A Composition Theorem for Simple Designs and Difference Families

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Abstract

A method is presented for constructing simple partially balanced designs from t-(v, k, λ) designs. When the component designs satisfy a compatibility condition the result is a simple balanced design. The component designs can even be trivial (with some exceptions) with the resulting design being nontrivial. The automorphism group of the composition is given in terms of the automorphism groups of the component designs. Some previously unknown simple designs are constructed, including an infinite family of 3-designs that are extremal with respect to an inequality of Cameron & Praeger. Some analogous theorems are given for difference families.

Introduction. Given a set $\mathcal{X}$ of size v, a t-(v, k, λ) *design* is a family $\mathcal{D}$ of k-sets of $\mathcal{X}$, called *blocks*, with the property that each set of t points of $\mathcal{X}$ is in exactly λ blocks. A design is *simple* when there are no repeated blocks. Let $b = |\mathcal{D}|$ be the number of blocks. A design is called *trivial* when b and λ are maximum;

$$b_{max} = \binom{v}{k} \text{ and } \lambda_{max} = \binom{v-t}{k-t}.$$

For any given v and k, there is always a t-(v, k, λ_{max}) design consisting of all the sets of size k. When $k \leq t$ or $v - t \leq k$ the only t-designs are the trivial ones. A t-design is an s-design for all $s \leq t$. Let r be the number of blocks containing any single point of $\mathcal{X}$. We need the relations [5, p. 10]:

$$r = \frac{bk}{v} \tag{1}$$

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$$\lambda = b\frac{\binom{k}{t}}{\binom{v}{t}}. \tag{2}$$

If Λ is a set of positive integers, a t-(v,k,Λ) *partially balanced design* $\mathcal{D}$ is a set of blocks of $\mathcal{X}$ where for each set of t points of $\mathcal{X}$ the number of blocks containing them is a number in Λ. We do not need the underlying association scheme structure in the usual definition of a partially balanced design [5, Chap. 11]; though this occurs in all the designs considered here. When $\Lambda = \{\lambda\}$ is a singleton, then $\mathcal{D}$ is called *balanced* and is a usual t-(v,k,λ) design as defined above.

Composition. Starting with $\mathcal{D}_e$, a t-(v_e,k_e,λ_e) design over the set $\mathcal{X}_e$, and $\mathcal{D}_i$, for i=0,1, t-(v_i,k_i,λ_i) designs with $v_0 = v_1$ and both over the set $\mathcal{X}_1$, define their *composition* $\mathcal{D} = (\mathcal{D}_0,\mathcal{D}_1)\circ\mathcal{D}_e$ as the following set of blocks over the set $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_e$:

$$\mathcal{D} = \left\{ \left(\bigcup_{j=1}^{v_e} B_j \times \{j\} \right)_B : \text{for } B \in \mathcal{D}_e,\ B_j \in \mathcal{D}_1 \text{ when } j \in B, \right.$$
$$\left. \text{and } B_j \in \mathcal{D}_0 \text{ when } j \notin B \right\}. \tag{3}$$

The blocks of $\mathcal{D}$ can be thought of as certain $v_1 \times v_e$ binary matrices A obtained as follows: Let $\mathcal{X}_1$ index the rows, and $\mathcal{X}_e$ the columns of A. For a fixed block $B \in \mathcal{D}_e$ fill the columns of A corresponding to the elements of B with characteristic vectors of blocks in design $\mathcal{D}_1$. Fill the columns of A corresponding to the complement of B, with characteristic vectors of blocks in design $\mathcal{D}_0$. This can be done in $b_1^{k_e} b_0^{v_e-k_e}$ ways for this block $B \in \mathcal{D}_e$. By taking all $B \in \mathcal{D}_e$, we obtain $b_e b_1^{k_e} b_0^{v_e-k_e}$ matrices and these, viewed as subsets of $\mathcal{X}_1 \times \mathcal{X}_e$, are the blocks of $(\mathcal{D}_0,\mathcal{D}_1)\circ\mathcal{D}_e$. If $\left(\bigcup_{j=1}^{v_e} B_j \times \{j\}\right)_B$ is a block of $\mathcal{D}$ we call the subsets $B_j \times \{j\}$ the *columns* of the block. The blocks of $\mathcal{D}$ are of size

$$k = k_0(v_e - k_e) + k_1 k_e$$

and the number of blocks is

$$b = b_e b_0^{v_e-k_e} b_1^{k_e}. \tag{4}$$

This construction is modeled on a construction given by Narayani [4].

In general, $\mathcal{D}$ is not a t-design but it is always a partially balanced design of a special type.

Compatibility. First, we specialize to the case of $t = 2$. Given two points p and q in $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_e$, we want to count the blocks of $\mathcal{D}$ containing p and q. There are two possibilities:

(A) when p and q are in the *same* column: the number of blocks is

$$r_e \lambda_1 b_0^{k'_e} b_1^{k_e - 1} + r'_e \lambda_0 b_0^{k'_e - 1} b_1^{k_e} \tag{5}$$

(B) when p and q are in *distinct* columns: the number of blocks is

$$\lambda_e r_1^2 b_0^{k'_e} b_1^{k_e - 2} + 2\eta_e r_0 r_1 b_0^{k'_e - 1} b_1^{k_e - 1} + \lambda'_e r_0^2 b_0^{k'_e - 2} b_1^{k_e} \tag{6}$$

where the primed variables are the corresponding parameters for the complementary designs [5, p. 45] : $k' = v - k$, $r' = b - r$, $\lambda' = b - 2r + \lambda$, and $\eta = r - \lambda$ is the number of blocks containing a given point and avoiding another. Let Λ be the set of the two quantities (5) and (6). They do not depend on particular p or q. We have the following result.

Theorem 1 *If $\mathcal{D}_e$ is a 2-(v_e, k_e, λ_e) design, and $\mathcal{D}_i$, for i=0,1, are 2-(v_i, k_i, λ_i) designs, then their composition $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1) \circ \mathcal{D}_e$ is a 2-(v, k, Λ) partially balanced design with $v = v_1 v_e$, $k = k_1 k_e + k_0(v_e - k_e)$, and Λ of size at most two.*

The composition is always a PBD with two association classes [5, §11.2]. To find out when the composition is balanced we equate the two quantities (5) and (6). Eliminating the λ's and r's with the relations (1) and (2), gives the following *compatibility condition*

$$\begin{aligned} & v_1(v_e - 1)\left[k_e \binom{k_1}{2} + k'_e \binom{k_0}{2}\right] \\ = \; & (v_1 - 1)\left[k_1^2 \binom{k_e}{2} + k_1 k_0 k_e k'_e + k_0^2 \binom{k'_e}{2}\right]. \end{aligned} \tag{7}$$

By symmetry and to avoid certain trivial cases we make the following restrictions:

$$0 < k_e < v_e, \tag{8}$$

$$0 \le k_0 < k_1 < v_1, \tag{9}$$

$$2 \le k_1. \tag{10}$$

$\mathcal{D}$ never contains all the k-sets since each block of $\mathcal{D}$ has at most two column sizes k_0 and k_1. The construction of $\mathcal{D}$ given in (3) shows that no blocks are repeated if $\mathcal{D}_0$ and $\mathcal{D}_1$ are disjoint; this is true since $k_0 < k_1$ by (9). We now have the following result.

Corollary 1 *If $\mathcal{D}_e$ is a 2-(v_e, k_e, λ_e) design, and $\mathcal{D}_i$, i=0,1, are 2-(v_i, k_i, λ_i) designs satisfying (8),(9),(10) and the compatibility condition (7), then their composition $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1) \circ \mathcal{D}_e$ is a nontrivial simple 2-(v, k, λ) design with $v = v_1 v_e$, $k = k_1 k_e + k_0(v_e - k_e)$, and λ as given by (2) and (4) above.*

The compatibility condition (7) is independent of λ_e and the λ_i's; it follows that trivial designs can be used to build nontrivial designs.

Given $\mathcal{D}_1$ and $\mathcal{D}_0$ when $k_0 = 0$, there are an infinite number of compatible $\mathcal{D}_e$'s; for example, the parameters given by $k_e = pn+1$ and $v_e = qn+1$ for all $n \geq 1$, where

$$\frac{p}{q} = \frac{v_1(k_1 - 1)}{k_1(v_1 - 1)}$$

and p/q is reduced.

The λ of a composition design is usually large; but this is often the problem issue when constructing *simple* designs. The automorphism group is often quite large, as shown by Narayani [4].

This composition is the basis for a construction of designs by D.K. Ray-Chaudhuri and Tianbao Zhu [7] that employs orthogonal arrays and substantially reduces the index λ.

Examples. The 2-(4, 2, 1) trivial design, as $\mathcal{D}_1$, and the 2-(4,0,0) trivial design, as $\mathcal{D}_0$, are compatible with all designs with parameters 2-$(3n+1, 2n+1, \lambda_e)$ for $n \geq 1$. Composing these gives 2-$(12n+4, 4n+2, \lambda)$ designs. Taking $n = 2$ gives a *simple 2-(28,10,19440)* design. According to recent tables in [2], this design is new.

Similarly, composing the 2-(5,2,1) design, as $\mathcal{D}_e$, with the 2-(5,3,3) design, as $\mathcal{D}_1$, and the 2-(5,1,0) design, as $\mathcal{D}_0$, we obtain a new *simple 2-(25,9,15000)* design.

Compose the 2-(3,1,0) design, as $\mathcal{D}_e$, and the 2-(7,1,0) design, as $\mathcal{D}_0$, with any of the 2-(7,3,s) designs, $s = 1, \ldots, 5$, as $\mathcal{D}_1$; this gives *2-(21,5,49s)* designs, $s = 1, \ldots, 5$. These are all simple designs and, for $s = 2, 3, 4, 5$, are new [2].

3-Designs. So far, only 2-designs have been considered; we now briefly treat 3-designs. When $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_e$ are 3-designs, the composition $(\mathcal{D}_0, \mathcal{D}_1) \circ \mathcal{D}_e$ is exactly as before in (3). It is a partially balanced design as in Theorem 1, but with a different Λ. If we choose three points from the underlying set $\mathcal{X}$, there are three possibilities:

(A) all three points are in the same column,
(B) two points are in one column and the third in a separate column,
(C) all three points are in separate columns.

In each of these cases we can count the number of blocks containing the three points; this gives three numbers λ_{A}, λ_{B} and λ_{C}. If $v_e \geq 3$, there are at least three columns and we set $\Lambda = \{\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}, \lambda_{\mathrm{C}}\}$. But, if $v_e = 2$, there are only two columns and case (C) doesn't occur and we set $\Lambda = \{\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}\}$. In any case, Λ is at most of size three.

The composition will be a balanced 3-design when

$$\lambda_{\mathrm{A}} = \lambda_{\mathrm{B}} = \lambda_{\mathrm{C}} \quad \text{for} \quad v_e \geq 3$$

or

$$\lambda_{\mathrm{A}} = \lambda_{\mathrm{B}} \quad \text{for} \quad v_e = 2.$$

These equations (whichever applies) are the compatibility conditions on the 3-designs $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_e$ for their composition to be balanced. These equations are ***homogeneous*** in the λ's of $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_e$, and when simplified involve only the v's and the k's. Thus we have the analogs of Theorem 1 and Corollary 1.

As an example, consider the case of only two columns, so that $v_e = 2$ and $k_e = 1$ and $\mathcal{D}_e$ is the 3-(2,1,0) trivial design. The compatibility condition is

$$\frac{\binom{k_0}{3} + \binom{k_1}{3}}{\binom{v_1}{3}} = \frac{k_1\binom{k_0}{2} + k_0\binom{k_1}{2}}{v_1\binom{v_1}{2}}.$$

It can be shown [4] that the solutions of this equation resulting in nontrivial composition are as follows:

$$\text{for any } n \geq 2$$

and

$$d \text{ any divisor of } \frac{n^2(n^2-1)}{4} \tag{11}$$

set

$$k_0 = \frac{n^2 - n}{2} + d$$

$$k_1 = \frac{n^2 + n}{2} + d$$

$$v_1 = \frac{k_0 k_1}{d}$$

with no restriction on λ_0 or λ_1. In this way we obtain an infinite number of simple 3-designs. It is shown later that an infinite subfamily of these designs satisfy a certain extremal condition.

For t larger than 3, there are variations on the present constructions that produce a number of new 5-designs [8].

Automorphism groups. A permutation on a set $\mathcal{X}$ induces a permutation on the k-sets of $\mathcal{X}$. An *automorphism* of a t-(v,k,λ) design $\mathcal{D}$ over the set $\mathcal{X}$ is a permutation on $\mathcal{X}$ that sends the blocks of $\mathcal{D}$ back into $\mathcal{D}$. Let $\mathcal{A}_i$ be the automorphism group of $\mathcal{D}_i$, for i=0,1, $\mathcal{A}_e$ that of $\mathcal{D}_e$, and $\mathcal{A}$ that of their composition $\mathcal{D} = (\mathcal{D}_0,\mathcal{D}_1)\circ\mathcal{D}_e$. For $\nu = \{\nu_j\}_{j=1}^{v_e} \subseteq \mathcal{A}_0 \cap \mathcal{A}_1$, and $\eta \in \mathcal{A}_e$ form the permutation $\nu \wr \eta$ that acts on the blocks of $\mathcal{D}$ given in (3) as follows

$$\nu \wr \eta \left(\bigcup_{j=1}^{v_e} B_j \times \{j\} \right) = \bigcup_{j=1}^{v_e} \nu_j(B_j) \times \{\eta j\};$$

that is, η permutes the columns and the ν_j's permute within the columns. The set of permutations of this type, written $(\mathcal{A}_0\cap\mathcal{A}_1)\wr\mathcal{A}_e$, is called a *wreath product* [3, p. 81]. It is clear they form a subgroup of $\mathcal{A}$; in fact we have the following result.

Theorem 2 $\quad$ *$\mathcal{A} = (\mathcal{A}_0\cap\mathcal{A}_1)\wr\mathcal{A}_e$.*

It follows that every automorphism sends columns to columns; these are called sets of *imprimitivity* of the group. The details of the proof are given by Narayani [4].

An automorphism group $\mathcal{A}$ of a design $\mathcal{D}$ is called *block-transitive* when any block of $\mathcal{D}$ can be mapped to any second block of $\mathcal{D}$ by a suitable automorphism.

Corollary 2 $\quad$ *If $\mathcal{A}_0\cap\mathcal{A}_1$ and $\mathcal{A}_e$ are block-transitive, then $\mathcal{A} = (\mathcal{A}_0\cap\mathcal{A}_1)\wr\mathcal{A}_e$ is block-transitive.*

Cameron & Praeger [6] have shown:
Any block-transitive, point-imprimitive 3-(v,k,λ) design satisfies

$$v \leq \binom{k}{2} + 1. \tag{12}$$

See the paper cited above for the definitions.

When $d = 1$ in (11), then for any $n \geq 2$, the resulting 3-(v,k,λ) design has parameters

$$\begin{aligned} v &= \frac{1}{2}\left((n^2+2)^2 - n^2\right) \\ k &= n^2 + 2 \end{aligned}$$

and it is easy to check that $v = \binom{k}{2}+1$. The automorphism group $\mathcal{A}_0\cap\mathcal{A}_1$ is often block-transitive on $\mathcal{D}_0$ and $\mathcal{D}_1$; when, for example, $\mathcal{D}_0$=$\mathcal{D}_1$ is trivial. So equality is attained in (12) for an infinite number of nontrivial simple 3-designs.

Difference families. Given an abelian group $\mathcal{G}$ of size v, a (v,k,λ) difference family $\mathcal{F}$ over the group $\mathcal{G}$ is a family of k-subsets of $\mathcal{G}$

$$\left(B_1,\ldots,B_s\right),$$

called *base blocks*, such that for each nonzero $d \in \mathcal{G}$, there are λ ordered triples (j,g_1,g_2) with the properties that $j \in \{1,2,\ldots,s\}$, $g_1,g_2 \in B_j$ and $g_1 - g_2 = d$. That is, there are exactly λ ways of expressing each nonzero element d in $\mathcal{G}$ in the form $d = g_1 - g_2$ where $g_1, g_2 \in B_j$ for some j with $1 \le j \le s$. Here, the blocks of $\mathcal{F}$ need not be distinct. This definition of a group difference family is more general than the one by Beth et al. [1], which assumes that the stabilizers of the blocks of $\mathcal{F}$ are trivial.

Let dev$\mathcal{F}$, called the *development* of $\mathcal{F}$, be the family of translates of the elements of $\mathcal{F}$ with any multiplicities counted. That is

$$\begin{aligned} \text{dev}\mathcal{F} &= \mathcal{F}+\mathcal{G} \\ &= \Big(B+g : B \in \mathcal{F},\, g \in \mathcal{G}, \text{ multiple blocks} \\ &\qquad\qquad \text{counted as distinct blocks}\Big). \end{aligned} \tag{13}$$

That is, if two translates of a block of $\mathcal{F}$ are equal, they are treated as distinct blocks and are counted separately.

The action of a group $\mathcal{G}$ on a set $\mathcal{X}$ is said to be *regular* if for any two elements of $\mathcal{X}$, there is exactly one element of $\mathcal{G}$ sending one to the other.

If $\mathcal{F}$ is a (v,k,λ) difference family over a group $\mathcal{G}$, then dev$\mathcal{F}$ is a 2-(v,k,λ) design, not necessarily simple, with a regular group of automorphisms induced by the action of $\mathcal{G}$ on itself.

Let b be the number of blocks in dev$\mathcal{F}$. Then the parameters of dev$\mathcal{F}$ satisfy the two following relations

$$b = sv \tag{14}$$

$$\lambda = b\frac{\binom{k}{2}}{\binom{v}{2}}. \tag{15}$$

Conversely, if $\mathcal{D}$ is a 2-(v,k,λ) design over $\mathcal{G}$, not necessarily simple, such that $\mathcal{D}$ is invariant under the action of $\mathcal{G}$ and if there is a subset $\mathcal{F}$ of $\mathcal{D}$ with $\mathcal{D} = \mathcal{F} + \mathcal{G}$ counting all multiplicities, then $\mathcal{F}$ is a (v,k,λ) difference family over the group $\mathcal{G}$ with $\mathcal{D}$ =dev$\mathcal{F}$.

From the composition of designs and this correspondence between difference families and 2-designs, and we define a composition of difference families as follows.

Given $\mathcal{F}_i$, for i=0,1, and $\mathcal{F}_e$ are (v_i,k_i,λ_i) and (v_e,k_e,λ_e) difference families, respectively, with $\mathcal{G}_0 \simeq \mathcal{G}_1$, we take their associated designs

$\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_e$ and, if they are compatible, form the composition design $\mathcal{D} = (\mathcal{D}_0,\mathcal{D}_1)\circ\mathcal{D}_e$. The underlying point set of $\mathcal{D}$ is $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_e$. Then $\mathcal{D}$ is invariant under the action of $\mathcal{G}$ and this action is regular.

Define the following subset of blocks from $\mathcal{D}$

$$\mathcal{F} = \left(\left(\bigcup_{j=1}^{v_e} (B_j + g_j) \times \{e_j\}\right)_B : B \in \mathcal{F}_e, g_j \in \mathcal{G}_1, \right.$$
$$\left. B_j \in \mathcal{F}_0 \text{ if } e_j \notin B \text{ and } B_j \in \mathcal{F}_1 \text{ if } e_j \in B \right) \tag{16}$$

with appropriate multiplicities. $\mathcal{F}$ is called the *composition* of the difference families $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_e$ and is written as

$$\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) \circ \mathcal{F}_e. \tag{17}$$

It can be shown that $\mathcal{F}$ is a (v, k, λ) difference family over the group $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_e$ and $\mathcal{D} = \text{dev}\mathcal{F}$.

We now have the following result.

Theorem 3 *Let $\mathcal{F}_i$, for i=0,1, be (v_i, k_i, λ_i) difference families over the groups $\mathcal{G}_0 \simeq \mathcal{G}_1$, and $\mathcal{F}_e$ a (v_e, k_e, λ_e) difference family over $\mathcal{G}_e$. If the compatibility condition (7) is satisfied, then their composition $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) \circ \mathcal{F}_e$ is a (v, k, λ) difference family over the group $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_e$ with $v = v_1 v_e$, $k = k_1 k_e + k_0(v_e - k_e)$, λ and b given by (2), (4) above.*

This result was suggested by Professor D.K. Ray-Chaudhuri. We remark that Theorem 3 also holds for non-abelian groups.

As an example, compose the 2-(3,1,0) design, as $\mathcal{D}_e$, and the 2-(7,1,0) design, as $\mathcal{D}_0$, with any of the 2-(7,3,s) designs, $s = 1, \ldots, 5$, as $\mathcal{D}_1$; this gives *2-(21,5,49s)* designs, $s = 1, \ldots, 5$. The designs $\mathcal{D}_e$ and $\mathcal{D}_0$ have block size one and are easily seen to be generated by difference families; the design $\mathcal{D}_1$, for $s = 1$, is a projective plane of order 2 and is generated by a difference set [1, p. 263]; $s = 2$ is a union of two disjoint difference sets, and $s = 3, 4, 5$ are complements. Thus the compositions are all generated by difference families. These are all simple designs and, for $s = 2, 3, 4, 5$, are new [2].

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