On Leech's Tree Labelling Problem

Yang Yuansheng, Zhang Chengxue *and* Ding Shanjing
Department of Computer Science & Engineering
Daling University of Technology
116024 Dalian
People's Republic of China

ABSTRACT. J. Leech has posed the following problem: For each integer n what is the greatest integer N such that there exists a labelled tree with n nodes in which the distance between the pairs of nodes include the consecutive values $1, 2, \ldots, N$? With the help of a computer, we get $B(n)$ (the number N for branched trees) for $2 \leq n \leq 10$ and lower bounds of $B(11)$ and $B(12)$. We also get $UU(n)$ (the number N for unbranched trees) for $2 \leq n \leq 11$ independently, confirming some results gotten by J. Leech.

Each edge of a finite tree is labelled with a positive integer, which we call its length. The distance between two nodes is the sum of the lengths on the (unique) path between the nodes. J. Leech [3] has posed the following **Problem**: For each integer n what is the greatest integer N such that there exists a labelled tree with n nodes in which the distances between pairs of nodes include the consecutive values $1, 2, \ldots, N$?

Let $\mathcal{T}_n$ be the set of all trees with n nodes. Let $e_1, \ldots, e_{n-1}$ denote the edges of $T_n \in \mathcal{T}_n$, and let e_i be labelled with integer a_i. A labelling of T_n is called M-labelling if the distances between pairs of nodes include the consecutive values $1, 2, \ldots, M$. hence we have

$$N = \text{Max}\{M : \text{there exists an } M\text{-labelling of } T_n \in \mathcal{T}_n\} \qquad (1)$$

As in [3], we let $UU(n)$ denote the number N for unbranched trees and $B(n)$ denote the number N for branched trees. J. Leech has shown in [3]

n	2	3	4	5	6	7	8	9	10	11	12
$UU(n)$	1	3	6	9	13	18	24		≥ 37	≥ 45	≥ 51
$B(n)$	1	3	6	9	15	≥ 20	≥ 26	≥ 31	≥ 38	≥ 45	≥ 52

Table 1

Now, by computer, we get the following results.

n	2	3	4	5	6	7	8	9	10	11	12
$UU(n)$	1	3	6	9	13	18	24	29	37	45	≥ 51
$B(n)$	1	3	6	9	15	20	26	34	41	≥ 48	≥ 55

Table 2

Let $(b_1, \ldots, b_{n-1})$ be a permutation of $(a_1, \ldots, a_{n-1})$ such that $b_1 \leq b_2 \leq \cdots \leq b_{n-1}$. We have

Lemma 1. *In an M-labelling of T_n, if $M \geq 1$ then $b_1 = 1$.*

Proof: If $b_1 > 1$ then all $a_i \geq b_1 > 1$, and the distance between any pair of nodes is greater than 1, a contradiction.

Lemma 2. $B(n) \geq B(n-1) + 2$.

Proof: From formula (1), there exists a $B(n-1)$-labelling of $T_{n-1} \in \mathcal{T}_{n-1}$. Let N_i be one of the nodes adjacent to the edge labelled 1. Add a new node N_n to T_{n-1} and join a new edge e_{n_i}, label it with $b(n-1)+1$, such that we get a $(B(n-1)+2)$-labelling of T_n, so $B(n) \geq B(n-1) + 2$.

Let $G \cdot e$ denote a graph obtained from the graph G by contraction of edge e, i.e. delete edge e from G and merge its two terminal nodes. e.g. in Figure 1, $G_1 = G \cdot e_1$, $G_3 = G \cdot e_3$.

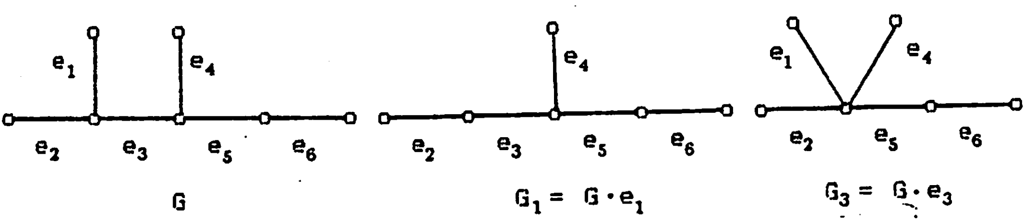

Figure 1. The contraction of an edge

It is easily seen that if there is an M-labelling of T_n with edges e_i labelled a_i and $a_i > M$, then there is an M-labelling of $T_n \cdot e_i$. Let $B(1) = 0$, we have

Lemma 3. *In an M-labelling of $T_n(M \geq B(n-1)+2)$, $b_i \leq B(i)+1$, $1 \leq i \leq n-1$.*

Proof: From Lemma 1, $b_1 = 1 = B(1)+1$. Suppose $b_i > B(j)+1$ for some j with $2 \leq j \leq n-1$, let $e_{i_k}(j \leq k \leq n-1)$ denote the edge labelled b_k. Since $M \geq B(n-1)+2 > B(j)+1$ and since an M-labelling of T_n is also a $(B(j)+1)$-labelling of T_n and since

$$b_{n-1} \geq \cdots \geq b_j > B(j)+1,$$

there exists a $(B(j)+1)$-labelling of $T_n \cdot e_{i_{n-1}}$. Continue in this fashion, generating a $(B(j)+1)$-labelling of $T_n \cdot e_{i_{n-1}} \cdot \ldots \cdot e_{i_j}$, which means that $B(j) \geq B(j)+1$, a contradiction.

Lemma 4. *In an M-labelling of T_n, if $T_n = T_r + e_i + T_{n-r}$ (see Fig 2.) and $M > \binom{n-1}{2}+1$ then $a_i \leq \binom{r}{2}+\binom{n-r}{2}+1$.*

Proof: Let $v_j(1 \leq j \leq r)$ denote the nodes in T_r and let $U_K(1 \leq k \leq n-r)$ denote the nodes in T_{n-r}. If $a_i > \binom{r}{2}+\binom{n-r}{2}+1$ then the distances between v_j and u_k are greater than $\binom{r}{2}+\binom{n-r}{2}+1$. So there are at most $\binom{r}{2}+\binom{n-r}{2}$ different distances no greater than $\binom{r}{2}+\binom{n-r}{2}+1$, in contradiction to $M > \binom{n-1}{2}+1 \geq \binom{r}{2}+\binom{n-r}{2}+1$.

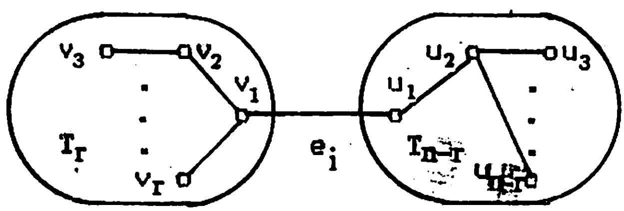

Figure 2. $T_n = T_r + e_i + T_{n-r}$

Furthermore, we have

Lemma 5. *In an M-labelling of T_n if $T_n = T_r + e_i + T_{n-r}$; $M > \binom{n-1}{2}+1$ and there are k distances between v_{j_1} and v_{j_2} or u_{k_1} and $u_{k_2}(1 \leq j_1 < j_2 \leq r, 1 \leq k_1 < k_2 \leq n-r)$ which are greater than $\binom{r}{2}+\binom{n-r}{2}+1-k$, then $a_i \leq \binom{r}{2}+\binom{n-r}{2}+1-k$.*

In order to find $B(n)$ we proceed as follows. First, choose T_n in $\mathcal{T}_n$, and arrange the edges and nodes of T_n so that $T_i = T_{i-1} + e_i + N_i$, where $2 \leq i \leq n$ and $T_0 = N_0$, where N_i is node i. (see Figure 3)

From Lemma 2 we see that we can begin our search for $B(n)$ by starting with $M = B(n-1)+2$. Once we find an M-labelling for T_n in $\mathcal{T}_n$, we will try to find an $(M+1)$-labelling. That is, we calculate the value of $B(n)$ by exhausting the possibilities.

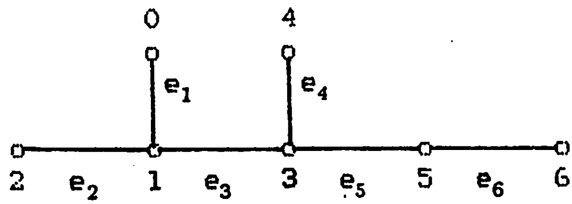

Figure 3. An arrangement of the edges of $T_7 \in \mathcal{T}_7$

In our algorithm for determining whether a particular T_n has an M-labelling, we shall try labelling edge e_i in our arrangement, where $1 \le i \le n-1$, with all possible positive integers a_i. Fortunately, Lemma 3 and Lemma 5 limit these possibilities. In particular, we let $a_{m_3}(i)$ and $a_{m_5}(i)$ be best upper bounds for the label a_i that we can assign to edge e_i, as dictated by Lemma 3 and Lemma 5, respectively. From Lemma 3 we know that there are at most k labels a_i greater than $B(n-1-k)+1$, where $1 \le j \le i-1$; if this is all the information we have then $1 \le a_i \le a_{m_3}(i) = B(n-1)+1$; however if there are k labels a_i greater than $B(n-1-k)+1$, where $1 \le j \le i-1$, then $a_{m_3}(i) = B(n-1-k)+1$. If $M \le \binom{n-1}{2}+1$, then define $a_{m_5}(i) = \infty$. Otherwise, from Lemma 5, we have: if $T_n = T_r + e_i + T_{n-r}$, $M > \binom{n-1}{2}+1$ and there are k distances between N_{j_1} and N_{j_2} $(0 \le j_1 < J_2 \le i-1)$ which are greater than $\binom{r}{2}+\binom{n-r}{2}+1-k$, then $a_{m_5}(i) = \binom{r}{2}+\binom{n-r}{2}+1-k$. Let

$$a_{\max}(i) = \mathrm{Min}(a_{m_3}(i), a_{m_5}(i)) \tag{2}$$

We have $1 \le a_i \le a_{\max}(i)$.

Let $f_d(1 \le d \le M)$ denote the number of the node pairs with distance d in an M-labelling of T_n, and f_{M+1} denote the number of node pairs with distance greater than M, then

$$\binom{n}{2} = M + f_{M+1} + \sum_{1 \le d \le M}^{t_d > 1} (f_d - 1)$$

Suppose after e_j is labelled $a_j(1 \le j \le i-1)$ and there are $f_d(i)$ node pairs with distance d and $f_{M+1}(i)$ node pairs with distance greater than M, then

$$\mathrm{sum}(i) = f_{M+1}(i) + \sum_{1 \le d \le M}^{t_d(i) > 1} (f_d(i) - 1) \tag{3}$$

$$0 \le \mathrm{sum}(1) \le \mathrm{sum}(2) \le \cdots \le \mathrm{sum}(n-1) = \binom{n}{2} - M.$$

That is, for a labelling of the edges of T_n as above, if after the $(n-1)^{\text{th}}$ iteration we have $\mathrm{sum}(n-1) \le \binom{n}{2} - M$, then that labelling is an M-labelling.

Now we construct Program CHECK(M) for determining whether there exists an M-labelling of $T_n \in \mathcal{T}_n$.

```
Program CHECK(M, T_n);
begin
    i := 0;
    TEST(i)
end.
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Procedure TEST(i);
begin
    i := i + 1;
    if i = n then writeln (a_1, ..., a_{n-1})
    else begin
        determine a_max(i); { see formula (2) }
        for a_i := 1 to a_max(i) do
        if sum(i) ≤ (n choose 2) − M then TEST(i); { see formula (3) }
    end
end.
```

We use Program CHECK(M) for all $T_n \in \mathcal{T}_n$ for n from 2 to 10 and some $T_{11} \in \mathcal{T}_{11}$, get $B(n)$ for $2 \leq n \leq 10$ and new lower bounds of $B(11)$ and $B(12)$. Table 3 shows all possible $B(n)$-labellings of T_n for $2 \leq n \leq 10$, and some 48-labelling of T_{11}, and some 55-labelling of T_{12} (the notation is that used in [3]).

Recently J. Leech has gotten 19 48-labelling of T_{11} and 124 55-labelling of T_{12} independently. Here are three of them:

$$(1,2,4,6)9(14)7(12,19,26,33) \quad (1,2,3,4,5)9(16)8(15,23,31)$$
$$(1,2,3,5,7)10(16)8(14,22,30)$$

We are confident that $B(11) = (48)$ and $B(12) = (55)$ can be improved. But our program is not efficient for $n \geq 11$, for example, $\mathcal{T}_{11}$ contains 235 nonisomorphic trees (see [6, p.190]), all of which must be arranged by hand as in figure 3, and the determination of the cardinality of is an open question ([1, p.95] or [6, p.190]), so we stop here.

By using a similar program, we get $UU(n)$ for $2 \leq n \leq 11$, confirming the results gotten by J. Leech, and showing that $UU(9) = 29$, $UU(10) = 37$, $U(11) = 45$. Thus the solutions found by Leech for $UU(n)$, $n = 8, 10, 11$ are found to be best possible by exhaustive search.

There is a related problem, the notion of edge-gracefulness found in references [2], [4], and [5]. In particular, a tree T is edge-graceful if its edges can be labelled consecutively from 1 to $n-1$ so that the nodes are labelled from 1 to n, where the label of a vertex is the sum modulo n of the labels of the incident edges. A lively conjecture is that all trees of the form T_{2n-1} are edge-graceful.

	n	$B(n)$	$B(n)$-labelling		n	$B(n)$	$B(n)$-labelling
1	2	1	1	19			1.10(3)5(2,4)
2	3	3	1.2	20			3.1(13)5(2,10)
3	4	6	1.3.2	21			(1,2)5(4,8,12)
4			(1,2,4)	22	8	26	3.1(2)7(15)5.5
5	5	9	1.1.4.3	23			5.5(15)7(1,2,4)
6			1.3.3.2	24			3.1(2)7(5,10,15)
7			2.5.1.3	25			2.13(1,11)4(3,6)
8			4.1.2.6	26			(1,2,4)7(5,10,15)
9			1.1(3,6)	27	9	34	(1,3)2(6)10(7,14,18)
10			1.2(4,5)	28	10	41	(1,2,3)7(12)6(11,17,23)
11			1.3(2,5)	29			1.1(3)7(12)6(11,17,23)
12			2.1(4,5)	30	11	(48)	(1,2,3,4)8(14)7(13,20,27)
13			3.4(1,2)	31			(1,2,4,6)9(14)7(12,19,26)
14			(1,2,3,6)	32			(1,2,7)4(10,12)20(3,13,16)
15			(1,2,4,7)	33			(1,2,3,5)9(7,19)13(15,21)
16	6	15	(1,2)5(4,8)	34	12	(55)	(1,2,3,4)8(14)7(13,20,27,34)
17	7	20	2.4.3(5)10.1	35			(1,2,3,4)8(14)7(20,27)13.21
18			4.4(12)5(1,2)				

Table 3. All $B(n)$-labellings of T_n for $2 \le n \le 10$ and some 48-labellings of T_{11} and some 55-labellings of T_{12}

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References

[1] N. Hartsfield and G. Ringel, "Pears in Graph Theory", Academic Press, Boston, 1990.

[2] S. Lee, A conjecture on edge-graceful trees, *Scientia* **3** (1989), 45–57.

[3] J. Leech, Another tree labelling problem, *American Mathematical Monthly*, Vol. 82, No. 9 (November 1975), 923–925.

[4] S. Lo, On edge-graceful labellings of graphs, *Congressus Numerantium* 50(1985), 231–241.

[5] J. Mitchem & A. Simoson, On edge-graceful and super-edge-graceful graphs, *Ars Combinatoria* **37**(1994), 97–111.

[6] R. Wilson and J. Watkins, "Graphs", John Wiley, New York, 1990.