CONSECUTIVE-INTEGER PARTITIONS

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Abstract. Functions $c(n)$ and $h(n)$ which count certain consecutive-integer partitions of a positive integer n are evaluated, and combinatorial interpretations of partitions with "$c(n)$ copies of n" and "$h(n)$ copies of n" are given.

Consider a generating function in the form

$$1+\sum_{n=1}^{\infty} A(n)p^n = \prod_{n=1}^{\infty}(1-q^n)^{-a_n} \qquad (1)$$

where n is a positive integer. Combinatorial interpretations of partitions with "a_n copies of n" have been given by defining appropriate sets of partitions having order $A(n)$ [3, Chap. 1]. This has has been done for $a_n = n$ by Agarwal and Andrews [1], for $a_n = d(n)$ (the number of positive integral divisors of n) by Agarwal and Mullen [2], for $a_n = \sigma(n)$ (the sum of the positive integral divisors of n) by Mitchell [6, Chap. 2], as well as for other a_n [4], [5]. In this paper, two functions, $c(n)$ and $h(n)$, which count certain consecutive-integer partitions are defined and evaluated; combinatorial interpretations of partitions with "$c(n)$ copies of n" and "$h(n)$ copies of n" are given by defining a set of plane partitions, C_n, and a set of solid partitions, H_n.

Let n be a positive integer. We define $c(n)$ as the number of linear partitions of n having consecutive-integer parts and at least two parts. If $n < 3$, $c(n) = 0$; $c(3) = 1$ $(3 = 2 + 1)$; $c(9) = 2$ $(9 = 5 + 4 = 4 + 3 + 2)$.

Theorem 1. *If n is a positive integer, $n = 2^w p_1^{e_1} \dots p_r^{e_r}$ (with $p_1, \dots, p_r$ distinct primes, $e_1, \dots, e_r$ positive integers, and w a nonnegative integer), then* $c(n) = (e_1+1)\dots(e_r+1) - 1 = d(n/2^w) - 1$.

Proof: A consecutive-integer partition of n with smallest part m and $k+1$ parts $(m \geq 1, k \geq 1)$ can be written as $n = (m+k) + (m+k-1) + \dots + (m+1) + m$, with $n = (k+1)m + k(k+1)/2 = (k+1)(m+k/2) = \frac{(k+1)}{2}(2m+k)$. If the number of parts, $k+1$, is odd then $k+1$ divides n and

$$n=\left(\frac{n}{k+1}+\frac{k}{2}\right)+\dots+\left(\frac{n}{k+1}+1\right)+\frac{n}{k+1}+\left(\frac{n}{k+1}-1\right)+\dots+\left(\frac{n}{k+1}-\frac{k}{2}\right). \qquad (2)$$

If the number of parts, $k+1$, is even then the sum of the middle two parts is the odd integer $\frac{n}{(k+1)/2}$, $2m+k$ and $2n/(k+1)$ divide n, and

$$n=\left(\frac{n}{k+1}+\frac{1}{2}+\left(\frac{k+1}{2}-1\right)\right)+\ldots+\left(\frac{n}{k+1}+\frac{1}{2}+1\right)+\left(\frac{n}{k+1}+\frac{1}{2}\right)$$
$$+\left(\frac{n}{k+1}-\frac{1}{2}\right)+\left(\frac{n}{k+1}-\frac{1}{2}-1\right) \tag{3}$$
$$+\ldots+\left(\frac{n}{k+1}-\frac{1}{2}-\left(\frac{k+1}{2}-1\right)\right).$$

Note that in (2), $m=\frac{n}{k+1}-\frac{k}{2}\geq 1$ so that $n\geq(k+1)(k+2)/2$ and $n\geq 6$ (since $k+1\geq 3$); and in (3), $m=\frac{n}{k+1}-\frac{1}{2}-\left(\frac{k+1}{2}-1\right)\geq 1$ so that $n\geq(k+1)(k+2)/2$ and $n\geq 3$ (since $k+1\geq 2$). Also, $c(2^w)=0$ for $w=0,1,2,\ldots$ (since 2^w has no odd integral divisors larger than 1). Let t be an odd divisor of n $(t>2)$. There is a unique consecutive-integer partition of n with t parts and smallest part $\frac{n}{t}-\frac{t-1}{2}$ (as in (2), $t=k+1$) provided that $\frac{n}{t}-\frac{t-1}{2}\geq 1$, $n>\frac{t(t-1)}{2}$. There is a unique consecutive-integer partition of n with $\frac{2n}{t}$ parts and smallest part $\frac{t}{2}-\frac{1}{2}-\left(\frac{n}{t}-1\right)$ (as in (3), $\frac{2n}{t}=k+1$) provided that $\frac{t}{2}-\frac{1}{2}-\left(\frac{n}{t}-1\right)\geq 1$, $n\leq\frac{t(t-1)}{2}$.

Let n be a positive integer. Let $P_n=\{v\colon v$ is a consecutive-integer linear partition of n of at least two parts$\}$ and let $T_n=\{t\colon t$ is an odd integral divisor of n, $t>2\}$. Define $F\colon P_n\to T_n$ by $F(v)$ is the number of parts of v if v has an odd number of parts and $F(v)$ is the sum of the largest and smallest parts of v if v has an even number of parts. Note that F is well-defined since if v has an odd number of parts, t, then t is an odd divisor of n $(t>2)$, and if v has an even number of parts, s, then the sum of the largest and smallest parts (which equals the sum of the two middle parts) is the odd integer $n/(s/2)=2n/s=t'$ with t' an odd divisor of n $(t'>2)$. Let t lie in T_n. If $n>t(t-1)/2$, there is an element v_1 of P_n having t parts and $F(v_1)=t$; if $n\leq t(t-1)/2$, there is an element v_2 of P_n having an even number $2n/t$ parts with t the sum of the two middle parts, and $F(v_2)=t$. And F is a surjection. Let v_1, v_2 lie in P_n with $F(v_1)=F(v_2)=t$. If $n>t(t-1)/2$ then both v_1 and v_2 have an odd number t of parts, and $v_1=v_2$; if $n\leq t(t-1)/2$ then both v_1 and v_2 have an even number of parts with t as the sum of the smallest and largest parts, and $v_1=v_2$. And F is an injection. F is a bijection with domain a finite set (or the null set if n has no odd integral divisors greater than 2). The number of consecutive-integer partitions of n of at least two parts equals the number of odd divisors of n greater than 2, and $c(n)=d(p_1^{e_1}\ldots p_r^{e_r})-1=(e_1+1)\ldots(e_r+1)-1=d(n/(2^w))-1$. ∎

Example 1: If $n=300=2^2\cdot 3\cdot 5^2$, $c(n)=2\cdot 3-1=5$, and the elements of T_n corresponding to $t_1=3, t_2=5, t_3=15, t_4=25, t_5=75$, respectively, are $v_1=101+100+99$ ($300>3(2)/2$, 3 parts), $v_2=62+61+60+59+58$ ($300>5(4)/2$, 5 parts), $v_3=27+\ldots+20+\ldots+13$ ($300>15(14)/2$, 15 parts), $v_4 = 24 + \ldots + 13 + 12 + \ldots + 1$ ($300 \leq 25(24)/2$, 24 parts), $v_5 = 41 + \ldots + 38 + 37 + \ldots + 34$ ($300 \leq 75(74)/2$, 8 parts). If $n = 16 = 2^4$, the sets P_{16} and T_{16} are empty and $c(16) = 0$.

For the positive integer n, $h(n)$ is defined as the number of identical-row plane partitions of n in which each of the u identical rows (u a positive integer) is a consecutive-integer linear partition of n/u of at least two parts. If n is an odd prime number then $h(n) = c(n) = 1$; $h(6) = 2$ $\left(\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix},\ \ 3\ 2\ 1\right)$.

Theorem 2. *If n is a positive integer, $n = 2^w p_1^{e_1} \ldots p_r^{e_r}$ (with $p_1, \ldots, p_r$ distinct odd primes, $e_1, \ldots, e_r$ positive integers, and w a nonnegative integer), then $h(n) = (w+1)(e_1+1)\ldots(e_r+1)\left(\left(1+\frac{e_1}{2}\right)\ldots\left(1+\frac{e_r}{2}\right) - 1\right)$ and $h(n) = (w+1)h\left(\frac{n}{2^w}\right)$.*

Proof: If n is odd, $w = 0$, then $h(n) = \sum_{u|n} c(n/u) = \sum_{u|n}(d(n/u) - 1) = \sum_{u|n} d(n/u) - d(n)$ by Theorem 1. Since $d(n)$ is multiplicative then $\sum_{u|n} d(n/u) = \sum_{u|n} d(u)$ is multiplicative [7, Chap. 4], and $\sum_{u|n} d(u) = \left(\sum_{u|p_1^{e_1}} d(u)\right) \ldots \left(\sum_{u|p_r^{e_r}} d(u)\right) = (d(1) + d(p_1) + \ldots + d(p_1^{e_1})) \ldots (d(1) + d(p_r) + \ldots + d(p_r^{e_r})) = (1 + 2 + \ldots + (e_1 + 1)) \ldots (1 + 2 + \ldots + (e_r + 1)) = \left(\frac{e_1+1}{2}\right)(e_1 + 2) \ldots \left(\frac{e_r+1}{2}\right)(e_r + 2) = (e_1 + 1) \ldots (e_r + 1)(e_1/2 + 1) \ldots (e_r/2 + 1)$, with $h(n) = d(n)((1 + e_1/2) \ldots (1 + e_r/2) - 1)$.

If n is even, $w \geq 1$, then $h(n) = \sum_{u|n} c(n/u) = \sum_{u|\frac{n}{2^w}} (c(u) + c(2u) + \ldots + c(2^w u)) = (w+1) \sum_{u|\frac{n}{2^w}} c(u) = (w+1)h\left(\frac{n}{2^w}\right)$. ∎

Example 2: If $n = 300 = 2^2 \cdot 3 \cdot 5^2$, $h(n) = 3(2)(3)((1+1/2)(1+2/2)-1) = 36$. Let S_n denote the set of identical-row plane partitions of n in which each row has consecutive-integer parts (with at least two parts). In S_{300}, the number of elements with u rows is $c(300/u)$:

u	1	2	4	3	6	12	5	10	20	15	30	60	25	50	100	75	150	300
$c(300/u)$	5	5	5	2	2	2	3	3	3	1	1	1	1	1	1	0	0	0.

And $\sum_{u|300} c(300/u) = 3(5 + 2 + 3 + 1 + 1) = 36 = h(300)$. Some elements of S_{300} are given:

$$\begin{matrix} 22 & 21 & 20 & 19 & 18 \\ 22 & 21 & 20 & 19 & 18 \\ 22 & 21 & 20 & 19 & 18 \end{matrix} \ (3\text{rows}), \quad \begin{matrix} 14 & 13 & 12 & 11 \\ 14 & 13 & 12 & 11 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 14 & 13 & 12 & 11 \end{matrix} \ (6\text{rows}),$$

$$\begin{matrix} 13 & 12 \\ 13 & 12 \\ \cdot\cdot & \cdot\cdot \\ 13 & 12 \end{matrix} \ (12\text{rows}), \quad \begin{matrix} 7 & 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 & 3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & 6 & 5 & 4 & 3 \end{matrix} \ (12\text{rows}), \quad \begin{matrix} 2 & 1 \\ 2 & 1 \\ \cdot & \cdot \\ 2 & 1 \end{matrix} \ (100\text{rows}).$$

A column replacement method is used to determine a set C_n of order $A(n)$ for $a_n = c(n)$ in (1); the elements of C_n are plane partitions. Replace a summand m of n by any of the $c(m)$ consecutive-integer columns of the type $\begin{array}{c} e \\ e-1 \\ \vdots \\ e-k \end{array}$ $(k \geq 1)$ having sum m; in the identical consecutive summand case, $m + m$, $\begin{array}{cc} e_i & e_j \\ e_i-1 & e_j-1 \\ \vdots & \vdots \\ e_i-k_i & e_j-k_j \end{array}$ (each with sum m and $e_i > k_i \geq 1$, $e_j > k_j \geq 1$) is an acceptable pair of consecutive integer column replacements if $e_i \geq e_j$. Define C_n to be the set of plane partitions of n in which the number of parts equal to $j \geq 2$ in the first row equals the number of parts equal to $j-1$ in the second row and the number of parts equal to $j \geq 1$ in row i $(i \geq 2)$ is not less than the number of parts equal to $j-1$ in row $i+1$. Each plane array obtained by replacement of the q summands in a linear partition $n = m_1 + m_2 + \ldots + m_q$ of n by suitable consecutive-integer columns corresponds to a unique q-column element in C_n; and each q-column element in C_n corresponds to exactly one plane array consisting of q consecutive-integer columns (in a proper summand replacement form). If $a_n = c(n)$ in (1), C_n has order $A(n)$.

Example 3: If $n = 101$, with $101 = 45 + 25 + 25 + 6$, $5 \cdot 3 \cdot 1 = 15$ plane arrays can be obtained by suitable column replacements (since $c(45) = 5$, $c(25) = 2$, $c(6) = 1$); one of these arrays is:

$$\begin{array}{cccc} 11 & 13 & 7 & 3 \\ 10 & 12 & 6 & 2 \\ 9 & & 5 & 1; \\ 8 & & 4 & \\ 7 & & 3 & \end{array}$$

it corresponds to

$$\begin{array}{cccc} 13 & 11 & 7 & 3 \\ 12 & 10 & 6 & 2 \\ 9 & 5 & 1 & \\ 8 & 4 & & \\ 7 & 3 & & \end{array}$$

in C_{101} (which also corresponds uniquely to the given array). No element in C_{101} corresponds to the partition $99 + 2$ of 101 (since $c(2) = 0$).

Let $a_n = h(n)$ in (1); a set H_n of order $A(n)$, and consisting of solid partitions, can be determined by using a layer-replacement method. A "rectangular" layer replacement of the type $R(r, s, L; k)$ with entry $k - v + 2$ at the point (x, y, L) for $x = v-1, v = 2, 3, \ldots, s+1, y = 1, \ldots, s$, and entry 0 at other points on layer L (with r, s, L, k, positive integers, $k \geq r$), and sum of entries m, can be used to replace a summand m of n; in the identical consecutive summand case, $m+m$, $R(r_1, s_1, L; k_i)$, $R(r_2, s_2, L+1; k_j)$ (each with sum m and $k_i \geq r_1$, $k_j \geq r_2$) is an acceptable pair of layer replacements if $k_i > k_j$, or if $k_i = k_j$ and $s_1 > s_2$, or if $k_i = k_j$ and $s_1 = s_2$ and $r_1 \geq r_2$. These are analogs of the "rectangular" identical-element layer replacements in [5]; analogs of form-D, form-C, form-B, and form-A arrays, and square and corner points, can be defined. And H_n can be defined as the set of solid partitions W of n having the following four properties. (i) If $(1, s, L)$ has entry k ($k \geq 2$) on layer L of W, then $(2, s, L)$ has entry $k-1$. (ii) The number of entries k on any line (r, s, L), $r \geq 2$, $s \geq 1$, $L = 1, 2, \ldots$, is at least as great as the number of entries $k-1$ on the line $(r+1, s, L)$, $L = 1, 2, \ldots$. (iii) For given $r \geq 1$, $s \geq 1$, there are as many entries k ($k \geq 1$) on a line (r, s, L), $L = 1, 2, \ldots$, as there are entries k on the line $(r, s+1, L)$, $L = 1, 2, \ldots$. (iv) The number of layers in W at which $k - r + 1$ occurs at points (r, s, L) is equal to the number of corner points (r', s', L') on layers with entry k ($k \geq 2$) at $(1, 1, L')$ and $r' \geq r$, $s' \geq s$, in the unique form-B array corresponding to W. Each form-D array obtained by replacement of the q summands in a linear partition $n = m_1 + m_2 + \ldots + m_q$ of n by suitable "rectangular" consecutive-integer layers corresponds to a unique q-layer element in H_n; and each element in H_n with q layers corresponds to a unique form-D array consisting of q "rectangular" consecutive-integer layers (in a proper summand replacement form). If $a_n = h(n)$ in (1), H_n has order $A(n)$.

Example 4: Let $n = 101$; there are $55 \cdot 5 \cdot 3 = 825$ form-D arrays which correspond to $101 = 30 + 30 + 21 + 20$ (since $h(30) = 10$, $h(21) = 5$, $h(20) = 3$). One of these form-D arrays is:

88	8	4444	8
77	7	3333	7
	6		6 ;
	5		
	4		

it corresponds to the form-C array

88	8	8	4444
77	7	7	3333
	6	6	,
	5		
	4		

the form-B array

88	8	8	4444
77	7	7	3333
66	6		,
5			
4			

and the form-A array

8844	84	8	4	
7733	73	7	3	
6	6			;
5				
4				

which is an element of H_{101}. Given this form-A array, we can find a unique corresponding form-D array (the one given above).

References

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