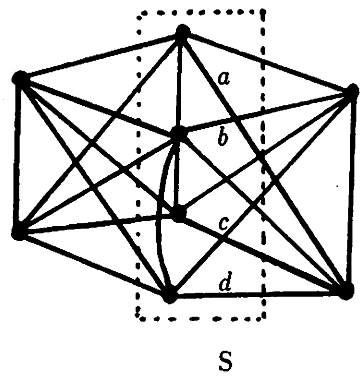


Figure 5

$S = \{a, b, c, d\}$ is the minimum vertex cutset, so $\kappa(G) = 4$. By Theorem 2.5, $2 = \lceil \frac{\kappa(G)}{2} \rceil \leq \Lambda(G) \leq \kappa(G) = 4$. Since <S> has no perfect matching, by Corollary 2.10, $\Lambda(G) \neq \frac{\kappa(G)}{2} = 2$. Since <S> is not a subgraph of 4 isolated vertices, by Corollary 2.11, $\Lambda(G) \neq \kappa(G) = 4$. Therefore $\Lambda(G) = 3$.

III. Graphs with the Minimum and Maximum Edge-Neighbor-Connectivity

For any fixed integers, n and m, $m \geq n + 1$, we give two classes of graphs with order m, connectivity n, and where one class of graphs has minimum edge-neighbor-connectivity and the other has maximum edge-neighbor-connectivity.

The Harary graph, $H_{n,m}$, is constructed as follows:

Case 1. n is even. Let $n = 2r$. Then $H_{2r,m}$ has vertices $0, 1, 2, ..., m-1$ and two vertices i and j are adjacent if $i - r \leq j \leq i + r$ (where addition is taken modulo m).
$H_{4,8}$ is shown in Figure 6.

$H_{4,8}$:

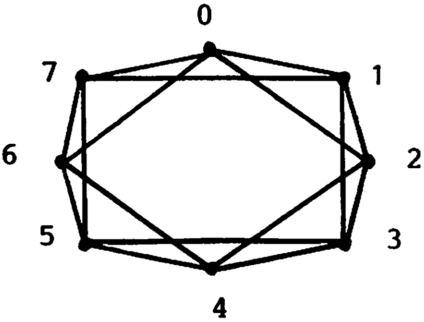


Figure 6

Case 2. n is odd ($n > 1$) and m is even. Let $n = 2r + 1$ ($r > 0$). Then $H_{2r+1,m}$ is constructed by first drawing $H_{2r,m}$, and then adding edges joining vertex i to vertex $i + \frac{m}{2}$ for $1 \leq i \leq \frac{m}{2}$.
$H_{5,8}$ is shown in Figure 7.

$H_{5,8}$:

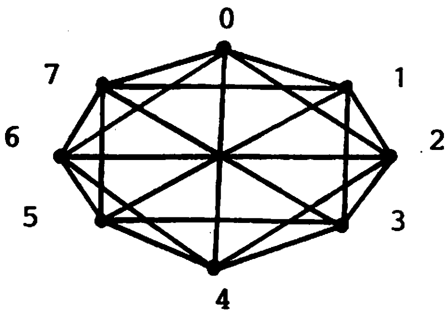


Figure 7

Case 3. n is odd ($n > 1$) and m is odd. Let $n = 2r + 1$ ($r > 0$). Then $H_{2r+1,m}$ is constructed by first drawing $H_{2r,m}$, and then adding edges $[0, \frac{m-1}{2}]$ and $[0, \frac{m+1}{2}]$, and $[i, i + \frac{m+1}{2}]$ for $1 \leq i < \frac{m-1}{2}$.
$H_{5,9}$ is shown in Figure 8.

$H_{5,9}$:

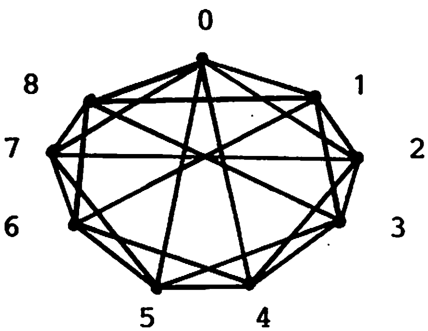


Figure 8

The Harary graph, $H_{n,m}$, is a graph with order m, connectivity n, and with minimum edge-neighbor-connectivity, as shown by the following result:

Theorem 3.1. $\Lambda(H_{n,m}) = \lceil \frac{n}{2} \rceil = \lceil \frac{\kappa(H_{n,m})}{2} \rceil$, for any integers $n, m, n \geq 4$ and $m \geq n+1$.

Proof: Let $H_{n,m}$ have vertices $0, 1, 2, ..., m-1$. Then, as shown in [4], $\kappa(H_{n,m}) = n$.

Case 1. n is even ($n \geq 4$).
Let $n = 2r$. Then the vertex set S = the set of neighbors of vertex $0 = \{1, 2, ..., r, m-1, m-2, m-3, ..., m-r\}$ is a minimum vertex cutset of $H_{n,m}$. By the construction of $H_{n,m}$, there is a perfect matching in <S>. If r is even, then $\{[1,2], [3,4], ..., [r-1, r], [m-1, m-2], ..., [m-(r-1), m-r]\}$ is a perfect matching in <S>. If r is odd, then $\{[r, r-1], [r-2, r-3], ..., [3,2], [1, m-1], [m-2, m-3], ..., [m-(r-1), m-r]\}$ is a perfect matching in <S>. By Corollary 2.10, $\Lambda(H_{n,m}) = \frac{n}{2} = \lceil \frac{\kappa(H_{n,m})}{2} \rceil$.

Case 2. n is odd and m is even ($n \geq 4$ and $m \geq n+1$).
Let $n = 2r+1$. Then the vertex set S = the set of neighbors of vertex $0 = \{1, 2, 3, ..., r, m-1, m-2, ..., m-r, \frac{m}{2}\}$ is a minimum vertex cutset of $H_{n,m}$. By the construction of $H_{n,m}$, the number of edges in a maximum matching M in <S> is $r = \lfloor \frac{\kappa(H_{n,m})}{2} \rfloor$. By Corollary 2.9, $\Lambda(H_{n,m}) = \lceil \frac{\kappa(H_{n,m})}{2} \rceil = \lceil \frac{n}{2} \rceil$.

Case 3. n is odd and m is odd ($n \geq 4$ and $m \geq n+1$).
Let $n = 2r+1$. Then the vertex set S = the set of neighbors of vertex $1 = \{2, 3, 4, ..., r+1, 0, m-1, m-2, ..., m-(r-1), \frac{m+3}{2}\}$ is a minimum vertex cutset of $H_{n,m}$. By the construction of $H_{n,m}$, the number of edges

in a maximum matching M in <S> is $r = \lfloor\frac{\kappa(H_{n,m})}{2}\rfloor$. Hence by Corollary 2.9, we have $\Lambda(H_{n,m}) = \lceil\frac{\kappa(H_{n,m})}{2}\rceil = \lceil\frac{n}{2}\rceil$. QED.

The cases of $n = 2$ and $n = 3$ about Theorem 3.1 are discussed in the appendix.

We have shown a class of graphs with minimum edge-neighbor-connectivity. If the edge-neighbor-connectivity of the graph G reaches the maximum value, κ(G), then is there a bound on the maximum value of the connectivity? Lemma 2.4 trivially answers this question:

Theorem 3.2. Let G be a graph with the order m. If $\Lambda(G) = \kappa(G)$, then $\kappa(G) \leq \lfloor\frac{m}{2}\rfloor$.

Proof: $\Lambda(G) \leq \lfloor\frac{m}{2}\rfloor$ by Lemma 2.4, so if $\kappa(G) = \Lambda(G)$, then $\kappa(G) \leq \lfloor\frac{m}{2}\rfloor$. QED.

For any fixed integers r, t, the complete bipartite graph $K_{r,t}$ is a simple bipartite graph with bipartition (X,Y) in which each vertex of X is joined to each vertex of Y, and $|X| = r, |Y| = t$. It is clear that if $r \leq t$ then $\kappa(K_{r,t}) = r$ and $\Lambda(K_{r,t}) = r$. Hence, for any fixed integers n, m, $n \leq \lfloor\frac{m}{2}\rfloor$, there exists a graph with order m, connectivity n, and maximum edge-neighbor-connectivity, as described in the following theorem.

Theorem 3.3. For any fixed integers n, m, $n \leq \lfloor\frac{m}{2}\rfloor$, there exists a graph with order m, connectivity n, and maximum edge-neighbor-connectivity.

Proof: For any fixed integers $n, m, n \leq \lfloor\frac{m}{2}\rfloor$, construct a complete bipartite graph $K_{n,m-n}$ whose order is m. Since $n \leq \lfloor\frac{m}{2}\rfloor$, $n \leq m - n$. Therefore, $\Lambda(K_{n,m-n}) = \kappa(K_{n,m-n}) = n$. That means the edge-neighbor-connectivity, Λ, reaches the maximum value. QED.

Appendix

The Cases of $n = 2$ and $n = 3$ about Theorem 3.1

For the case of $n = 2$: The Harary graph $H_{n,m} = H_{2,m} = C_m$ (m-cycle).

(1) $m = 3$. $\Lambda(H_{2,3}) = \Lambda(C_3) = 1 = \lceil\frac{\kappa(H_{2,3})}{2}\rceil$.

(2) $m \geq 4$. $\Lambda(H_{2,m}) = \Lambda(C_m) = 2 \neq \lceil\frac{\kappa(H_{2,m})}{2}\rceil = 1$.

For the case of $n = 3$:

(1) $m = 4$. $H_{n,m} = H_{3,4} = K_4$. It is clear that $\Lambda(H_{3,4}) = \Lambda(K_4) = 2 = \lceil\frac{\kappa(H_{3,4})}{2}\rceil$.

(2) $m = 6$. Any set of two edges cannot be an edge-cut-strategy of $H_{3,6}$, so $\Lambda(H_{3,6}) > 2$. By Lemma 2.4, $\Lambda(H_{3,6}) \leq \lfloor\frac{|V(H_{3,6})|}{2}\rfloor = 3$. Therefore $\Lambda(H_{3,6}) = 3 \neq \lceil\frac{\kappa(H_{3,6})}{2}\rceil = 2$.

(3) m is even and $m \geq 8$. The edge set $\{[i, i + \frac{m}{2}], [i + 2, i + 2 + \frac{m}{2}]\}$, for $i = 0, 1, 2, ..., \frac{m}{2} - 3$, is an edge-cut-strategy of $H_{3,m}$, so $\Lambda(H_{3,m}) \leq 2$. By Theorem 2.5, $\Lambda(H_{3,m}) \geq \lceil\frac{\kappa(H_{3,m})}{2}\rceil = 2$. Therefore $\Lambda(H_{3,m}) = \lceil\frac{\kappa(H_{3,m})}{2}\rceil$.

(4) m is odd and $m \geq 5$. The edge set $\{[i, i+\frac{m+1}{2}], [i+2, i+2+\frac{m+1}{2}]\}$, for $i = 0, 1, 2, ..., (\frac{m-1}{2}) - 3$, is an edge-cut-strategy of $H_{3,m}$, so $\Lambda(H_{3,m}) \leq 2$. By Theorem 2.5, $\Lambda(H_{3,m}) \geq \lceil\frac{\kappa(H_{3,m})}{2}\rceil = 2$. Therefore $\Lambda(H_{3,m}) = \lceil\frac{\kappa(H_{3,m})}{2}\rceil$.

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