On Resolvable BIBDs with Block Size Five[1]

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Abstract. It is shown that a resolvable BIBD with block size five and index two exists whenever $v \equiv 5 \pmod{10}$ and $v \geq 50722395$. This result is based on an updated result on the existence of a BIBD with block size six and index unity, which leaves 88 unsolved cases. A construction using difference families to obtain resolvable BIBDs is also presented.

1. Introduction

A *group divisible design* (or GDD) is a tiple $(X, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

(1) X is a finite set of *points*;
(2) $\mathcal{G}$ is a partition of X into subsets called *groups*;
(3) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and every pair of points from distinct groups occurs in exactly λ blocks.

The *group-type* (or *type*) of a GDD$(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$.

A GDD with block sizes from a positive integer set K is called a (K, λ)-GDD. If $K = \{k\}$, we simply write (k, λ)-GDD. A (k, λ)-GDD of type m^k is called a *transversal design* (or TD) and denoted by TD(k, m). A (K, λ)-GDD $(X, \mathcal{G}, \mathcal{A})$ with type 1^v is called a *pairwise balanced design* (or PBD) and denoted by $B(K, \lambda; v)$ and also by $(X, \mathcal{A})$. When $K = \{k\}$, a $B(\{k\}, \lambda; v)$ is called a *balanced incomplete block design* (or BIBD), denoted simply by $B(k, \lambda; v)$. Let $B(K, \lambda)$ denote the set of positive integers v such that a $B(K, \lambda; v)$ exists.

A GDD or a BIBD is said to be *resolvable* if its blocks can be partitioned into parallel classes each of which partitions the set of points. We denote them by (K, λ)-RGDD and RB$(k, \lambda; v)$ respectively.

It is well known that the following are the necesary conditions for the existence of a $B(k, \lambda; v)$:

(1) $\lambda(v-1) \equiv 0 \pmod{k-1}$,
(2) $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

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For the existence of an $B(k,\lambda;v)$, a further condition (3) is necessary:

(3) $v \equiv 0 \pmod{k}$.

In fact, (1) and (3) are the necessary conditions for the existence of an RB($k,\lambda;v$) since (2) is implied by (1) and (3). In this paper we shall investigate the sufficiency of (1) and (3) for block size five.

When $k = 3$ and $\lambda = 1$, the existence of an RB($3,1;v$) is famous Kirkman's schoolgirl problem, which was solved by Ray-Chaudhuri and Wilson [12]. The case when $k = 4$ and $\lambda = 1$ was solved in [7]. For $\lambda \geq 1$ and $k = 3,4$, several authors discussed the existence problem and a complete solution was provided in [15], that is, the condition (1) and (3) are also sufficient with the exception of $k = 3, v = 6$ and $\lambda \equiv 2 \pmod{4}$.

When $k = 5$, there are three basic cases: $\lambda = 1,2$ and 4. The necessary conditions (1) and (3) become the following:

(4) $v \equiv 5 \pmod{20}$ for $\lambda = 1$;
(5) $v \equiv 5 \pmod{10}$ for $\lambda = 2$;
(6) $v \equiv 0 \pmod{5}$ for $\lambda = 4$.

In [13] Ray-Chaudhuri and Wilson conjectured that (4) is also sufficient. This conjecture has been verified to be true in [5], [21] and [22] with 109 values of v undecided where 7845 is the largest. In [10] Miao showed the sufficiency of (6) with one exception and 73 possible exceptions of v where 1535 is the largest. The asymptotic result in [8] guarantees the existence of an integer v_0 such that (5) is sufficient whenever $v \geq v_0$. However, this result does not provide any specific value of v_0. The purpose of this paper is to provide such a value, namely $v_0 = 50722395$. As a consequence, (1) and (3) are also sufficient for $k = 5$ and any $\lambda \geq 1$ whenever $v \geq v_0$.

Since we shall need the result on $B(6,1;v)$, we first update the result on its spectrum in Section 2. The number of possible exception of v can be reduced to 88. In Section 3 difference families are used to generate some RB($5,2;v$)'s, especially for $v = 55$. In Section 4 and 5 we use recursive constructions to prove our main result, that is, an RB($5,2;v$) exists whenever $v \equiv 5 \pmod{10}$ and $v \geq 50722395$.

For concepts and results on design theory which are not mentioned in this paper, we refer the reader to [1].

2. Spectrum of $B(6,1;v)$'s

In this section we use the Baer subplane of a projective plane to give some direct constructions for GDDs and then improve the existence results of $B(6,1;v)$'s.

A *projective plane* is considered to be a BIBD $B(k,1;v)$ such that the BIBD contains v blocks (lines). The *order* of the plane is $k-1$. A projective plane $\mathbf{S} = (U,\mathbf{A})$ is said to be a *subplane* of a projective plane $\mathbf{D} = (V,\mathbf{B})$ if $U \subseteq V$ and for each $A \in \mathbf{A}$, there exists a unique $B \in \mathbf{B}$ such that $A \subseteq B$. If $\mathbf{D}$ has order n and S has order $\sqrt{n}$, the subplane S is called a *Baer subplane*. Such a subplane has the following property [2].

Lemma 2.1. *A Baer subplane intersects each line in either $q+1$ points or one point.*

Theorem 2.2. *If q is a prime power, then there exists a $(\{h, q+h\}, 1)$-GDD of type $(q^2-q)^{q+1}(q^2)^h$, in which each block intersects any group of size q^2, where $1 \le h \le q^2 - q$.*

Proof: Let **D** be a projective plane of order q^2 with a Baer subplane S of order q. Choose a point c in S. Let $L_0, L_1, \ldots, L_q$ be the lines of S through c, and $L_{q+1}, \ldots, L_{q+h}$ the other $h(1 \le h \le q^2 - q)$ lines through c. Let L_i' $(i = 0, 1, \ldots, q)$ be the set of $q^2 - q$ points of L_i which are not in S, and $L_i'' = L_i - \{c\}$ for $i = q+1, \ldots, q+h$. Put

$$X = L_0' \cup \cdots \cup L_q' \cup L_{q+1}'' \cup \cdots \cup L_{q+h}'',$$
$$\mathrm{G} = \{L_0', \ldots, L_q', L_{q+1}'', \ldots, L_{q+h}''\}.$$

It is clear that G forms a partition of X, then we call the members of G groups. Let **B** be the set of lines of **D**. Put

$$\mathrm{A} = \{B \cap X : B \in \mathbf{B}, c \notin B\}.$$

The triple $(X, \mathrm{G}, \mathrm{A})$ is a GDD of type $(q^2-q)^{q+1}(q^2)^h$. Now let us show that any block $A \in \mathrm{A}$ has either $q+h$ points or h points. Suppose A^l is a line in **B** containing A. By Lemma 2.1 A^l contains at least one point of the Baer subplane S. If A^l contains one more point of S, then it must contain $q+1$ points of S. A^l intersects $L_0', \ldots, L_q'$ in no point but intersects $L_{q+1}'', \ldots, L_{q+h}''$ in exactly one point each. Thus A contains h points. Otherwise, if A^l contains exactly one point of S, say a point on L_0, then A^l intersects L_0' in no point but intersects other groups in eactly one point each. In this case, A contains $q+h$ points. In either case, A intersects any group of size q^2. The proof is complete. ∎

We could construct some resolvable GDDs by deleting groups of this GDD.

Corollary 2.3. *If q is a prime power, then there exists a $(\{h, q+h\}, 1)$-RGDD of type $(q^2-q)^{q+1}(q^2)^h$, where $0 \le h \le q^2 - q - 1$.*

Proof: Each block intersects the goups of size q^2 of the GDD. Delete one such group. ∎

Corollary 2.4. *If q is a prime power, then there exists a $(q+1, 1)$-GDD of type $(q^2-q)^{q+1}(q^2)^1$.*

Proof: Take $h = 1$ in Theorem 2.2. ∎

Corollary 2.5. *If q is a prime power, then there exists a $(q, 1)$-RGDD of type $(q^2 - q)^{q+1}$.*

Proof: Take $h = 0$ in Corollary 2.3. ∎

If we take $h = q^2 - q$ and add one point c to each group, we obtain a PBD $B(\{q+1, q^2 - q + 1, q^2 + 1\}, 1; q^3 + q^2 - q + 1)$ obtained originally by Wilson [18].

Now we use these GDDs to construct some new BIBDs. We first state the following "Filling in Holes Construction" without proof.

Lemma 2.6. *Suppose there exists a (k, λ)-GDD of type $\{n_1, n_2, \ldots, n_h\}$. Suppose for each i, $1 \leq i \leq h$, there exists a $(k, \lambda; n_i + d)$ containing a subdesign $B(k, \lambda; d)$. Then there exists a $B(k, \lambda; d + \sum_{i=1}^{h} n_i)$.*

Using Wilson's "Fundamental Construction" [18], we get

Lemma 2.7. *Let $d = 1$, or 6. Suppose there exists a $B(6, 1; v)$ with $v = 20t + d$, or $v = 25m + d$, where $0 \leq m \leq t$. If $N(t) \geq 5$, then there exists a $B(6, 1; 120t + 25m + d)$.*

Proof: Give weight 25 to m points of the last group of a TD$(7, t)$, and weight 0 to the remaining points of the group. Give weight 20 to other points of this TD. The input design $(6, 1)$-GDD of type $20^6 25^1$ comes from Corollary 2.4 when taking $q = 5$. Another input design TD$(6, 20)$ comes from four mutually orthogonal latin squares of order 20 [17]. Using Fundamental Construction we get a $(6, 1)$-GDD of type $(20t)^6 (25m)^1$. The conclusion then follows from Lemma 2.6. ∎

For the existence of $B(6, 1; v)$ there are 4 unsolved values of v, see for example [20]. Greig [6] constructed three of them.

Lemma 2.8. $\{246, 486, 5901\} \subseteq B(6, 1)$.

We can further delete three more.

Lemma 2.9. $\{1186, 1516, 1546\} \subseteq B(6, 1)$.

Proof: Apply Lemma 2.7 with $(t, m, d) = (9, 4, 6)$, $(12, 3, 1)$ and $(12, 4, 6)$. Note that $\{76, 106, 186, 241, 246\} \subseteq B(6, 1)$.

Combining all these we can update the result as follows.

Theorem 2.10. *Let $v \geq 6$ be a positive integer. The condition $v \equiv 1, 6 \pmod{15}$ is necessary and sufficient for the existence of a BIB $B(6, 1; v)$, except three nonexiting designs $B(6, 1; 16)$, $B(6, 1; 21)$ and $B(6, 1; 36)$, and 88 possible exceptions $B(6, 1; v)$ where the values of v are shown in Table 2.1.*

46	51	61	81	141	166	171	196	201	226
231	256	261	276	286	291	316	321	336	346
351	376	406	411	436	441	466	471	496	501
526	561	591	616	621	646	651	676	706	711
736	741	766	771	796	801	831	886	891	916
946	1071	1096	1101	1131	1141	1156	1161	1176	1191
1221	1246	1251	1276	1396	1401	1456	1461	1486	1491
1521	1611	1641	1671	1816	1821	1851	1881	1971	2031
2241	2601	3201	3471	3501	4191	4221	5391		

Table 2.1

3. Construction using difference families

There are many constructions using difference families from rings. The following one is basic in this respect. By a ring R we shall mean a commutative ring with an identity in which the identity does not equal to zero. Recall that $U(R)$, the units of R, form a group under ring multiplication.

Let $F = \{B_1, \dots, B_s\}$ be a family of subsets of R. If $B_i = \{b_{i1}, \dots, b_{ik}\}$, define the *development* of B_i and F as follows

$$\text{dev } B_i = \{B_i + g : g \in R\},$$

$$\text{dev } F = \bigcup_{i=1}^{s} \text{dev } B_i.$$

where $B_i + g = \{b_{i1} + g, \dots, b_{ik} + g\}$ for $1 \leq i \leq s$. If $(R, \text{dev } F)$ is a BIBD $B(k, \lambda; v)$, F is called a (v, k, λ)*-difference family*, and denoted by DF$(k, \lambda; v)$. The subset $B_1, \dots, B_s$ are called *base blocks* of the BIBD.

Theorem 3.1. *Let $\lambda \leq k - 1$. Suppose there is a DF$(k, \lambda; q)$, $\{A_1, \dots, A_s\}$, over a ring R such that the base blocks are mutually disjoint. If there are k distinct units $u_i, 0 \leq i \leq k - 1$, such that the differences $u_i - u_j$ $(0 \leq i < j \leq k - 1)$ are all units of R. Then there exists an RB$(k, \lambda; kq)$ containing a subdesign RB$(k, \lambda; k)$.*

Proof: (a) The number s of base blocks A_i is $\lambda(q-1)/k(k-1)$; hence $\sum_{i=1}^{s} |A_i| = s \cdot k = \lambda(q - 1)/(k - 1) < q$ since $\lambda \leq k - 1$, and we may, without loss of generality, assume that $o \notin A_i$ for $i = 1, 2, \dots, s$. Put $V = R \times I_k$ with $I_k = \{0, 1, \dots, k - 1\}$. As some base blocks for an RB$(k, \lambda; kq)$ we choose

$$B_j^i = A_j \times \{i\} = \{(a_i^j, i), \dots, (a_k^j, i)\} \quad (i \in I_k; j = 1, 2, \dots, s),$$

where $A_j = \{a_i^j, \dots, a_k^j\}$. Up to now we have $ks = \lambda(q-1)/(k-1)$ base blocks, but we need $\lambda(kq - 1)/(k - 1)$ ones.

(b) In order to get further base blocks we put

$$C_x = \{(u_0,0),(u_i,1),\dots,(u_{k-1},k-1)\}\cdot x, x \in R.$$

Here, of course, $(u,i)\cdot x$ means (ux,i). We have q new base blocks and in total there are $\lambda(q-1)/(k-1)+\lambda q = \lambda(kq-1)/(k-1)$ base blocks B_j^i and λC_x, as desired, where λC_x means that each base block C_x is taken λ times. Now replace the base blocks B_j^i by $u_i B_j^i$, we have the set S of new base blocks

$$S = \{u_i B_j^i : i \in I_k, j = 1,2,\dots,s\} \cup \{\lambda C_x : x \in R\}.$$

The pure differences arise all from the blocks $u_i B_j^i$, and the mixed differences all from λC_x. By hypothesis,

$$\sum_j \Delta_{ii}(u_i B_j^i) = \sum_j u_i \Delta_{ii} B_j^i = u_i \sum_j \Delta A_j = \lambda(R-\{0\}).$$

Furthermore, for $i < j$,

$$\Delta_{ij}\{\lambda C_x : x \in R\} = (u_i - u_j)\cdot(\lambda R) = \lambda R.$$

Hence $\Delta_{ii}S = \lambda(R-\{0\})$, $\Delta_{ij}S = \lambda R$ for $i \neq j$, and $(V, \text{dev } S)$ is a BIBD $B(k,\lambda;kq)$. It remains to show that the BIBD is resolvable.

(c) We have to partition the blocks into $r = \lambda(kq-1)/(k-1)$ parallel classes. As first parallel class P_0 take all blocks $u_i B_j^i$ and the blocks C_x where x is distinct from all $a_i^j (i = 0,\dots,k-1;\ j = 1,\dots,s)$. The number of blocks in P_0 is $ks + (q-ks) = q$.

The points in these blocks are

$$(u_i a_i^j, i),\dots,(u_i a_k^j, i) \quad (i = 0,\dots,k-1; j = 1,\dots,s) \text{ and}$$
$$(u_0 x, 0),\dots,(u_{k-1}x, k-1) \quad \text{where } x \neq a_j^j \text{ for all } i,j.$$

Obviously every point of $R \times I_k$ occurs exactly once; i.e., P_0 is a parallel class. Hence some parallel classes are given by

$$P_g = \tau_g P_0 \text{ with } \tau_g : (x,i) \mapsto (x+g,i), g \in R.$$

That is $P_g = \{\tau_g(B) : B \in P_0\}$.

We construct still more parallel classes $Q_x = \{\tau_g C_x : g \in R\}$ with $x \in \bigcup_{j=1}^s A_j$ and $R_x = \{\tau_g C_x : g \in R\}$ with $x \in R - \bigcup_{j=1}^s A_j$. Obviously either Q_x or R_x is a parallel class. The total number of parallel classes P_x, Q_x and R_x is $q + \lambda ks + (\lambda-1)(q-ks) = \lambda q + ks = \lambda(kq-1)/(k-1) = r$, as desired. Furthermore, $(C_0, \lambda C_0)$ is a subdesign RB$(k,\lambda;k)$. This completes the proof. ∎

In the special case of a field F, $U(F) = F - \{0\}$, and we have

Theorem 3.2. *Let q be a prime power and $\lambda \leq k-1$. Suppose there is a DF$(k,\lambda;q)$ $\{A_1,\ldots,A_s\}$ in GF(q) such that the base blocks are mutually disjoint. Then there exists an RB$(k,\lambda;kq)$ containing a subdesign RB$(k,\lambda;k)$.*

Proof: In this case, we choose k distinct elements $u_0,\ldots,u_{k-1}$ of GF$(q)-\{0\}$ to define C_x and the proof of Theorem 3.1 remains valid. ∎

It should be noted that Schellenberg [14] proved Theorem 3.1 in the case $\lambda = 1$ and Ray-Chaudhuri and Wilson [13] proved Theorem 3.2 also in the case $\lambda = 1$.

Now we turn to find such block-disjoint difference families. The following examples were first given by Wilson [19] without mentioning the block-disjointness.

Lemma 3.3. *Let $q = ke+1$ be a prime power. Let ω be a primitive element and H the multiplicative subgroup of order k of GF(q). Then $\{A_0,\ldots,A_{e-1}\}$ form a block-disjoint DF$(k,k-1;q)$ where $A_j = \omega^j H$, $j = 0,\ldots,e-1$.*

Lemma 3.4. *Let k be odd and $q = 2ks+1$ a prime power. Let ω be a primitive element and H the multiplicative subgroup of order k of GF(q). Then $\{A_1,\ldots,A_s\}$ form a block-disjoint DF$(k,(k-1)/2;q)$ where $A_j = \omega^j H$, $j = 1,\ldots,s$.*

Corollary 3.5. *There is an RB$(6,5;42)$.*

Proof: Apply Theorem 3.2 with $q = 7$. The required DF$(6,5;7)$ comes from Lemma 3.3. ∎

Corollary 3.6. *Let $q \equiv 1 \pmod{10}$ be a prime power. Then there is an RB$(5,2;5q)$ containing a subdesign RB$(5,2;5)$.*

Proof: Apply Theorem 3.2 with $k = 5$. The required DF$(5,2;q)$ comes from Lemma 3.4. ∎

Corollary 3.7. *There is an RB$(5,2;55)$ with a subdesign RB$(5,2;5)$.*

Proof: Apply Corollary 3.6 with $q = 11$. ∎

Note that the existence of RB$(6,5;42)$ and RB$(5,2;55)$ were listed as unknown cases in [9].

4. Existence of RB$(5,2;v)$ for $v \equiv 5,25 \pmod{30}$

In this section we shall prove the existence of an RB$(5,2;v)$ for $v \equiv 5,25 \pmod{30}$ and $v \geq 50722395$. We shall work modulo 300. When $v \equiv 5 \pmod{20}$, the conclusion comes from the result on RB$(5,1;v)$. For other residue classes we first consider the case $v \equiv 55,155 \pmod{300}$, then the remaining classes.

Let $(X,\mathrm{G},\mathrm{A})$ be a (k,λ)-GDD of type T. For $G \in \mathrm{G}$, let P_G be a subset of A such that the blocks in P_G form a partition of $X-G$. P_G is called a *holey parallel class with hole G*. The GDD is called a *(k,λ)-frame* with type T if A can be partitioned into holey parallel classes.

The following constructions can be found in [10], [16].

Lemma 4.1 (Fundamental Frame Construction). *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a GDD with* $\lambda = 1$, *and let* $w : X \mapsto Z^{+} \cup \{0\}$ *be a weight function. For each* $A \in \mathcal{A}$, *suppose that we have a* (k, λ)*-frame of type* $\{w(x) : x \in A\}$. *Then there exists a* (k, λ)*-frame of type* $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.

Lemma 4.2 (Inflation by TDs). *Suppose there exists a* (k, λ)*-frame of type* T *and an RTD*(k, m). *Then there is a* (k, λ)*-frame of type* $\{mt : t \in T\}$.

(k, λ)-frames can be used to obtain RBIBDs as follows [20, Construction 3.9].

Lemma 4.3. *Suppose there exists a* (k, λ)*-frame of type* $\{t_1, \ldots, t_n\}$, *and let* $w > 0$. *For each* i, $1 \leq i \leq n-1$, *suppose there is an RB*$(k, \lambda; t_i + w)$ *containing a subdesign RB*$(k, \lambda; w)$. *If an RB*$(k, \lambda; t_n + w)$ *also exists, then there is an RB*$(k, \lambda; v)$ *containing a subdesign RB*$(k, \lambda; u)$ *where* $v = w + \sum_{i=1}^{n} t_i$, *and* $u = t_n + w$. *Furthermore, if the RB*$(k, \lambda; t_n + w)$ *also contains a subdesign RB*$(k, \lambda; w)$, *then the resulting RBIBD contains more subdesigns as* u *may take* w *or* $w + t_i$ *for* $1 \leq i \leq n$.

Lemma 4.4. *The following GDDs exist:*

(1) *a* $(6, 1)$*-GDD of type* 5^{24};
(2) *a* $(6, 1)$*-GDD of type* 5^{25};
(3) *a* $(\{6, 25\}, 1)$*-GDD of type* $5^{24}24^{1}$; *and*
(4) *a* $(\{6, 25\}, 1)$*-GDD of type* $5^{25}24^{1}$.

Proof: Deleting one point from a B$(6, 1; 121)$ and a B$(6, 1; 126)$ respectively to get the GDDs in (1) and (2). Adding one point to groups of a TD$(6, 24)$ and then deleting another point of TD$(6, 24)$, we get the GDD in (3). The last GDD is obtained by deleting one point from a TD$(6, 25)$. ∎

Lemma 4.5. *There exist* $(5, 2)$*-frames of type* 2^{6}, 10^{6} *and* 10^{25}.

Proof: $(5, 2)$-frame of type 2^{6} exists from [11]. Giving weight 5 to each point of this frame and applying Lemma 4.2, we obtain a $(5, 2)$-frame of type 10^{6} since an RTD$(5, 5)$ exists. Also giving weight 2 to each point of a $(6, 1)$-GDD of type 5^{25} and applying Lemma 4.1, we obtain a $(5, 2)$-frame of type 10^{25} since a $(5, 2)$-frame of type 2^{6} exists. ∎

Then we have

Theorem 4.6. *Let* $u \equiv 55, 155 \pmod{300}$ *and* $u \notin A$. *Then there is an RB*$(5, 2; u)$ *with a subdesign RB*$(5, 2; 5)$, *where* $A = \{455, 1955, 2255, 2555, 2755, 3355, 4355, 4655, 6755, 7055, 7955, 8855, 9455, 10955, 12455, 14555\}$.

Proof: Deleting one point from a B$(6, 1; v)$ to obtain a $(6, 1)$-GDD of type $5^{(v-1)/5}$. Give weight 10 to each point of this GDD. Then a $(5, 2)$-frame of type $50^{(v-1)/5}$ exists from Lemma 4.1 since a $(5, 2)$-frame of type 10^{6} exists. By Corollary 3.7 we have an RB$(5, 2; 55)$ with a subdesign RB$(5, 2; 5)$. Apply Lemma 4.3, we obtain an RB$(5, 2; 10v - 5)$ containing a subdesign RB$(5, 2; 5)$.

When $v \equiv 6, 16 \pmod{30}$, and $v \geq 36$, Theorem 2.10 guarantees the existence of an RB$(5,2; 10v-5)$ except v =*36*, 46, *166*, 196, 226, 256, 276, *286*, *316*, 336, *346*, *376*, *406*, 436, 466, *496*, *526*, *616*, *646*, 676, 706, *736*, *766*, 796, 886, *916*, 946, 1096, *1156*, *1176*, 1246, *1276*, *1396*, 1456, *1486*, *1816*. By Corollary 3.6, an RB$(5,2; 10v-5)$ also exists for the italic v. Since an RB$(5,2; 55)$ exists from Corollary 3.7 and an RB$(5,2; 155)$ exists from Corollary 3.6, the conclusion then follows. ∎

Now we use this result to consider the remaining residue classes modulo 300, but we need some preliminaries. Let $N(t)$ be the maximum number of mutually orthogonal Latin squares of order t.

Lemma 4.7. *Let* $N(t) \geq 24$. *Then there is a* $(\{6,25\},1)$*-GDD of type* $(5t)^{24}(5u)^1(24v)^1$, *where* $0 \leq u, v \leq t$.

Proof: Give weight 24 to v points of the last group of a TD$(26,t)$ and weight 0 to other points of the group. Give weight 5 to u points of the second last group and weight 0 to the other points of the group. The remaining points of the TD all receive weight 5. Apply Wilson's Fundamental Construction with the ingredient GDDs from Lemma 4.4. We obtain the required GDD. ∎

Lemma 4.8. *Let* $N(t) \geq 24$. *Then there is a* $(5,2)$*-frame of type* $(50t)^{24}(50u)^1(240v)^1$, *where* $0 \leq u, v \leq t$.

Proof: Give weight 10 to each point of the GDD in Lemma 4.7. Apply Lemma 4.1 with the input frames frum Lemma 4.5. ∎

For ease of notation we write $v \in \mathrm{RB}_w(k,\lambda)$ for the fact that there is an RB$(k,\lambda; v)$ containing a subdesign RB$(k,\lambda; w)$. We also write $v \in \mathrm{RB}(k,\lambda)$ for the fact that an RB$(k,\lambda; v)$ exists.

Lemma 4.9. *Suppose* $N(t) \geq 24$ *and* $0 \leq u, w \leq t$. *If* $50t+5 \in \mathrm{RB}_5(5,2)$, *one of* $50u+5$ *and* $240w+5$ *is in* $\mathrm{RB}_5(5,2)$ *and the other is in* $\mathrm{RB}(5,2)$, *then* $v = 1200t + 50u + 240w + 5 \in \mathrm{RB}(5,2)$. *Furthermore, if both* $50u+5$ *and* $240w+5$ *are in* $\mathrm{RB}_5(5,2)$, *then* $v \in \mathrm{RB}_5(5,2)$.

Proof: Apply Lemma 4.3 with the $(5,2)$-frame from Lemma 4.8. ∎

Lemma 4.10. $50u+5 \in RB_5(5,2)$ *whenever* $u \equiv 1 \pmod{6}$ *and* $u \geq 73$, *or* $u \equiv 3 \pmod{6}$ *and* $u \geq 225$.

Proof: See Theorem 4.6. ∎

Let $O_r = \max\{v : v \text{ odd and } N(v) < r\}$.

Lemma 4.11. $O_{30} < 60000$.

Proof: See [4]. ∎

We are now in a position to handle the remaining classes.

Lemma 4.12. *Suppose there is an infinite series of positive integers* $\{t_i\}_{i=0,1,2,\ldots}$ *such that for each* $i, i = 0,1,2,\ldots$

(1) $N(t_i) \geq 24, t_i \equiv 1 \pmod 6, t_i \geq 73;$
(2) $50t_i + 5 \in RB_5(5,2);$
(3) $t_i \leq t_{i+1} \leq (25t_i - 73)/24.$

If $240w + 5 \in RB(5,2)$, *then* $v \in RB(5,2)$ *whenever* $v \equiv 240w + 55 \pmod{300}$ *and* $v \geq 1200t_0 + 3655 + 240w$.

Proof: For any given $t \in \{t_i\}$, take $u \equiv 1 \pmod 6$ and $73 \leq u \leq t$. Apply Lemma 4.9, we obtain $v = 1200t + 50u + 240w + 5 \in RB(5,2)$ since all the required RBIBDs exist from the assumption and Lemma 4.10. For fixed t and w, if we let u take all values for $73 \leq u \leq t$, we get an interval $[1200t + 50\cdot 73 + 240w + 5, 1250t + 240w + 5]_{300}^{240w+55}$ contained in $RB(5,2)$, where $[m,n]_a^b = \{x \in Z : m \leq x \leq n, x \equiv b \pmod a\}$. Condition (3) gurantees that any two consecutive integers overlap. Hence $v \in RB(5,2)$ for $v \equiv 240w + 55 \pmod{300}$ whenever $v \geq 1200t_0 + 50\cdot 73 + 240w + 5$. The proof is complete. ∎

We define a series $\{t_i\}_{i=0,1,2,\ldots}$ as in Appendix 1 for $t_i \leq 60001$ and for $t_i \geq 60001$ we recursively define $t_{i+1} = t_i + 6$. With the help of [3] it is readily checked that the series satisfies the conditions of Lemma 4.12, where $t_0 = 601$.

Theorem 4.13. *There exists an* $RB(5,2;v)$ *for* $v \equiv 25 \pmod{30}$ *whenever* $v \geq 726295$.

Proof: By Lemma 4.6 and the result on $RB(5,1;v)$, we need only to consider the cases $v \equiv 115, 175, 235$ and $295 \pmod{300}$. Apply Lemma 4.12 with $w = 4,3,2$ and 6, respectively. Notice that $\{965, 725, 485, 1445\} \subseteq RB(5,1)$ (see [22]). ∎

We have the following lemma similar to Lemma 4.12.

Lemma 4.14. *Suppose there is an infinite series of positive integers* $\{t_i\}_{i=0,1,2,\ldots}$ *such that for each* $i, i = 0,1,2,\ldots$

(1) $N(t_i) \geq 24, t_i \equiv 3 \pmod 6, t_i \geq 225;$
(2) $50t_i + 5 \in RB_5(5,2);$
(3) $t_i \leq t_{i+1} \leq (25t_i - 225)/24.$

If $240w + 5 \in RB(5,2)$, *then* $v \in RB(5,2)$ *whenever* $v \equiv 240w + 155 \pmod{300}$ *and* $v \geq 1200t_0 + 240w + 11255$.

Proof: Similar to that of Lemma 4.12, noticing in this case $u \equiv 3 \pmod 6$ and $225 \leq u \leq t_i$. The existence of an $RB_5(5,2;50u)$ comes from Lemma 4.10, and $v = 1200t_i + 50u + 240w + 5$ gives $v \equiv 240w + 155 \pmod{300}$. ∎

We also define a series $\{t_i\}_{i=0,1,2,\ldots}$ as follows: for $t_i \leq 60003$ we take t_i in turn to be

$$41175 = 27 \times 25 \times 61, \quad 42309 = 27 \times 1567, \quad 43983 = 181 \times 243,$$
$$45225 = 27 \times 25 \times 67, \quad 46899 = 193 \times 243, \quad 47331 = 27 \times 1753,$$
$$51219 = 27 \times 1897, \quad 53325 = 27 \times 1975, \quad 55431 = 27 \times 2053,$$
$$57699 = 27 \times 2137, \quad 60003$$

and for $t_i \geq 60003$ we recursively define $t_{i+1} = t_i + 6$. It is readily checked that the series satisfies the conditions of Lemma 4.14, where $t_0 = 41175$.

Theorem 4.15. *There exists an* RB$(5,2;v)$ *for* $v \equiv 5 \pmod{30}$ *whenever* $v \geq 50722395$.

Proof: By Lemma 4.6 and the result on RB$(5,1;v)$, we need only to consider the cases $v \equiv 35, 95, 215$ and $275 \pmod{300}$. Apply Lemma 4.14 with $w = 2, 6, 4$ and 3, respectively.

5. Existence of RB$(5,2;v)$ for $v \equiv 15 \pmod{30}$

We first consider the case $v \equiv 255 \pmod{300}$, and then the remaining cases. The following construction is a slight generalization of a theorem due to Harrison see, for example, [20]).

Theorem 5.1. *If there exist an* RB$(k,1;kq_1)$, *an* RB$(k,\lambda;kq_2)$ *and an* RTD(k,q_2), *then there exists an* RB$_{kq_2}(k,\lambda;kq_1q_2)$.

Corollary 5.2. RB$_5(5,2) \supseteq \{4455, 37455, 47355\}$.

Proof: Apply Theorem 5.1 with $k = 5$, $\lambda = 2$, $q_1 = 81, 681, 861$ and $q_2 = 11$. ∎

A subset of blocks in a BIBD is called a *partial parallel class* if the subset consists of pairwise disjoint blocks. The following construction can be found in [20].

Theorem 5.3. *Suppose* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *are an* RB$(k,\lambda;v)$ *and a* B$(k,\lambda;v)$ *respectively. Suppose* $\mathcal{B}$ *can be partitioned into* s *disjoint partial parallel classes where* $s \leq \lambda(u+v-2)/k-1)$. *If there is an* RTD$(k,v)$, *then there exists an* RB$(k,\lambda;uv)$ *containing a subdesign* RB$(k,\lambda;u)$.

Corollary 5.4. $1155 \in$ RB$_5(5,2)$.

Proof: Let $X = Z_{21}$, and the disjoint base blocks be

$$\{8,11,12,17,19\}, \{15,9,14,18,7\}.$$

Then we have a $B(5,2;21)$ with 21 partial parallel classes. Apply Theorem 5.3 with $1155 = 55 \times 21$. Note that $N(21) \geq 4$ from [3] and $21 \leq 2 \times (21 + 55 - 2)/(5-1) = 37$. ∎

We also need a preliminary result.

Lemma 5.5. *There is an* RB$(5,2;240w+5)$ *containing a subdesign* RB$(5,2;5)$ *whenever* $w \equiv 0 \pmod 5$.

Proof: Theorem 4.3. ∎

The following lemma is similar to Lemma 4.12.

Lemma 5.6. *Suppose there is an infinite series of positive integers* $\{t_i\}_{i=0,1,2,\ldots}$ *such that for each* i, $i = 0,1,2,\ldots$

(1) $N(t_i) \geq 24$, $t_i \equiv 1 \pmod 6$, $t_i \geq 73$;
(2) $50t_i + 5 \in \mathrm{RB}_5(5,2)$;
(3) $t_i \leq t_{i+1} \leq (6t_i - 4)/5$.

Then $v \in$ RB$(5,2)$ *whenever* $v \equiv 50u + 5 \pmod{1200}$ *and* $v \geq 1200t_0 + 50u + 5$.

Proof: For fixed u and $t \in \{t_i\}$ we change w such that $0 \leq w \leq t-4$ and $w \equiv 0 \pmod 5$. Apply Theorem 4.6, we obtain $[1200t + 50u + 5, 1200t + 50u + 240(t-4) + 5]_{1200}^{50u+5} \subseteq$ RB$(5,2)$. Since $t_i \leq t_{i+1} \leq (6t_i - 4)/5$, we have $1200t_{i+1} + 50u + 5 \leq 1200t_i + 50u + 5 + 240(t_i - 4)$. Hence the assertion. ∎

Theorem 5.7. $v \in$ RB$(5,2)$ *whenever* $v \equiv 255 \pmod{300}$ *and* $v \geq 1200555$.

Proof: Apply Lemma 5.6 with $50u + 5 = 37455, 47355, 4455, 1155$, and the series before Theorem 4.13, where $t_0 = 961$. ∎

For the remaining cases, we first prove.

Lemma 5.8. *Suppose there is an infinite series of positive integers* $\{t_i\}_{i=0,1,2,\ldots}$ *such that for each* i, $i = 0,1,2,\ldots$

(1) $N(t_i) \geq 24$, $t_i \equiv 5 \pmod 6$, $t_i \geq 24011$;
(2) $50t_i + 5 \in \mathrm{RB}_5(5,2)$;
(3) $t_i \leq t_{i+1} \leq (25t_i - 24011)/24$.

If $240t + 5 \in$ RB$(5,2)$, *then* $v \in$ RB$(5,2)$ *whenever* $v \equiv 255 + 240w \pmod{300}$ *and* $v \geq 1200t_0 + 240w + 1200555$.

Proof: Apply Lemma 4.9 with $t = t_i$, $i = 0,1,2,\ldots, 24011 \leq u \leq t_i$, $u \equiv 5 \pmod 6$ to obtain $[1200t_i + 240w + 1200555, 1250t_i + 240v + 5]_{300}^{240w+255} \subseteq$ RB$(5,2)$. Since $t_i \leq t_{i+1} \leq (25t_i - 24011)/24$, we have $1200t_{i+1} + 240w + 1200555 \leq 1250t_i + 240v + 5$. Hence the assertion. ∎

We again define a series $\{t_i\}_{i=0,1,2,\ldots}$ as follows: when $t_i \leq 60005$, see Appendix 2; when $t_i \geq 60005$, we recursively define $t_{i+1} = t_i + 6$. Then it is readily checked that this series satisfies the conditions of Lemma 5.8 where $t_0 = 41627$.

Theorem 5.9. *There exists an* RB$(5,2;v)$ *for* $v \equiv 15 \pmod{30}$ *whenever* $v \geq 50722395$.

Proof: By Theorem 5.7 and the result on RB$(5,1;v)$, we need only to consider the cases $v \equiv 15, 75, 135$ and $195 \pmod{300}$. Apply Lemma 5.8 with $w = 4, 3, 2$ and 6, repectively. ∎

6. Conclusion

Combining Theorems 4.13, 4.15 and 5.9 we obtain the main result of this paper.

Theorem 6.1. *The necessary condition for the existence of an* RB$(5,2;v)$, *namely* $v \equiv 5 \pmod{10}$, *is also sufficient whenever* ≥ 50722395.

With the result on RB$(5,1;v)$ [22] and RB$(5,4;v)$ [10] we further obtain the following

Theorem 6.2. *The necessary conditions for the existenxce of an* RB$(5,\lambda;v)$, *namely* $v \equiv 0 \pmod 5$ *and* $\lambda(v-1) \equiv 0 \pmod 4$, *are also sufficient for any positive integer* λ *whenever* $v \geq 50722395$.

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