A Completion of the Spectrum For Large Sets of Disjoint Ordered Triple Systems*

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Abstract. In this paper, we completely solve the existence problem of LOTS(v) (i. e. large set of pairwise disjoint ordered triple systems of order v).

1. Introduction

In what follows, an ordered pair will always be an ordered pair (x, y) where $x \neq y$. A *transitive triple* is a collection of three ordered pairs of the form $\{(x,y),(y,z),(x,z)\}$, which we will always denote by (x,y,z). Similarly a *Mendelsohn triple* is a collection of three ordered pairs of the form $\{(x,y),(y,z),(z,x)\}$, denoted by $\langle x,y,z\rangle$ (or $\langle y,z,x\rangle$, or $\langle z,x,y\rangle$), where x,y,z are distinct elements. An *ordered triple* means either a transitive or a Mendelsohn triple.

An *ordered triple system* (briefly OTS(v)) is a pair (X,B) where X is a set containing v elements and B is a collection of ordered triples of elements of X such that every ordered pair of elements of X belongs to exactly one ordered triple of B. In particular, if the triples in B are all transitive or all Mendelsohn, then (X,B) is said to be a transitive triple system (TTS) or Mendelsohn triple system (MTS) respectively. The number $|X| = v$ is called the order of the OTS(X,B). It is easy to show that $|B| = v(v-1)/3$. It is well known that the spectrum for OTSs and TTSs is the set of all $v \equiv 0$ or $1 \pmod 3$, and for MTSs the set of all $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$.

If $|X| = v$, and $O(X)$ is set of all ordered triples of X, then $|O(X)| = 4v(v-1)(v-2)/3$. Thus it is natural to ask whether it is always possible to partition $O(X)$ into $4(v-2)$ subsets $B_1, B_2, \ldots, B_{4(v-2)}$ so that each of (X,B_1), $(X,B_2), \ldots, (X,B_{4(v-2)})$ is an OTS(v), for all $v \equiv 0$ or $1 \pmod 3$. Such a collection of OTS(v)s is called a *large set of pairwise disjoint* OTS(v)s. It is denoted by LOTS(v). The corresponding questions of partitioning all transitive and Mendelsohn triples into large sets of TTS(v)s and MTS(v)s respectively have been considered. See [2], [3] and [4], it is denoted by LTTS(v) or LMTS(v) respectively. If LTTS(v) and LMTS(v) both exist, then there exists an LOTS(v) too. For $v = 6$, [1] directly construct a large set of pairwise disjoint OTSs of order 6. In this paper we will give the spectrum of LOTS(v).

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2. The Construction of LOTS($n+2$)

Let $n \equiv 1,5 \pmod 6$ and $Z_n = \{0,1,\ldots,n-1\}$. Similar to the construction of LTTS($n+2$) in [4], we immediately obtain the fact

Lemma 1. *There exists a partition of all ∞_s-triples (triples containing ∞_s, but none of ∞_{3-s}, $s = 1,2$) from the set $\{\infty_1, \infty_2\} \cup Z_n$:*

$$\bigcup_{k=1}^{3} \bigcup_{i=0}^{n-1} C_i^k$$

such that ordered pairs of triples in each C_i^k are just

$$\{(\infty_s, u), (u, \infty_s); \quad u \in Z_n \setminus \{i\}, s = 1,2\}$$
$$\cup \{(x, 3i-2x), (3i-2x, x); \quad x \in Z_n \setminus \{i\}\}.$$

Now we will construct $4n$ subsets J_i^k $(i \in Z_n, k \in \{0,1,2,3\})$ on the set $X = \{\infty_1, \infty_2\} \cup Z_n$ as follows

$J_i^0 (i \in Z_n)$ consists of the following Mendelsohn triples:

(1) $\langle \infty_1, \infty_2, i\rangle, \langle \infty_2, \infty_1, i\rangle$

(2) $\langle x, y, z\rangle$, $\langle y, x, z\rangle$ where $\{x,y,z\}$ satisfying the equation $x + y + z \equiv 3i \pmod n$.

(3) $\langle \infty_1, x, 3i-2x\rangle, \langle \infty_2, 3i-2x, x\rangle$.

The construction of $\mathbf{J_i^k} (i \in Z_n, k \in \{1,2,3\})$: Denote by $T(i)$ the set of all triples $\{x,y,z\}$ such that x,y,z are distinct and $x+y+z \equiv 3i \pmod n$. For each triple $t \in T(i)$, denote by $\{(t, t(k,i)); k = 1,2,3\}$ a large set of pairwise disjoint TTS(3)s defined on t. Finally, denote by L_i^1, L_i^2, L_i^3 a large set of pairwise disjoint TTS(3)s defined on $\{\infty_1, \infty_2, i\}$. And set

$$T_i^k = \bigcup_{t \in T(i)} t(k,i)$$

$$J_i^k = T_i^k \bigcup L_i^k \bigcup C_i^k$$

Theorem 1. *For every positive integer $n \equiv 1,5 \pmod 6$, there exists an LOTS($n+2$).*

Proof. From the proofs of Theorem 6 in [2] and Lemma 2.1 in [4], we have known that $\{(X, J_i^0); i \in Z_n\}$ is an LMTS($n+2$), and $\{(X, J_i^k); i \in Z_n, k \in \{1,2,3\}\}$ is an LTTS($n+2$). Thus $\{(X, J_i^k); i \in Z_n, k \in \{0,1,2,3\}\}$ forms an LOTS($n+2$). ∎

3. The Construction of LOTS($mn+2$), $m \neq 2$.

Let $n \equiv 1,5 \pmod 6$, $Q = \{0,1,\ldots,m-1\}$ and $m > 3$. Let $(Q,\cdot)$ be an idempotent quasigroup of order m. Define set $d_{x,y}(u,v) =$

$$\{((x,u),(x,v),(y,u\cdot v)),((x,u),(y,u\cdot v),(x,v)),((y,u\cdot v),(x,u),(x,v))\}$$

where $x \neq y \in Z_n$, $u \neq v \in Q$. According to [1], $D(x,y) = \bigcup_{u \neq v \in Q} d_{x,y}(u,v)$ can be partitioned into three sets $D^1(x,y)$, $D^2(x,y)$, $D^3(x,y)$ such that

(1) The three triples in each $d_{x,y}(u,v)$ belong to different $D^k(x,y)$; $k = 1,2,3$.
(2) If $u \neq v \in Q$, then each pair of ordered pairs $((x,u),(y,v))$ and $((y,v),(x,u))$ belongs to exactly one triple in each of $D^k(x,y)$, $k = 1,2,3$.

Taking $\infty_1,\infty_2 \notin Z_n \times Q$, $X = \{\infty_1,\infty_2\} \cup (Z_n \times Q)$ and α be a cycle of length m on Q. Let $\{(\{\infty_1,\infty_2\} \cup Q, L_j^k); j \in Q, k \in \{0,1,2,3\}\}$ be a LOTS($m+2$). And denote by M_j^k the set of all Mendelsohn triples in L_j^k, and by $L_j^k - M_j^k$ the set of all transitive triples in L_j^k. Let $\{(\{\infty_1,\infty_2\} \cup Z_n, B_i^k); i \in Z_n, k \in \{0,1,2,3\}\}$ be a LOTS($n+2$) given above section. Now we will construct $4mn$ systems $\Omega_{ij}^k (i \in Z_n, j \in Q, k \in \{0,1,2,3\})$ as follows.

$\Omega_{ij}^0 (i \in Z_n, j \in Q)$ consists of the following ordered triples:

(1) $\langle (x,u),(y,v),(z,(u\cdot v)\alpha^j) \rangle$ where $\langle x,y,z \rangle \in B_i^0$, $x,y,z \in Z_n$, $u,v \in Q$.
(2) $\langle (x,u),(x,v),(y,(u\cdot v)\alpha^j) \rangle$ where $\langle \infty_1,x,y \rangle \in B_i^0$, $x,y \in Z_n$, $u,v \in Q$.
(3) $\langle \infty_1,(x,u),(y,u\alpha^j) \rangle$, $\langle \infty_2,(y,u\alpha^j),(x,u) \rangle$ where $\langle \infty_1,x,y \rangle \in B_i^0$, $x,y \in Z_n$, $u \in Q$.
(4) $\langle (i,u),(i,v),(i,w) \rangle$ provided $\langle u,v,w \rangle \in M_i^0$; And $((i,u),(i,v),(i,w))$ provided $(u,v,w) \in L_i^0 - M_i^0$. Whenever ∞_1 or ∞_2 appears for u,v,w omit the first coordinate i.

$\Omega_{ij}^k (i \in Z_n, j \in Q, k \in \{1,2,3\})$ consists of the following ordered triples:

(1) $\langle (x,u),(y,v),(z,(u\cdot v)\alpha^j) \rangle$ where $\langle x,y,z \rangle \in B_i^k$, $x,y,z \in Z_n$, $u,v \in Q$.
(2) The collection of transitive triples obtained by replacing (y,w) with $(y,w\alpha^j)$ in each triple of $\{D^k(x,y);$ where $y \equiv 3i-2x \pmod n\}$.
(3) The collection of transitive triples obtained by replacing the unique pair $(x,3i-2x)$ with $((x,u),(3i-2x,u\alpha^j))$ in each ∞_1-triple of C_i^k (in Lemma 1); and by replacing the unique pair $(3i-2x,x)$ with $((3i-2x,u\alpha^j),(x,u))$ in each ∞_2-triple of C_i^k, where $u \in Q$.
(4) $\langle (i,u),(i,v),(i,w) \rangle$ provided $\langle u,v,w \rangle \in M_i^k$; And $((i,u),(i,v),(i,w))$ provided $(u,v,w) \in L_i^k - M_i^k$. Whenever ∞_1 or ∞_2 appears for u,v,w omit the first coordinate i.

Theorem 2. *For every $n \equiv 1,5 \pmod 6$, if there exists an LOTS$(m+2)$, $m > 3$, then there exists an LOTS$(mn+2)$.*

Proof.

1. Each $\Omega_{ij}^k (i \in Z_n, j \in Q, k \in \{0,1,2,3\}$ is an OTS$(mn+2)$.
   Direct calculation shows that each Ω_{ij}^k contains $(mn+2)(mn+1)/3$ ordered triples, just the number we expected. Therefore, we only need to show that every ordered pair P of distinct elements of X is contained in some triple in Ω_{ij}^k. For example, we only verify that $\Omega_{ij}^k (i \in Z_n, j \in Q$, $k \in \{1,2,3\}$ is an OTS$(mn+2)$. Similar to $\Omega_{ij}^k (i \in Z_n, j \in Q)$.
   (1) $P = (\infty_s, \infty_{3-s}), (\infty_s, (i,u)), ((i,u), \infty_s)$ and $((i,u),(i,v))$ contained in (4) of Ω_{ij}^k, where $s = 1,2$, $u \neq v \in Q$.
   (2) $P = (\infty_s, (x,u))$ and $((x,u), \infty_s)$ in (3) of Ω_{ij}^k, where $s = 1,2$, $x \in Z_n \setminus \{i\}$ and $u \in Q$.
   (3) $P = ((x,u),(x,v)), x \in Z_n \setminus \{i\}, u \neq v \in Q$. Let $y \equiv 3i - 2x \pmod n$, we have $y \in Z_n \setminus \{i\}$. So, P is covered by each triple of $d_{x,y}(u,v)$. Thus, P in (2) of Ω_{ij}^k.
   (4) $P = ((x,u),(y,v)), x \neq y \in Z_n \setminus \{i\}, u, v \in Q$.
      (i) If $y \equiv 3i - 2x \pmod n$ or $x \equiv 3i - 2y \pmod n$, then there exists $w \in Q$ such that $(u \cdot w)\alpha^j = v$. If $w = u$, i. e. $v = u\alpha^j$, then P in (3). Or else $((x,u),(y,(u \cdot w))$ is covered by some transitive triple of $D^k(x,y)$. Thus, $P = ((x,u),(y,(u \cdot w)\alpha^j))$ in (2).
      (ii) If $x \not\equiv 3i - 2y \pmod n$ and $y \not\equiv 3i - 2x \pmod n$, then there exists $z \in Z_n$ such that (x,y,z) or (x,z,y) or $(z,x,y) \in B_i^k$. Thus, P contained in (1).
2. $\{(X, \Omega_{ij}^k); i \in Z_n, j \in Q, k \in \{0,1,2,3\}\}$ is an LOTS$(mn+2)$.
   We only need to show that every ordered triple T from X is contained in some Ω_{ij}^k above. Next we only verify that every transitive triple T from X is contained in some Ω_{ij}^k (similar to prove every Mendelsohn triple T from X is contained in some Ω_{ij}^k). All the possibilities are exhausted as follows:
   (1) $T = (\infty_s, \infty_{3-s}, (i,u)), (\infty_s, (i,u), \infty_{3-s})$ or $((i,u), \infty_s, \infty_{3-s})$, $s = 1,2$, $i \in Z_n$, $u \in Q$. There exists $k \in \{1,2,3\}$ and $j \in Q$ such that $(\infty_s, \infty_{3-s}, u)$, $(\infty_s, u, \infty_{3-s})$ or $(u, \infty_s, \infty_{3-s}) \in L_j^k$, thus $T \in$ (4) of Ω_{ij}^k.
   (2) $T = (\infty_s, (x,u), (y,v))$ (or $((x,u), \infty_s, (y,v))$ or $((x,u),(y,v), \infty_s))$, $s = 1,2$, $(x,u) \neq (y,v) \in Z_n \times Q$. If $x = y$ and (∞_s, u, v) (or (u, ∞_s, v) or $(u, v, \infty_s)) \in L_j^k$, then T in (4) of Ω_{ij}^k. Or else, let $i = (2x+y)/3 \pmod n$ $(t = 1)$ or $i = (x+2y)/3 \pmod n$ $(t = 2)$, there exists $j \in Q$ such that $v = u\alpha^j$. Then T in (3) of $\bigcup_{k=1}^3 \Omega_{ij}^k$.
   (3) $T = ((i,u),(i,v),(i,w)), i \in Z_n, u \neq v \neq w \neq u \in Q$. There exist

j and k such that $(u,v,w) \in L_j^k$. Then T in (4) of Ω_{ij}^k.

(4) $T = ((x,u),(x,v),(y,w))$ (or $((x,u),(y,w),(x,v))$ or $((y,w),(x,u),(x,v))$), $x \neq y \in Z_n, u,v,w \in Q, u \neq v$. Let $i = (2x+y)/3 \pmod n$, there exists j such that $(u\cdot v)\alpha^j = w$. Then T in (2) of $\bigcup_{k=1}^3 \Omega_{ij}^k$.

(5) $T = ((x,u),(y,v),(z,w))$, $x \neq y \neq z \neq x \in Z_n$, $u,v,w \in Q$. There exist i,j and k such that $(x,y,z) \in B_i^k$ and $(u\cdot v)\alpha^j = w$. Then T in (1) of Ω_{ij}^k.

This completes the proof. ∎

4. The Main Results

Lemma 2. *For every positive integer $n \equiv 1,5 \pmod 6$, there exists an LOTS$(2n+2)$.*

Proof. From [2] and [4], for $n \equiv 1,5 \pmod 6$, there exists an LMTS$(2n+2)$ and an LTTS$(2n+2)$. Thus, there exists an LOTS$(2n+2)$. ∎

Lemma 3. *For any integer $u \geq 1$, there exists an LOTS(2^u+2).*

Proof. When $u \geq 3$ there exists a LMTS(2^u+2) by [3] and a LTTS(2^u+2) by [5]. Thus, we have LOTS(2^u+2). When $u = 1,2$, we have LOTS(4), LOTS(6) by directly constructions (see [1]). ∎

Theorem 3. *For any $v \equiv 0$ or $1 \pmod 3$, there exists an LOTS(v).*

Proof. Let $v = 2^u n + 2$, where $n \equiv 1$ or $5 \pmod 6$. If $u = 0$, by Theorem 1, there exists an LOTS(v). If $u \geq 2$, there exists an LOTS(2^u+2) by Lemma 3. Noticing that $2^u > 3$, thereby we have an LOTS(v) by Theorem 2. If $u = 1$, we have known by Lemma 2.

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