Graceful Paths to Edge-Graceful Trees

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Graceful and edge-graceful graph labelings are dual notions of each other in the sense that a graceful labeling of the vertices of a graph G induces a labeling of its edges, whereas an edge-graceful labeling of the edges of G induces a labeling of its vertices. In this paper we show a connection between these two notions, namely, that the graceful labeling of paths enables particular trees to be labeled edge-gracefully. Of primary concern in this enterprise are two conjectures: that a path can be labeled gracefully starting at any vertex label, and that all trees of odd order are edge-graceful. We give partial results for the first conjecture and extend the domain of trees known to be edge-graceful for the second conjecture.

Graceful Paths

Let G be a graph with $|V(G)| = p$ and $|E(G)| = q$. Let (ℓ, ℓ^*) be a function pair mapping the vertices and edges into the set of integers; that is,

$$\ell : V(G) \to Z \text{ and } \ell^* : E(G) \to Z.$$

Let V and E be sets of integers of size p and q, respectively. We say that G is *VE-graceful* if ℓ is onto V, ℓ^* is onto E and

$$\ell^*(uv) = |\ell(u) - \ell(v)|,$$

for all uv in $E(G)$. (Note: the expression *VE-graceful* is synonymous with the expression *(V,E)-graceful.*) Let $V_p = \{1, \ldots, p\}, E_q = \{1, 2, \ldots, q\}$. A graph that is (V_p, E_q)-graceful is said to be *graceful,* as coined by Golomb [2] and as initially posed by Ringel [6].

For the rest of this section let G be a path. We say that G is *VE_r-graceful* if G is VE-graceful and one of the two end vertices of G is labeled

*Much of this work was done while the first author was a Fulbright Scholar at the University of Botswana in 1991, and the second author—who found many of the patterns in *Theorem 9* —was 10 years old.

r. The following proposition is clear. (As before, the term *VEr-graceful* is synonymous with *(V,E,r)-graceful*.)

Proposition 1. *If G is VEr-graceful and $a \in Z$, then G is $(V+a, E, r+a)$-graceful and $(-V, E, -r)$-graceful.*

If $V = \{0, 1, \ldots, n\}$ and $E = \{1, \ldots, n\}, 0 \leq r \leq n$, and if the corresponding path is VEr-graceful, we say that r ***threads*** n. As a sewn seam starts with the threading of a needle, the terminology, r *threading* n, was chosen to mean that a path can be woven through the marks along a ruled line at the labels in V starting with r, leaving edges of lengths in E. For example, figure 1 demonstrates that 2 threads 9. We make the following conjecture.

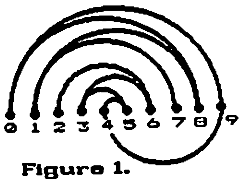

Figure 1.

Conjecture 2. *For all n and r, with $0 \leq r \leq n$, r threads n.*

Some notation helpful in grappling with this conjecture is the following. If G is VEr-graceful, let S be a sequence of vertex labels beginning with r, corresponding to a VEr-graceful path labeling of G. We say that S is a ***strand*** for this labeling. For example, a strand for 2 threads 9 is $\{2, 6, 3, 5, 4, 9, 0, 8, 1, 7\}$. For simplicity we consider a strand as both a string of integers and as a set of integers. Let S and T be strings of integers with r being the last term of S and the first term of T. Let T' be the string obtained from T by deleting its first term. Then S *join* T, denoted $S \vee T$, is the concatenation of S with T'. If the notation $a_1b_1a_2b_2\ldots c$ appears in a string then $a_1a_2a_3\ldots$ and $b_1b_2b_3\ldots$ are consecutive integers and there exists an integer $j \geq 3$ with either $a_j = c$ or $b_j = c$.

Proposition 3: Thread Symmetry. *If r threads n then $n-r$ threads n.*

Proof: If S is a strand for r thread n, then by Proposition 1, $n - S$ is a strand for $n - r$ thread n, where

$$n - S = n - \{r, s_1, \ldots, s_n\} = \{n - r, n - s_1, \ldots, n - s_n\}.\square$$

To determine whether a certain path is threadable it is sometimes useful to look at a shorter path, as given by the following theorem.

Theorem 4: Path Reduction. *If r threads $n-2r-1$ then r threads n.*

Proof: Since r threads $n-2r-1$ then $n-3r-1$ threads $n-2r-1$ by Proposition 3. Let S_0 be the strand for this threading. Let $S = S_0+(r+1)$, so that the first term in S is $n-2r$, and the labels in S range from $r+1$ to $n-r$. Let

$$T = \{r, n-r+1, r-1, n-r+2, \ldots, 1, n, 0, n-2r\},$$

whose edge labels are $\{n-2r+1, n-2r+2, \ldots, n, n-2r\}$. Hence $T \vee S$ is a strand for r threading n. See figure 2.□

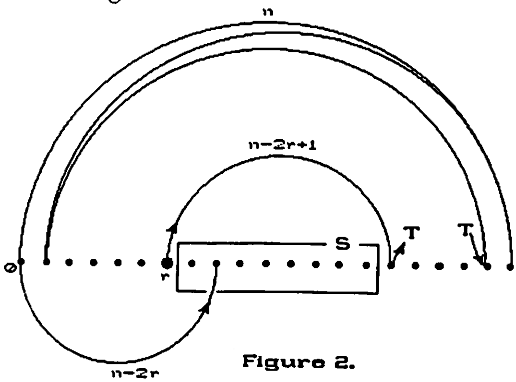

Figure 2.

Theorem 4 allows us to focus attention on a critical range of ordered pairs of integers (r, n) with respect to conjecture 2, as the following corollary makes precise.

Corollary 5. *If r threads n for all integers n with $r \leq n \leq 3r$, then r threads n for all $n \geq r$.*

Proof: A strand for 0 threads n is

$$S = \{0, n, 1, n-1, \ldots, \lfloor \frac{n+1}{2} \rfloor\}.$$

Hence, let $r > 0$. Let $m > 3r$. Choose the least positive integer k such that

$$m - k(2r+1) \leq 3r.$$

Observe that $m-k(2r+1) \geq r$, which follows because $m-(k-1)(2r+1) > 3r$. By hypothesis, r threads $m-k(2r+1)$. Apply Theorem 4 a total of k times to yield the desired result. □

The next proposition further refines this critical range of ordered pairs.

Proposition 6. *For all $r \geq 0$, r threads $2r, 2r+1, 3r$.*

Proof: A strand for r threads $2r$ is

$$S = \{r, r+1, r-1, r+2, \ldots, 2r, 0\}.$$

A strand for r threads $2r+1$ is $S \vee \{0, 2r+1\}$. Strands S_r for r thread $3r$ can be taken as follows.

$$S_0 = \{0\}, S_1 = \{1, 2, 0, 3\}, S_2 = \{2, 4, 3, 6, 0, 5, 1\},$$

$$S_3 = \{3, 6, 4, 5, 9, 0, 8, 1, 7, 2\},$$

$$S_4 = \{4, 8, 5, 3, 9, 2, 10, 1, 11, 0, 12, 7, 6\},$$

$$S_5 = \{5, 10, 7, 6, 8, 4, 11, 3, 12, \ldots, 15, 9\},$$

$$S_6 = \{6, 12, 8, 9, 7, 10, 5, 13, 4, 14, \ldots, 18, 11\},$$

$$S_7 = \{7, 14, 9, 11, 10, 13, 21, 0, 20, 1, \ldots, 6, 12, 8\},$$

$$S_8 = \{8, 16, 10, 11, 13, 9, 14, 7, 17, 6, 18, \ldots, 24, 15, 12\}.$$

To find S_r for $r \geq 9$, observe that 2 threads n for all $n \geq 2$. To see this, note that 2 threads 2 since $\{2, 0, 1\}$ is its strand, 2 threads 3 since 1 threads 3 by S_1 above, 2 threads 4 since r threads $2r$, 2 threads 5 since r threads $2r+1$, 2 threads 6 by S_2 above, and 2 threads n for all $n \geq 7$ by Corollary 5. Therefore, let T_0 be a strand for 2 threads $r-7$. Let $T = T_0 + (r+3)$, so that T starts with $r+5$. Let

$$S = \{r, 2r, r+2, 2r-3, r+1, 2r-2, r-1, 2r+1, r-2, 2r+2, \ldots, 0, 3r, 2r-1, r+5\}.$$

Then $S_r = S \vee T$. See figure 3. □

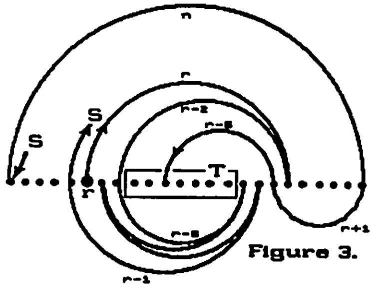

Figure 3.

The following corollary gives an inductive approach to refining this critical range of integer pairs.

Corollary 7: Inductive Threading. *If r threads n for all integers n with $2r+1 < n < 3r$ and if s threads n for all $n \geq s$ for all s with $0 \leq s < r$, then r threads n for all $n \geq r$.*

Proof: By the hypothesis and Proposition 6, r threads n with $2r \leq n \leq 3r$. Let m be any integer with $r \leq m < 2r$. Then $0 \leq m - r < r$. By hypothesis $m - r$ threads m. So r threads m by Proposition 3. Hence r threads n for all n with $r \leq n \leq 3r$. By Corollary 5, r threads n for all $n \geq r$. □

The following lemma enables further refining, as will be made clear in the next theorem, the main result of this paper.

Lemma 8: Refining the Critical Range.

a. *Let* $n = 3r - 2k, r, k > 0$. *If* $2k - 1$ *threads* $r - 4k$ *then* r *threads* n.

b. *Let* $n = 3r - (2k - 1), r, k > 0$. *If* $2k - 2$ *threads* $r - 4k$ *then* r *threads* n.

Proof: Since $2k - 1$ threads $r - 4k$ then $r - 6k + 1$ threads $r - 4k$ by Proposition 3. Let D_0 be a strand for this latter threading. Let $D = D_0 + (r + k + 1)$ making D's first term $2r - 5k + 2$. Let

$$A = \{r, 0, n, 1, n - 1, \ldots, 2r - k + 1, r - k\},$$

$$B = \{r - k, 2r - 3k + 2, r + k, 2r - 3k + 3, r + k - 1, \ldots, r + 1, 2r - 2k + 2\},$$

$$C = \{2r - 2k + 2, r - 1, 2r - 2k + 3, r - 2, \ldots, 2r - k, r - k + 1, 2r - 5k + 2\}.$$

Let $S = A \vee B \vee C \vee D$. The component paths of S have been labeled in figure 4a; for example, A has been labeled both near its beginning and its ending; initial and final vertices for each component path are enlarged.

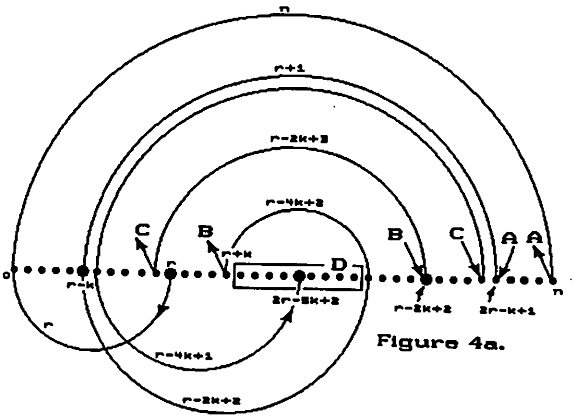

Figure 4a.

It is straight forward to check that S is a strand for r threads n. That is, from A we get the edge labels from r to n; from B we get the edge labels from $r - 4k + 2$ to $r - 2k + 2$; from C we get the edge labels from $r - 2k + 3$ to $r - 1$ as well as $r - 4k + 1$; and from D we get the edge labels from 1 to $r - 4k$.

Let D_0 be a strand for $2k-2$ threads $r-4k$. Let $D = D_0 + r + k + 1$, making D's first term $r + 3k - 1$. Let

$$A = \{r, 0, n, 1, n-1, \ldots, r-k, 2r-k+1\},$$

$$B = \{2r-k+1, r+k, 2r-3k+2, r+k-1, 2r-3k+3, \ldots, r+1, 2r-2k+1\},$$

$$C = \{2r-2k+1, r-1, 2r-2k+2, r-2, \ldots, r-k+1, 2r-k, r+3k-1\}.$$

Then as before, $S = A \vee B \vee C \vee D$ is a strand for r threads n. See figure 4b. □

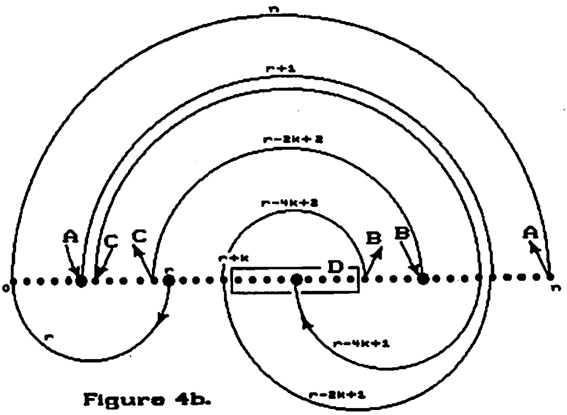

Figure 4b.

Theorem 9. *For $0 \le r \le 20$, r threads n for all $n \ge r$.*
Proof: Let

$$A_{r,n} = \begin{cases} \{r, 0, n, 1, n-1, \ldots, \frac{n-r}{2}\}, & \text{if } n-r \text{ is even,} \\ \{r, 0, n, 1, n-1, \ldots, \frac{n+r+1}{2}\}, & \text{if } n-r \text{ is odd.} \end{cases}$$

Consider the list of strings $T_{r,n}$ given at the end of this proof for various integer pairs (r, n).

Strands, $S_{r,n}$, for r threads n for the indicated integer pairs are given by

$$S_{r,n} = A_{r,n} \vee T_{r,n}.$$

For $r = 0, 1, 2$, there is no n such that $2r + 1 < n < 3r$, so r threads n for all $n \ge r$, for $r = 0, 1, 2$, by Corollary 7. For $r = 3$ the only integer n with $2r + 1 < n < 3r$ is $n = 8$. A strand for 3 threads 8 is $S_{3,8}$. Hence 3 threads n for all $n \ge 3$.

Since 0 threads $r - 4$ for all $r \ge 4$ then r threads $3r - 1$ for all $r \ge 4$, by Lemma 8b, with $k = 1$. For $r = 4$ the only n with $2r + 1 < n < 3r - 1$ is $n = 10$. A strand for 4 threads 10 is $S_{4,10}$. So 4 threads n for all $n \ge 4$.

Since 1 threads $r-4$ for all $r \geq 5$, then 1 threads $3r-2$ for all $r \geq 5$, by Lemma 8a, with $k=1$. For $r=5$ the only n with $2r+1 < n < 3r-2$ is $n=12$. A strand for 5 threads 12 is $S_{5,12}$. Hence 5 threads n for all $n \geq 5$.

By Lemma 8 and the previous work, we can conclude that

$$\begin{aligned}
&r \text{ threads } 3r-1 \text{ for all } r \geq 4,\\
&r \text{ threads } 3r-2 \text{ for all } r \geq 5,\\
&r \text{ threads } 3r-3 \text{ for all } r \geq 10,\\
&r \text{ threads } 3r-4 \text{ for all } r \geq 11,\\
&r \text{ threads } 3r-5 \text{ for all } r \geq 16,\\
&r \text{ threads } 3r-6 \text{ for all } r \geq 17.
\end{aligned}$$

Thus for $r=6$ since the only integer n such that $2r+1 < n < 3r-2$ is $n=14$ and since $S_{6,14}$ demonstrates that 6 threads 14, then 6 threads n for all $n \geq 6$. We can use the same argument to demonstrate that r threads n for all integers r and n with $7 \leq r \leq 20$, $n \geq r$, and we are done.

(r,n)	$T_{r,n}$
(3,8)	{6,4,5}
(4,10)	{3,6,5,7}
(5,12)	{9,6,4,8,7}
(6,14)	{4,8,9,7,10,5}
(7,16)	{12,9,8,10,6,11,5}
(7,17)	{5,10,9,11,8,12,6}
(8,18)	{5,9,6,13,7,12,10,11}
(8,19)	{14,10,11,9,12,7,13,6}
(8,20)	{6,11,10,12,9,13,7,14}
(9,20)	{15,8,14,6,11,7,10,12,13}
(9,21)	{6,11,7,15,8,14,13,10,12}
(9,22)	{16,10,15,7,14,11,13,12,8}
(9,23)	{7,14,8,16,12,15,10,11,13}
(10,22)	{6,11,7,16,8,15,9,12,14,13}
(10,23)	{17,13,16,7,15,8,14,9,11,12}
(10,24)	{7,13,8,17,9,16,15,11,14,12}
(10,25)	{18,13,12,14,11,15,9,16,8,17}
(10,26)	{8,16,14,13,9,18,11,17,12,15}
(11,24)	{18,13,14,10,12,9,15,8,16,7,17}
(11,25)	{7,12,8,18,9,17,10,16,13,15,14}
(11,26)	{19,13,18,8,17,9,16,15,12,14,10}
(11,27)	{8,15,14,16,13,17,12,18,10,19,9}
(11,28)	{20,13,14,19,9,18,10,16,12,15,17}
(11,29)	{9,18,10,20,14,19,12,15,13,17,16}

(12, 26)	{7, 13, 17, 10, 18, 9, 19, 8, 11, 16, 14, 15}
(12, 27)	{20, 16, 15, 13, 8, 19, 9, 18, 10, 17, 11, 14}
(12, 28)	{8, 14, 9, 20, 10, 19, 11, 18, 15, 13, 17, 16}
(12, 29)	{21, 16, 20, 9, 19, 10, 18, 11, 17, 14, 15, 13}
(12, 30)	{9, 17, 10, 21, 11, 20, 18, 14, 19, 13, 16, 15}
(12, 31)	{22, 15, 21, 10, 20, 11, 19, 18, 13, 17, 14, 16}
(13, 28)	{21, 15, 16, 20, 8, 19, 9, 18, 10, 17, 12, 14, 11}
(13, 29)	{8, 14, 9, 21, 10, 20, 11, 19, 12, 16, 17, 15, 18}
(13, 30)	{22, 17, 21, 9, 20, 10, 19, 11, 18, 12, 15, 14, 16}
(13, 31)	{9, 16, 10, 22, 11, 21, 12, 20, 19, 14, 18, 15, 17}
(13, 32)	{23, 17, 14, 19, 12, 20, 11, 21, 10, 22, 18, 16, 15}
(13, 33)	{10, 19, 11, 23, 12, 22, 15, 21, 16, 14, 18, 17, 20}
(13, 34)	{24, 16, 23, 11, 22, 12, 21, 18, 17, 15, 19, 14, 20}
(14, 30)	{8, 15, 9, 22, 10, 21, 11, 20, 12, 16, 13, 18, 19, 17}
(14, 31)	{23, 17, 15, 16, 13, 18, 22, 9, 21, 10, 20, 11, 19, 12}
(14, 32)	{9, 15, 10, 23, 11, 22, 12, 21, 13, 20, 16, 19, 17, 18}
(14, 33)	{24, 18, 13, 20, 12, 21, 11, 22, 10, 23, 19, 16, 15, 17}
(14, 34)	{10, 18, 11, 24, 12, 23, 13, 22, 21, 15, 20, 16, 19, 17}
(14, 35)	{25, 18, 15, 21, 13, 22, 12, 23, 11, 24, 19, 17, 16, 20}
(14, 36)	{11, 21, 17, 12, 25, 13, 24, 15, 23, 16, 22, 19, 18, 20}
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(15, 37)	{11, 20, 19, 21, 18, 22, 17, 23, 16, 24, 14, 25, 13, 26, 12}
(15, 38)	{27, 19, 20, 18, 21, 17, 22, 16, 23, 14, 24, 13, 25, 12, 26}
(15, 39)	{12, 23, 18, 22, 20, 21, 24, 17, 25, 16, 26, 14, 27, 13, 19}
(15, 40)	{28, 18, 22, 27, 13, 26, 14, 25, 16, 24, 17, 23, 20, 19, 21}
(16, 34)	{9, 15, 10, 25, 11, 24, 12, 23, 13, 22, 14, 21, 17, 20, 18, 19}
(16, 35)	{26, 20, 25, 10, 24, 11, 23, 12, 22, 13, 21, 14, 17, 15, 19, 18}
(16, 36)	{10, 17, 11, 26, 12, 25, 13, 24, 14, 23, 15, 20, 19, 21, 18, 22}
(16, 37)	{27, 21, 26, 11, 25, 12, 24, 13, 23, 14, 22, 15, 19, 18, 20, 17}
(16, 38)	{11, 19, 12, 27, 13, 26, 14, 25, 15, 24, 21, 20, 18, 22, 17, 23}
(16, 39)	{28, 21, 18, 22, 27, 12, 26, 13, 25, 14, 24, 15, 23, 17, 19, 20}
(16, 40)	{12, 22, 13, 28, 14, 27, 15, 26, 18, 25, 19, 24, 23, 21, 17, 20}
(16, 41)	{29, 20, 18, 24, 17, 25, 15, 26, 14, 27, 13, 28, 23, 19, 22, 21}
(16, 42)	{13, 25, 14, 29, 15, 28, 22, 21, 23, 20, 24, 19, 26, 18, 27, 17}

(17, 36)	{27, 20, 26, 10, 25, 11, 24, 12, 23, 13, 22, 14, 19, 15, 16, 18, 21}
(17, 37)	{10, 18, 11, 27, 12, 26, 13, 25, 14, 24, 15, 21, 16, 19, 23, 22, 20}
(17, 38)	{28, 21, 27, 11, 26, 12, 25, 13, 24, 14, 23, 15, 20, 16, 18, 19, 22}
(17, 39)	{11, 18, 12, 28, 13, 27, 14, 26, 15, 25, 16, 24, 19, 23, 20, 22, 21}
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(20, 43)	{32, 24, 31, 12, 30, 13, 29, 14, 28, 15, 27, 16, 26, 17, 23, 18, 19, 21, 25, 22}
(20, 44)	{12, 21, 13, 32, 14, 31, 15, 30, 16, 29, 17, 28, 18, 25, 19, 23, 26, 27, 22, 24}
(20, 45)	{33, 25, 32, 13, 31, 14, 30, 15, 29, 16, 28, 17, 27, 18, 24, 19, 21, 22, 26, 23}
(20, 46)	{13, 21, 14, 33, 15, 32, 16, 31, 17, 30, 18, 29, 19, 28, 22, 27, 23, 26, 24, 25}
(20, 47)	{34, 26, 33, 14, 32, 15, 31, 16, 30, 17, 29, 18, 28, 19, 23, 21, 27, 22, 25, 24}
(20, 48)	{14, 24, 15, 34, 16, 33, 17, 32, 18, 31, 19, 30, 27, 23, 25, 26, 21, 29, 22, 28}
(20, 49)	{35, 26, 34, 15, 33, 16, 32, 17, 31, 18, 30, 19, 29, 25, 28, 21, 27, 22, 24, 23}
(20, 50)	{15, 27, 16, 35, 17, 34, 18, 33, 19, 32, 28, 25, 24, 26, 21, 31, 22, 30, 23, 29}
(20, 51)	{36, 25, 35, 16, 34, 17, 33, 18, 32, 19, 31, 30, 21, 29, 22, 28, 23, 27, 24, 26}
(20, 52)	{16, 30, 17, 36, 18, 35, 19, 34, 27, 33, 21, 32, 22, 31, 23, 24, 29, 25, 28, 26}
(20, 53)	{37, 24, 36, 17, 35, 18, 34, 19, 33, 28, 32, 21, 31, 22, 30, 23, 29, 26, 27, 25}

For this last set of $T_{r,n}$, see figure 5 to get a visual flavor of the nature of these paths. The upward arrow indicates the vertex labeled 20. □

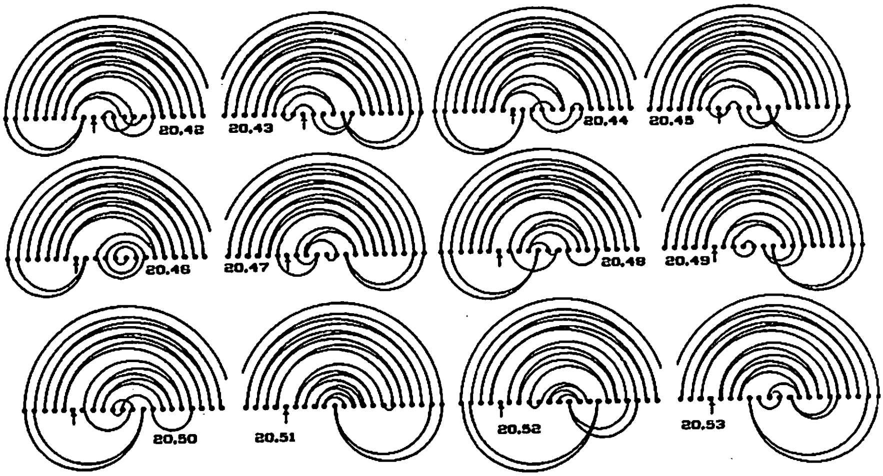

Figure 5.

The following class of graceful paths will also be useful in dealing with the dual problem of labeling trees. The path P_{2n} is said to be *ve(r)y-graceful* if it has an r strand such that for all uv in $E(P_{2n})$ with $\ell(u) < \ell(v)$ we have

$$\ell(u) \leq n < \ell(v).$$

Intuitively, every edge of a ve(r)y-graceful path spans the midpoint, $n + \frac{1}{2}$, of the vertex set $\{0, 1, \ldots, 2n+1\}$. The path of figure 1 is ve(2)y-graceful.

Conjecture 10. *P_{2n+2} is ve(r)y-graceful for all r and n where* $0 \leq r \leq 2n+1$.

The following results for Conjecture 10 parallel the preceding results for Conjecture 2, albeit on a less grand scale.

Theorem 11. *If $2r < n$ and $P_{2(n-r)}$ is ve(r)y-graceful, then $P_{2(n+1)}$ is ve(r)y-graceful.*

Proof: Note that the vertices of P_{2n+2} need to be labeled 0 through $2n+1$. Let T_0 be a very-graceful strand for r threads $2(n-r)-1$. Let

$T = T_0 + (r+1)$. Let

$$S = \{r, 2n-r+1, r-1, 2n-r+2, \ldots, 2n, 0, 2n+1, 2r+1\}.$$

With the condition $2r < n$, it is easy to see that $S \vee T$ is a strand for r threads $2n+1$. See figure 6.□

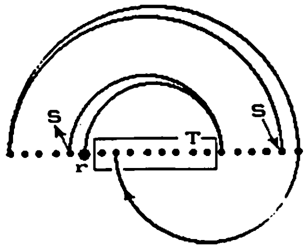

Figure 6.

Corollary 12. *If the path P_{2n+2} is ve(r)y-graceful for all n with $r \le n \le 2r$, then P_{2n+2} is ve(r)y-graceful for all n with $n \ge r$.*

Proof: Let $m > 2r$. Let k be the least positive integer with

$$m - k(r+1) \le 2r.$$

Then $m - k(r+1) \ge r$ since $m - (k-1)(r+1) > 2r$. So $P_{2(m-k(r+1))+2}$ is very-graceful by hypothesis. Apply Theorem 11 a total of k times, showing that P_{2m+2} is very-graceful.□

Proposition 13. *P_{2r+2} is ve(r)y-graceful for all $r \ge 0$.*

Proof: A very-graceful strand is

$$\{r, r+1, r-1, r+2, \ldots, 0, 2r+1\}.□$$

Theorem 14. *P_{2n+2} is ve(r)y-graceful for $r = 0, 1, 2, 3$, for all n with $n \ge r$.*

Proof: Proposition 13 and Corollary 12 along with the integer pairs (r, n) below are enough to prove the theorem.

(1,2)	{1,5,0,3,2,4}
(2,3)	{2,5,3,4,0,7,1,6}
(2,4)	{2,6,3,5,4,9,0,8,1,7}
(3,4)	{3,5,4,7,1,6,2,9,0,8}
(3,5)	{3,7,4,6,5,10,0,11,2,8,1,9}
(3,6)	{3,9,4,8,5,7,6,13,0,12,1,11,2,10}□

Proposition 15. *Let $V = \{0, 1, \ldots, n\} \cup \{n+k, n+k+1, \ldots, 2n+k\}$ where $k \ge 1$; let $E = \{k, k+1, \ldots, 2n+k\}$. If P_{2n+2} is ve(r)y-graceful with $n \ge r$, then the path is also $VEr-$graceful and $(V, E, 2n+k-r)$-graceful.*

Proof: Immediate.

Edge-Graceful Trees

Let G be a graph with $|V(G)| = p, |E(G)| = q$, and (ℓ, ℓ^*) be a function pair mapping the vertices and edges into the set of integers, as before. Following Lo [4], define G as *edge-graceful* if there is a function pair (ℓ, ℓ^*) such that ℓ is *onto* $\{0, 1, \ldots, p-1\}$, ℓ^* is *onto* $\{1, 2, \ldots, q\}$, and

$$\ell(u) = \left(\sum_{uv \in E(G)} \ell^*(uv) \right) \bmod p.$$

Of particular edge-graceful interest is the following conjecture.

Conjecture 16: Lee's Conjecture. *Every tree of odd order is edge-graceful.*

The current domain of odd ordered trees shown to be edge-graceful include all trees in which each vertex is of odd degree [5], all regular spiders [1,5,7], various trees obtained by performing certain transformations on regular spiders [5], and various trees which avoid having adjacent vertices of degree 2 [1].

A notion we find useful in studying this conjecture is the following. Let

$$P = \begin{cases} \{\pm 1, \ldots, \pm \frac{p}{2}\}, & \text{if } p \text{ is even,} \\ \{0, \pm 1, \ldots, \pm \frac{p-1}{2}\}, & \text{if } p \text{ is odd.} \end{cases}$$
$$Q = \begin{cases} \{\pm 1, \ldots, \pm \frac{q}{2}\}, & \text{if } q \text{ is even,} \\ \{0, \pm 1, \ldots, \pm \frac{q-1}{2}\}, & \text{if } q \text{ is odd,} \end{cases}$$

We say that G is *super-edge-graceful* if there is a function pair (ℓ, ℓ^*) such that ℓ is onto P, ℓ^* is onto Q and

$$\ell(u) = \sum_{uv \in E(G)} \ell^*(uv).$$

It is an easy exercise (see Theorem 1 of [5]) to see that if G is a tree of odd order and is super-edge-graceful, then G is edge-graceful. Hence to prove a tree is edge-graceful it is sufficient to show it is super-edge-graceful; which is the technique we use throughout this section.

A tree G is a *spider* graph if it has at most 1 vertex of degree greater than 2; such a vertex of a spider is called its *core*. If all the *legs* of a spider are of equal length, the spider is said to be *regular.* We say that a graph is a *paired-k*-spider, if the spider has k legs, $\{L_i\}_{i=1}^{k}$, where L_i and L_{k-i+1} have the same lengths for $1 \leq i \leq \frac{k}{2}$. Hence if a spider has an odd number of legs there is at most one leg of a unique length. If Lee's conjecture is true, it seems reasonable that it should be relatively easy to show that all spiders of odd degree having a few legs are super-edge-graceful. In particular we should be able to resolve the conjecture for spiders having only 4 legs.

Proposition 17. *If Conjecture* 1 *is true then every paired-4-spider is super-edge-graceful.*

Proof: Let G be a paired-4-spider with legs of length m and n where $1 \leq m \leq n$. Let $s = \lfloor \frac{m}{2} \rfloor$. Let

$$t = \begin{cases} s, & \text{if } m \text{ is odd,} \\ s-1, & \text{if } m \text{ is even.} \end{cases}$$

Hence $m = s + t + 1$. Let

$$V_1 = \{1, 2, \ldots, s\} \cup \{m+n-t, m+n-t+1, \ldots, m+n\},$$

and $E_1 = \{n+1, \ldots, m+n-1\}$. Let $S_1 = \{e_1, e_2, \ldots, e_m\} = \{m+n, 1, m+n-1, 2, \ldots, u\}$ be a $\{V_1, E_1, m+n\}$-graceful strand for P_m where u is either s or $m+n-t$. Let

$$L_1 = \{e_1, -e_2, e_3, -e_4, \ldots, (-1)^{m+1} e_m\};$$

$$M_1 = \{e_1, e_1 - e_2, -e_2 + e_3, \ldots, (-1)^m e_{m-1} + (-1)^{m+1} e_m\}.$$

Then $M_1 = \{n+1, n+2, \ldots, m+n\}$. Let

$$V_2 = \{s+1, s+2, \ldots, m+n-t-1\} \text{ and } E_2 = \{1, \ldots, n-1\}.$$

Since s threads $n-1$ by hypothesis then $n-s-1$ threads $n-1$. Let S_0 be a strand for $n-s-1$ threads $n-1$. Let $S_2 = S_0 + (s+1)$, so that S_2 starts with n. Furthermore S_2 is a $\{V_2, E_2, n\}$-graceful strand. Obtain L_2 and M_2 from S_2 in the same way that L_1 and M_1 were obtained from S_1. Let $L_4 = -L_1$ and $L_3 = -L_2$. Interpret $L_i, 1 \leq i \leq 4$, as the edge labels of the four legs of the spider from the outer edges to the core. Then the vertices of the spider are precisely $\pm M_1 \cup \pm M_2 \cup \{0\}$, where the outer vertices are labeled $\pm n, \pm(m+n)$ and the core is labeled 0. □

Theorem 18. *If G is a paired-4-spider whose shortest pair of legs is no more than* 41, *then G is super-edge-graceful.*

Proof: In the above proof, take $s \leq 20$, then $m \leq 41$. By Theorem 9 we are done. □

To illustrate the above theorem consider a paired-4-spider with leg pair lengths of 5 and 7. Since $s = 2 = t$, label the short leg $\{12, -1, 11, -2, 10\}$

from exterior to the core. Since 2 threads 6, then 4 threads 6 with strand $\{4,2,3,6,0,5,1\}$, making $S_2 = \{7,5,6,9,3,8,4\}$. Label the other leg

$$\{7,-5,6,-9,3,-8,4\}$$

from exterior to core. On the other two legs use the inverse labels, resulting in a super-edge-graceful labeling.

Proposition 19. *Let $\{d_i\}_{i=1}^k$ be a sequence of positive even integers with $d_j \geq \sum_{i=1}^{j-1} d_i$. Let G be a paired-2k-spider whose leg lengths form this sequence. If Conjectures 2 and 10 are true then G is super-edge-graceful.*

Proof: For $1 \leq j \leq k$, let $s_j = \dfrac{d_j}{2}$ and

$$V_j = \{\sum_{i=1}^{j-1} s_i + 1, \ldots, \sum_{i=1}^{j} s_i\} \cup \{\sum_{i=1}^{k} d_i - \sum_{i=1}^{j} s_i + 1, \ldots, \sum_{i=1}^{k} d_i - \sum_{i=1}^{j-1} s_i\}.$$

It follows that

$$\sum_{i=j}^{k} d_i \in V_j \text{ for all } j, 1 \leq j \leq k.$$

Observe also that $\{V_j\}_{j=1}^k$ is a pairwise disjoint partition of the integers from 1 to $\sum_{i=1}^k d_i$. Let

$$E_j = \{\sum_{i=j+1}^{k} d_i + 1, \ldots, \sum_{i=j}^{k} d_i - 1\}.$$

Observe that $\{E_j\}_{j=1}^k$ is a pairwise disjoint partition of the integers from 1 to $\sum_{i=1}^k d_i$, excluding those integers of the form $\sum_{i=j}^k d_i$. (We interpret $\sum_{i=k+1}^k d_i$ as 0.) By hypothesis let S_j be a $(V_j, E_j, \sum_{i=j}^k d_i)$-graceful strand for P_{d_j}. Obtain L_j and M_j from S_j in the manner as described in the proof of Theorem 17. As sets, observe that $M_j = E_j$. For each $j, 1 \leq j \leq 2k$, let $L_{2k-j+1} = -L_j$. Interpret $L_j, 1 \leq j \leq 2k$, as the edge labels of the edges of the spider of length d_j, from the outer edge towards the core. Then the vertices of the spider are precisely $\cup_{j=1}^k \pm M_j \cup \{0\}$, where the outer vertices are labeled $\pm \sum_{i=j}^k d_i$ and the core is labeled 0. □

Theorem 20. *The following paired-k-spiders are super-edge-graceful.*

1. *The paired-6-spider with* $d_1 = 2, 4 \leq d_2 \leq 38, d_3 \geq d_2 + 4$.
2. *The paired-6-spider with* $d_1 = 4, 6 \leq d_2 \leq 36, d_3 \geq d_2 + 6$.
3. *The paired-6-spider with* $d_1 = 6, 8 \leq d_2 \leq 34, d_3 \geq d_2 + 8$.
4. *The paired-8-spider with* $d_1 = 2, d_2 = 4, 8 \leq d_3 \leq 34, d_4 \geq d_3 + 8$.

Proof: Let us only prove case 3, since the arguments for the other cases are quite similar. Let $d = d_1 + d_2 + d_3$.Let

$$L_1 = \{d, -1, d-1, -2, d-2, -3\},$$

so that $M_1 = \{d, d-1, d-2, d-3, d-4, d-5\}$. On L_2 we want to use the labels

$$V_2 = \{-4, -5, \ldots, -\frac{d_2+d_1}{2}\} \cup \{d - \frac{d_2+d_1}{2} + 1, \ldots, d-6, d-5, d-4, d-3\},$$

starting with $d-6$. Since P_{2n} is ve(r)y-graceful for $r = 3$ for all $n > 3$, let L_2 be such a strand, so that M_2 as a set is $\{d-6, \ldots, d-d_1-d_2+1\}$. On L_3 we shall use the labels

$$V_3 = \{-\frac{d_2+d_1}{2} + 1, \ldots, -\frac{d}{2}, \frac{d}{2} + 1, \ldots, d - \frac{d_2+d_1}{2}\},$$

starting with the label $d - d_1 - d_2 = d_3$ so that M_3 as a set is $\{1, \ldots, d_3\}$; note that by Proposition 19, $d_3 \in V_3$. We can label L_3 accordingly when $(d - \frac{d_1+d_2}{2}) - d_3 \leq 20$ by Theorem 9, which means that $d_1 + d_2 \leq 40$. □

Theorem 21. *Let G be a paired-2k-spider with legs of length $\{2^i\}_{i=i}^{k}$. Then G is super-edge-graceful.*

Proof: Following Proposition 19, let $d_i = 2^i$. Then

$$V_j = \{2^{j-1}, \ldots, 2^j - 1\} \cup \{2^{k+1} - 2^j, \ldots, 2^{k+1} - 2^{j-1} - 1\}.$$

Furthermore,

$$\sum_{i=j}^{k} d_i = 2^{k+1} - 2^j \text{ and } E_j = \{2^{k+1} - 2^{j+1} + 1, \ldots, 2^{k+1} - 2^j - 1\}.$$

By Propositions 13 and 15, P_{2^j} is $(V_j, E_j, 2^{k+1} - 2^j)$-graceful for all j. Now use this graceful labeling to super-edge-graceful label G as described in the proof of Proposition 19.□

For example consider the paired-8-spider with leg pair lengths of 2,4,8,16. Using the labeling scheme as given by the theorem results in labeling the four sets of legs as follows, where the labeling sequence proceeds from exterior to core:

$$\{30, -1\}, \ \{-30, 1\},$$

$$\{28, -3, 29, -2\}, \ \{-28, 3, -29, 2\},$$

$$\{24, -7, 25, -6, 26, -5, 27, -4\}, \ \{-24, 7, -25, 6, -26, 5, -27, 4\},$$

$$\{16, -15, 17, -14, 18, -13, 19, -12, 20, -11, 21, -10, 22, -9, 23, -8\},$$

$\{-16, 15, -17, 14, -18, 13, -19, 12, -20, 11, -21, 10, -22, 9, -23, 8\}$.

Furthermore, theorems 18, 20, 21 can be used to construct other trees not previously known to be edge-graceful. That is, suppose G is a super-edge-graceful paired-k-spider with core c. With the Cut & Paste Algorithm of [5], dissect the spider into $\lfloor k/2 \rfloor$ leg pairs each with a copy of the core vertex c, which now has label 0. Starting with the tree consisting of any leg pair (or of the single odd leg if the spider has an odd number of legs), append to this tree any leg pair by affixing its copy of the core to any vertex of the tree. Continue appending leg pairs in this fashion until there are no unattached leg pairs. The resulting tree is super-edge-graceful.

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