Greedy Ordered Sets With No Four-Cycles

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Abstract. We characterize "effectively" all greedy ordered sets, relative to the jump number problem, which contain no four-cycles. As a consequence, we shall prove that $O(P) = G(P)$ whenever P is a greedy ordered set with no four-cycles.

Introduction

Throughout this paper, P denotes a finite ordered set. *A linear extension* L of P is a total ordering of the elements of P such that $x < y$ in P implies $x < y$ in L. Suppose that $L = \{x_1 < x_2 < \dots\}$ is a linear extension of P. A *jump* in L is a pair of consecutive elements in L, (x_i, x_{i+1}) which are noncomparable in P. We denote $s(P, L)$ the *number of jumps* in L. The *jump number* of P, denoted by $s(P)$, is the minimum of $s(P, L)$, over all linear extensions L of P. The problem of minimizing jumps, called *the jump number problem*, has been widely studied, in part due to its applications to scheduling. A linear extension L of P is called *optimal* if $s(P, L) = s(P)$. We denote $O(P)$ the set of all optimal linear extensions of P. One of the ways of constructing a linear extension of P, which comes to mind is to minimize the number of jumps in each step of the construction. Choose x_1 any minimal element in P. Suppose $x_1, x_2, \dots, x_i$ are already defined, choose x_{i+1} minimal in $P - \{x_1, \dots, x_i\}$ such that $x_{i+1} \geq x_i$ in P, whenever possible. A linear extension constructed by such method is called a *greedy linear extension*. We denote by $G(P)$ the set of all greedy linear extensions of P. Call an ordered set P *greedy* if every greedy linear extension of P is optimal. Kierstead [4] has proved that the problem of deciding whether or not an ordered set P is greedy is NP-complete. (See also Bouchitté and Habib [1]) On the other hand there are classes of ordered sets for which the subclasses of greedy ordered sets are effectively characterized. For instance Rival [5] proved that for an N-free ordered set P, $O(P) = G(P)$. Rival and Zaguia [6] gave a simple characterization of greedy ordered sets of length one. Also Ghazal, Sharary and Zaguia [3] characterized greedy ordered sets which are interval orders.

Our first theorem in this paper contains a characterization of the greedy ordered sets which contain no subsets isomorphic to a four-cycle as illustrated in Figure 1(a). These are the greedy ordered sets with no four-element subsets $\{a, b, c, d\}$ such that $a < c$, $a < d$, $b < c$ and $b < d$ are the only comparabilities among these elements. As a consequence of this characterization we shall prove the following.

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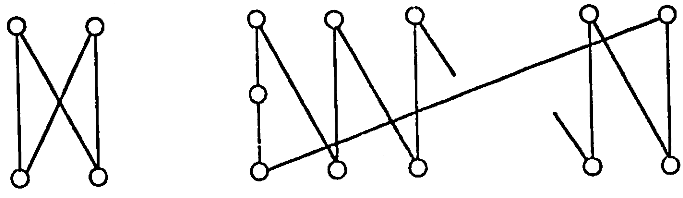
Theorem 2. *Let P be an ordered set with no four-cycle. Then P is greedy if and only if $O(P) = G(P)$.*

The main tool used to prove our results is the "chain interchange" technique developed by Rival and Zaguia [6]. Also, notice that Theorem 2 is much related to a result of El-Zahar and Rival [2] which states that if an ordered set P does not contain cycles as illustrated in Figure 1(b), then $G(P) \supseteq O(P)$.



(a) (b)

Figure 1.

For a and b in P, we say that a covers b, denoted by $a \succ b$, if $a > b$ in P and if $a > c \geq b$ then $b = c$; we call a an *upper cover* of b. (Also $b \prec a$ and b is a *lower cover* of a.) Let P and S be ordered sets and let E be a subset of the edges of the diagram of S; say that $S_E = (S, E)$ is a *subdiagram* of P if there is a subset of P isomorphic to S in which each of the edges corresponding to E is a covering edge in P. Usually we write S instead of S_E. Whenever we write $N = \{a, b, c, d\}$, $W = \{a, b, c, d, e\}$, $M = \{a, b, c, d, e, f\}$, $X = \{a, b, c, d, e\}$ we shall mean the diagrams as illustrated in Figure 2. Let P and T be ordered sets and let S be a subdiagram of P. We say that S has a *T-completion* in P if there is a subdiagram T' of P isomorphic to T and containing S. We say that a subdiagram $N = \{a, b, c, d\}$ of an ordered set P has a *good W-completion*, if it has a W-completion $\{a, b, x, c, d\}$ (with $x > a$) which, in turn, has no M-completion $\{y, a, b, x, c, d\}$, with $y < x$ in P.

Theorem 1. *Let P be an ordered set with no four-cycles. Then P is greedy if and only if every subdiagram N of P has a good W-completion.*

Notice that the same characterization holds for bipartite greedy ordered sets. (See Rival and Zaguia [6].) Let L be a linear extension of P. The jumps of L induce a decomposition of L into chains. Thus $L = C_1 \oplus C_2 \oplus \ldots$, where $(\sup C_i, \inf C_{i+1})$, $i = 1, 2, \ldots$, are the jumps of L. Call an element a of P *accessible* if $\{x \in P : x \leq a\}$ is a chain in P. Say that an element a is *maximal accessible* if it is accessible and every upper cover of a is not. Notice that if $L = C_1 \oplus C_2 \oplus \ldots$ is a greedy linear extension of P then $\sup C_i$ is maximal accessible in $\cup_{k \geq i} C_k$. An easy fact which we shall use frequently, is that whenever P is greedy and $L = C_1 \oplus C_2 \oplus \ldots$ is a greedy linear extension, then $\cup_{k \geq i} C_k$ is greedy for every

$i = 1, 2, \ldots$. For chains C, C' in P we say that an $N = \{a, b, c, d\}$ in $C \cup C'$ is *minimal* if $b, d \in C$, $a, c \in C'$ and, for every $N = \{a_1, b_1, c_1, d_1\}$ in $C \cup C'$ then $c < c_1$ and $b < b_1$ in P. Finally, we shall use the term C_4*-free* to call ordered sets with no four-cycles.

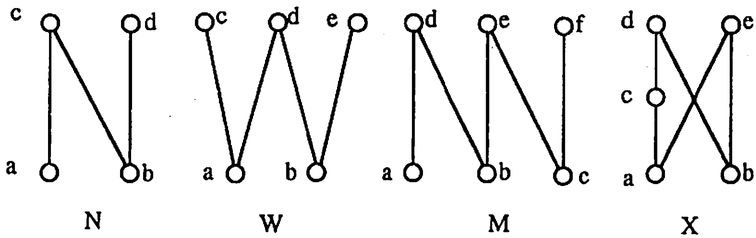


Figure 2.

Now before we get to the proofs of our main results, here are some preliminary lemmas. But first, we state the following technical theorem which is important in our proofs.

Theorem 3. *(Rival and Zaguia [6]). An ordered set P is greedy if and only if every subdiagram N minimal in $C_i \cup C_{i+1}$, for some greedy linear extension $L = C_1 \oplus C_2 \oplus \ldots$ of P, has either a W-completion or an X-completion in $C_i \cup C_{i+1} \cup \{x\}$, for some minimal element x in $\cup_{k>i+1} C_k$ such that either $x > \sup C_i$ or x does not cover y in P for every y in C_i.*

Lemma 1. *Let P be a C_4-free greedy ordered set. Then every subdiagram N of P has a W-completion.*

Proof of Lemma 1: We proceed by induction on, $|P|$, the cardinality of P. Let $\{a, b, c, d\}$ be an N subdiagram of P. Suppose that there exists a greedy chain C of P which contains neither a nor b. By induction hypothesis, $\{a, b, c, d\}$ has a W-completion in $P - C$, and therefore in P. ($|P - C| < |P|$ and $P - C$ is a C_4-free greedy ordered set.) Thus, we may assume that every greedy chain of P contains either a or b. We consider three different cases.

case 1. c has a lower cover c' in P, with $c' \neq a$ and $c' \neq b$.

Obviously c' is noncomparable to a, b and d. (If $c' < d$ then $\{c', b, c, d\}$ is a four-cycle in P.) Let C_a be any greedy chain of P containing a. Since $\{c', b, c, d\}$ is an N in $P - C_a$, then by induction hypothesis it has a W-completion $\{c', b, y, c, d\}$ in $P - C_a$. Now consider any greedy chain C_b in P, containing b. It is easy to show that $\{a, c', c, y\}$ is an N in $P - C_b$, and thus it has a W-completion $\{a, c', x, c, y\}$ in $P - C_b$. Since x is noncomparable to c, b and d. (If $x > b$ or $x > d$ then $\{a, b, c, x\}$ is a C_4; if $x \leq b$ then $a \leq b$ and if $x \leq d$ then $a \leq d$.) then $\{a, b, x, c, d\}$ is a W-completion in P.

case 2. d has a lower cover d' with $d' \neq b$.
Notice first that d' is noncomparable to a and b. (b and d' are both lower covers of d, and if $d' \geq a$ then $d \geq a$, also $d' < a$ implies that $\{d', b, c, d\}$ is a C_4.) Since we are assuming that every greedy chain in P contains either a or b, then d' is not accessible. Thus $D(d') = \{x \in P : x \leq d'\}$ is not a chain in P. So, let y_1 and y_2 be two noncomparable elements in $D(d')$. We may assume that y_1 and y_2 are accessible in P. Since very greedy chain in P contains either a or b, then each of y_1 and y_2 is comparable to either a or b. In both cases it will imply that $y_1 < c$ and $y_2 < c$. But then $\{y_1, y_2, c, d\}$ is a four-cycle in P. Which is a contradiction.
case 3. a and b are the only lower covers of c, and b is the only lower cover of d. In this case d is accessible in P, and thus let C_1 be a greedy chain in P, containing $\{b, d\}$. Also c is accessible in $P - C_1$, and thus let C_2 be a greedy chain in $P - C_1$ containing $\{a, c\}$. Consider any greedy linear extension $L = C_1 \oplus C_2 \oplus \ldots$ of P starting with $C_1 \oplus C_2$. Obviously $\{a, b, c, d\}$ is a minimal N in $C_1 \cup C_2$. Therefore, according to Theorem 3, there exists x such that $\{a, b, x, c, d\}$ is a W-completion of $\{a, b, c, d\}$ in P. This completes the proof of Lemma 1.

Lemma 2. *Let P be a C_4-free greedy ordered set. Then for every subdiagram $N = \{a, b, c, d\}$ of P, all the lower covers of c and d are accessible.*

Proof of Lemma 2: According to Lemma 1, $\{a, b, c, d\}$ has a W-completion $\{a, b, x, c, d\}$ in P. If b is not accessible, then there exist two noncomparable elements b_1, b_2 such that $b_1 < b$ and $b_2 < b$. Therefore $\{b_1, b_2, c, d\}$ is a four-cycle in P. Similarly, if a is not accessible then there exist two noncomparable elements a_1 and a_2 such that $a_1 < a$ and $a_2 < a$. But then $\{a_1, a_2, x, c\}$ will be a four-cycle. Now if c' is any lower cover of c then the same argument applied to the $N = \{c', b, c, d\}$ shows that c' is accessible. Similarly, if d' is a lower cover of d then by considering the $N = \{d', b, d, c\}$ we conclude that d' is accessible.

Proof of Theorem 1: We proceed by induction on $|P|$, the cardinality of P. We assume that if $|P'| < |P|$ and P' is a C_4-free greedy ordered set then every N in P' has a good W-completion.
Claim 1. If $N = \{a, b, c, d\}$ has no good W-completion in P then the only lower covers of c in P, are a and b.
To prove the claim, we assume for a contradiction that $y \prec c$ with $y \neq a$ and $y \neq b$. According to Lemma 2, y is accessible. Let C_y be a greedy chain containing y. By the induction hypothesis $\{a, b, c, d\}$ has a good W-completion $\{a, b, x, c, d\}$ in $P - C_y$. Since this W-completion is not good in P, then there exists x' such that $\{x', a, b, x, c, d\}$ is an M subdiagram of P. Thus x' is in C_y. Now if $x' \leq y$ then $\{x', a, x, c\}$ is a four-cycle and if $x' > y$ then $\{y, a, x, c\}$ is a four-cycle, which is a contradiction.
Claim 2. If $N = \{a, b, c, d\}$ has no good W-completion in P, then there is $x \prec d$ with $x \neq b$. We assume for a contradiction that b is the only lower cover of d. Notice that a and b are the only lower covers of c. Since d covers only b and since

b is accessible (by Lemma 2), then d is accessible. Thus we consider a greedy chain C_1 containing $\{b, d\}$. Now in $P - C_1$, c is accessible and thus we consider a greedy chain C_2 in $P - C_1$, containing $\{a, c\}$. Let $L = C_1 \oplus C_2 \oplus \ldots$ be a greedy linear extension of P starting with $C_1 \oplus C_2$. According to Theorem 3, there must exist an element x minimal in $\cup_{i>2} C_i$ such that $\{a, b, x, c, d\}$ is a W-completion of $N = \{a, b, c, d\}$. Since it is not a good W-completion, then there is $y \prec x$ in P such that $\{y, a, b, x, c, d\}$ is an M subdiagram of P. But x is minimal in $\cup_{i>2} C_i$, thus $y \in C_1 \cup C_2$. Therefore either $y \in C_1$ and y is comparable to b, or $y \in C_2$ and y is comparable to a. Both are contradictions.

Now we return to the proof of Theorem 1. Suppose that $N = \{a, b, c, d\}$ has no good W-completion in P. According to Claim 1 and Claim 2, the only lower covers of c are a and b, and there is $d' \prec d$ such that $d' \not> b$. Let $C_{d'}$ be a greedy chain in P containing d'. By induction hypothesis, there is a good W-completion $\{a, b, x_1, c, d\}$ of $N = \{a, b, c, d\}$ in $P - C_{d'}$. This W-completion is not good in P, thus there is $y_1 \prec x_1$ such that $\{y_1, a, b, x_1, c, d\}$ is an M subdiagram of P. Thus $y_1 \in C_{d'}$ and so it is comparable to d', in fact $y_1 > d'$. (If $y_1 \leq d'$ then $y_1 < d$ and this contradicts that $\{y_1, a, b, x_1, c, d\}$ is an M.) Let C_b be any greedy chain in P containing b. Neither a nor d' are in C_b. (a and d' are noncomparable to b.) In $P - C_b$, the $N = \{y_1, a, x_1, c\}$ has a good W-completion $\{y_1, a, t_1, x_1, c\}$, (see Figure 3). Now suppose that t_1 has a lower cover u with $u \not> y_1$. If $u \geq b$ then $t_1 \geq b$ and thus $\{d', b, t_1, d\}$ is a four-cycle. Therefore u is noncomparable to b and thus $u \in P - C_b$. However $\{y_1, a, t_1, x_1, c\}$ is a good W-completion of $\{y_1, a, x_1, c\}$ in $P - C_b$, and thus $u < x_1$ or $u \leq c$. If $u < x_1$ then $\{y_1, u, t_1, x_1\}$ produces a four-cycle, and $u \lessdot c$ would imply that $u < a$ or $u < b$. [Since we are assuming that the only lower covers of c are a and b.] And in both cases we produce a four-cycle. Therefore, we conclude that the only lower cover of t_1 is y_1, and according to Claim 2, $\{a, y_1, x_1, t_1\}$ has a good W-completion $\{a, y_1, x, t_1, x_1\}$ in P. Moreover $x \not> c$, for otherwise $b < x_1$ or $b < t_1$, and in both cases we produce a four-cycle. Finally, we claim that $\{a, b, x, c, d\}$ must be a good W-completion of $N = \{a, b, c, d\}$ in P. Assume that there is $u \prec x$ such that $u \not> a$. Then either $u < t_1$ or $u < x_1$. But $u < x_1$ implies that $\{u, a, x, x_1\}$ is a four-cycle, and $u < t_1$ implies $u < y_1$ (since the only lower cover of t_1 is y_1), and thus $u < x_1$. Therefore $\{a, b, x, c, d\}$ is a good W-completion of $N = \{a, b, c, d\}$ in P, which is a contradiction.

Finally, notice that if every N in P has a good W-completion then according to Theorem 3, P is greedy. This ends the proof of Theorem 1.

Before we get to the proof of Theorem 2, here is a Lemma which we shall use.

Lemma 3. *Let P be an ordered set. Then $G(P) \not\supseteq O(P)$ if and only if there is an optimal greedy linear extension $L = C_1 \oplus \cdots \oplus C_m$ of P and $1 \leq i < j \leq m$ such that:*

(i) $x = supC_i \prec y = infC_j$ *in P, and*
(ii) *x is noncomparable with t in P, for every t such that $x < t < y$ in L.*

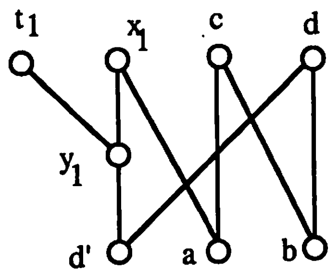


Figure 3.

Proof of Lemma 3: Assume that $L \in O(P) - G(P)$. We shall construct inductively a sequence $L = L_0, L_1, \ldots, L_n = L'$ of optimal linear extensions of P, such that $L' \in G(P)$ while $L_i \notin G(P)$ for every $i < n$. Suppose that $L_i = \{z_1 < z_2 < \ldots\}$, if L_i is greedy then set $n = i$. Otherwise let k be the least index such that z_{k+1} is not chosen in a greedy way, that is, z_k is noncomparable to z_{k+1} and there is z_q minimal in $\{z_{k+1}, z_{k+2}, \ldots\}$ such that $q > k+1$ and $z_q \succ z_k$ in P. We transform L_i into a new linear extension L_{i+1} by putting z_q between z_k and z_{k+1} in L_i.

$$L_{i+1} = \{z_1 < \cdots < z_k < z_q < z_{k+1} < \cdots < z_{q-1} < z_{q+1} < \ldots\}.$$

Assume that $L_{n-1} = C'_1 \oplus \cdots \oplus C'_m$. According to the construction described above, L_n is obtained from L_{n-1} by removing an element x of C'_j as an upper cover of some element u in C'_i, that is

$$L_n = C'_1 \oplus \cdots \oplus (C'_i \cup \{x\}) \oplus \cdots \oplus (C'_j - \{x\}) \oplus \cdots .$$

Clearly $x = \sup(C'_i \cup \{x\})$. Now denote $L_n = C_1 \oplus \cdots \oplus C_m$ where $C_i = C'_i \cup \{x\}$, $C_j = C'_j - \{x\}$ and $C_k = C'_k$ for every $k \notin \{i, j\}$. It is easy to see that $C_j \neq \phi$, for otherwise L_n will have less jumps than L_{n-1} and this contradicts the optimality of L_{n-1}. Moreover $\inf C_j \succ x = \sup C_i$ in P, and $j > i+1$ for otherwise C_i and C_{i+1} will form the same chain. Thus L_n satisfies the properties of the Lemma. To prove the converse, suppose that P has an optimal greedy linear extension $L = C_1 \oplus \cdots \oplus C_m$ satisfying the conditions (i) and (ii) of the Lemma. By moving x from C_i to just before C_j, we obtain a nongreedy optimal linear extension of P.

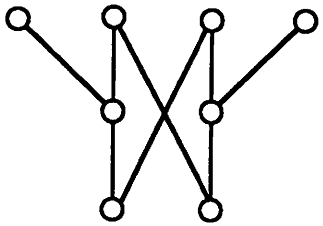
Proof of Theorem 2: Suppose that P is greedy and $O(P) - G(P) \neq \phi$. Let $L = C_1 \oplus \cdots \oplus C_m$ be a linear extension of P satisfying the properties of Lemma 3. We may choose L with the smallest possible value of $j - i$.

If $y = \inf C_j$ is noncomparable to every element in C_{j-1}, then y is minimal in $\cup_{k \geq j-1} C_k$. So consider a greedy linear extension $C'_{j-1} \oplus C'_j \oplus \cdots \oplus C'_m$ of $\cup_{k \geq j-1} C_k$, such that $y = \inf C'_{j-1}$. But then the greedy linear extension $C_1 \oplus \cdots \oplus C_{j-2} \oplus C'_{j-1} \oplus \cdots \oplus C'_m$ satisfies the properties of Lemma 3, for $x = \sup C_i$

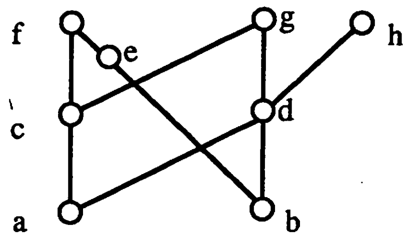
and $y = \inf C'_{j-1}$ and moreover x and y are now closer than in L [since $(j-1)-i < j-i$], which contradicts the choice of L.

Therefore, there is $z \in C_{j-1}$ such that $y \succ z$ in P. Necessarily $z \neq \sup C_{j-1}$, for otherwise L cannot be a greedy linear extension. Thus, $z \prec u$ in P for some u in C_{j-1}. The subdiagram $\{x, z, y, u\}$ is an N in P. According to Theorem 1, this N has a good W-completion $\{x, z, v, y, u\}$. Since $v \succ x$ in P then $v \in \cup_{k \geq j} C_k$. [That is because x is noncomparable to t, whenever $x < t < y$ in L.] Also v must be minimal in $\cup_{k>j} C_k$. Indeed if there exists $t \in \cup_{k>j} C_k$ such that $t \prec v$ in P then either $t < y$ or $t < u$. [Since $\{x, z, v, y, u\}$ is a good W-completion of $\{x, z, y, u\}$.] Which is a contradiction, since $y < t$ and $u < t$ in L.

Moreover, v is noncomparable with every element in C_j. Indeed if $v > t$ for some t in C_j, then $v > y$. Thus v is minimal in $\cup_{k \geq j-1} C_k$. Also v is noncomparable to every element in C_{j-1}, for otherwise $\{x, \inf C_{j-1}, y, v\}$ will be a four-cycle. Consider a greedy linear extension $C''_{j-1} \oplus C''_j \oplus \cdots \oplus C''_m$ of $\cup_{k \geq j-1} C_k$, such that $v = \inf C''_{j-1}$. But then the greedy linear extension $L'' = C_1 \oplus \cdots \oplus C_{j-2} \oplus C''_{j-1} \oplus \cdots \oplus C''_m$ has the properties of Lemma 3 for the elements $x = \sup C_i$ and $v = \inf C''_{j-1}$. Moreover x and v are now closer than x and y in L. Which contradicts the choice of L. This completes the proof of Theorem 2.



(a)



(b)

Figure 4.

Notice that the ordered set illustrated in Figure 4(a) contains a four-cycle and still $O(P) = G(P)$. Moreover it is not enough to forbid only four-cycles $\{x, y, z, t\}$ in which $x \prec t$ and $y \prec z$. For instance, the ordered set P illustrated in Figure 4(b) is greedy and does not contain any of these four-cycles. Nevertheless, the optimal linear extension $\{b, a, d, h, c, g, e, f\}$ is not greedy and so $O(P) \neq G(P)$.

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