Existence of Self-orthogonal Latin Squares ISOLS($6m+2, 2m$)*

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Abstract

An incomplete self-orthogonal latin square of order v with an empty subarray of order n, an ISOLS(v, n), can exist only if $v \geq 3n+1$. It is well known that an ISOLS(v, n) exists whenever $v \geq 3n+1$ and $(v, n) \neq (6m+2, 2m)$. In this paper we show that an ISOLS($6m+2, 2m$) exists for any $m \geq 24$.

1 Introduction

A *self-orthogonal* latin square of order v, an SOLS(v), is a latin square of order v which is orthogonal to its transpose. It is well known that an SOLS(v) exists for all values of v, $v \neq 2, 3$ or 6.

An *incomplete* self-orthogonal latin square of order v is a $v \times v$ latin array $A = (a_{ij})$ with row and column indices and entries taken from the set $I_{v-n} \cup X$, $I_{v-n} = \{1, 2, \cdots, v-n\}$, $X = \{x_1, x_2, \cdots, x_n\}$, and with an empty subarray of order n so that

$$\begin{aligned}(I_{v-n} \times I_{v-n}) \cup (I_{v-n} \times X) \cup (X \times I_{v-n}) \\ = \{(a_{ij}, a_{ji}) : (i,j) \in (I_{v-n} \times I_{v-n}) \cup (I_{v-n} \times X) \cup (X \times I_{v-n})\}.\end{aligned}$$

We denote such an array by ISOLS(v, n).

Simple counting shows that a necessary condition for the existence of an ISOLS(v, n) is $v \geq 3n+1$. There has been considerable work done on

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the existence of such arrays [3], [5–6], [9–11], [13–15]. The known results can be summarized as follows. There exists an ISOLS(v, n) for all values of v and n satisfying $v \geq 3n+1$, except for $v = 6$ and $(v, n) = (8, 2)$ and perhaps excepting $(v, n) = (6m+2, 2m), m \geq 2$. In this paper we show that an ISOLS($6m+2, 2m$) exists for any integer $m \geq 24$.

In section 2 we define SOLS with holes and present the "Filling in Holes" construction. In section 3 we state Inflation Construction using various kinds of transversals and in section 4 we present an ISOLS(11; 3, 2) and use it to find the first example of an ISOLS(38, 12). To obtain an ISOLS($6m+2, 2m$) from the ISOLS(38, 12) we need a new type of SOLS which is discribed in section 5. Our main result is given in section 6, where we also conjecture that an ISOLS($6m+2, 2m$) exists for $2 \leq m \leq 23$.

For concepts used but not defined in this paper the reader is referred to the book of Beth, Jungnickel and Lenz [1].

2 Holey SOLS

We begin by defining a holey SOLS.

Let S be a set and let $\mathbf{H}$ be a set of subsets of S. A *holey latin square* having *hole set* $\mathbf{H}$ is a $|S| \times |S|$ array, L, indexed by S, which satisfies the following properties :

1. every cell of L either is empty or contains a symbol of S,
2. every symbol of S occurs at most once in any row or column of L,
3. the subarrays $H \times H$ are empty, for every $H \in \mathbf{H}$ (these subarrays are referred to as *holes*),
4. symbol $s \in S$ occurs in row or column t if and only if $(s, t) \in (S \times S) \setminus \cup_{H \in \mathbf{H}} (H \times H)$.

The *order* of L is $|S|$.

Two holey latin squares on symbol set S and hole set $\mathbf{H}$, say L_1 and L_2, are said to be *orthogonal* if their superposition yields every ordered pair in $(S \times S) \setminus \cup_{H \in \mathbf{H}} (H \times H)$. We shall use the notation IMOLS($s; s_1, \cdots, s_n$) to denote a pair of orthogonal holey latin squares on symbol set S and hole set $\mathbf{H} = \{H_1, \cdots, H_n\}$, where $s = |S|$ and $s_i = |H_i|$ for $1 \leq i \leq n$.

If $\mathbf{H} = \{H\}$, we simply write IMOLS(s, $|H|$) for the orthogonal pair of holey latin squares. We write IMOLS(s; $|H_1|$, $|H_2|$) for the case $\mathbf{H} = \{H_1, H_2\}$, and $H_1 \cap H_2 = \emptyset$.

If L_1 and L_2 form IMOLS(s; $s_1, \cdots, s_n$) such that L_2 is the transpose of L_1, then we call L_1 a *holey* SOLS, denoted by ISOLS(s; $s_1, \cdots, s_n$). We shall now identify several particular special cases of holey SOLS that will be useful in recursive constructions. First, if $\mathbf{H} = \emptyset$, then a holey SOLS is just an SOLS of order $|S|$. Also, if $\mathbf{H} = \{H\}$, then a holey SOLS is an ISOLS($|S|$, $|H|$). If $\mathbf{H} = \{H_1, H_2\}$, then we write ISOLS($|S|$; $|H_1|$, $|H_2|$) for the case ä $H_1 \cap H_2 = \emptyset$.

If $\mathbf{H} = \{S_1, \cdots, S_n\}$ is a partition of S, then a holey SOLS is called a *frame* SOLS. The *type* of the frame SOLS is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We shall use an "exponential" notation to describe types : so type $t_1^{u_1} t_2^{u_2} \cdots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$, in the multset.

We observe that existence of an SOLS(n) is equivalent to existence of a frame SOLS of type 1^n, and existence of an ISOLS(n, s) is equivalent to existence of a frame SOLS of type $1^{n-s} s^1$.

If $\mathbf{H} = \{S_1, \cdots, S_n, T\}$, where $\{S_1, \cdots, S_n\}$ is a partition of S, then a holey SOLS is called an *incomplete frame* SOLS or an *I-frame* SOLS. The *type* of the I-frame SOLS is defined to be the multiset $\{(|S_i|, |S_i \cap T|) : 1 \leq i \leq n\}$. We may also use an "exponential" notation to describe types of I-frame SOLS.

We now discuss the idea of Filling in Holes.

Construction 2.1 (Filling in Holes) [13, Lemma 1.11] Suppose there is a frame SOLS of type $\{s_i : 1 \leq i \leq n\}$, and let $a \geq 0$ be an integer. For $1 \leq i \leq n-1$, suppose there is an ISOLS($s_i + a, a$). Then there is an ISOLS($s + a, s_n + a$), where $s = \sum_{1 \leq i \leq n} s_i$.

We now give a generalization of Filling in Holes which starts with an I-frame SOLS.

Construction 2.2 (Generalized Filling in Holes) Suppose there is an I-frame SOLS of type $\{(s_i, t_i) : 1 \leq i \leq n\}$, and let $a \geq 0$ be an integer. For $1 \leq i \leq n-1$, suppose there is an ISOLS($s_i + a$; t_i, a). Also, suppose there is an ISOLS($s_n + a, t_n$). Then there is an ISOLS($s + a, t$), where $s = \sum s_i$ and $t = \sum t_i$.

3 Transversals and inflation constructions

If T is the type $t_1^{u_1} t_2^{u_2} \cdots t_k^{u_k}$ and m is an integer, then mT is defined to be the type $(mt_1)^{u_1}(mt_2)^{u_2}\cdots(mt_k)^{u_k}$. The following recursive construction is referred to as the Inflation Construction. It essentially "blows up" every occupied cell of a frame SOLS into a latin square such that if one cell is filled with a certain latin square, then its symmetric cell is filled with the transpose of an orthogonal mate of the latin square. We mention the work [4], [16] which can be thought of as sources of the Inflation Construction.

Construction 3.1 (Inflation Construction) Suppose there is a frame SOLS of type T, and suppose m is a positive integer, $m \neq 2$ or 6. Then there is a frame SOLS of type mT.

To obtain an I-frame SOLS from a frame SOLS we may "blow up" every occupied cell into a latin square with one hole.

Construction 3.2 Suppose there is a frame SOLS of type $t_1^{u_1} t_2^{u_2} \cdots t_k^{u_k}$, and suppose there is an IMOLS$(m+a, a)$. Then there is an I-frame SOLS of type $\prod_{1\leq i\leq k}(t_i(m+a), t_i a)^{u_i}$

Suppose F is a frame SOLS with holes $S_1, \cdots, S_n$, and $S = \cup S_i$. A *transversal* is a set T of $|S|$ occupied cells in F such that every symbol is contained in exactly one cell of T and the cells in T intersect each row and each column in exactly one cell. We call two transversals *disjoint* if they have no cell in common. A transversal T is *symmetric* if $(i,j) \in T$ implies $(j,i) \in T$. A pair of transversals T_1 and T_2 are *symmetric* if $(i,j) \in T_1$ implies $(j,i) \in T_2$. Here is another generalization of the Inflation Construction.

Construction 3.3 Suppose there is a frame SOLS of type t^g which has $2k+l$ disjoint transvcersals, l of them being symmetric and the rest being k symmetric pairs. For $1 \leq i \leq l$ and $1 \leq j \leq k$, let $u_i \geq 0$ and $v_j \geq 0$ be integers. Let m be a positive integer, $m \neq 2$ or 6, and suppose there exist IMOLS$(m+u_i, u_i)$ for $1 \leq i \leq l$ and IMOLS$(m+v_j, v_j)$ for $1 \leq j \leq k$. Then there is a frame SOLS of type $(mt)^g(u+2v)^1$, where $u = \sum u_i$ and $v = \sum v_j$.

Suppose F is a frame SOLS with holes $S_1, \cdots, S_n$, where $S = \cup S_i$. A *holey transversal* with hole S_1 is a set T of $|S| - |S_1|$ occupied cells in F such that every symbol of $S \backslash S_1$ is contained in exactly one cell of T and the $|S| - |S_1|$ cells in T intersect each row and each column indexed by $S\backslash S_1$ in exactly one cell. A holey transversal T is *symmetric* if $(i,j) \in T$ implies $(j,i) \in T$. A holey symmetric transversal will be referred to as an HS

transversal. The following construction is a modification of Construction 3.3, in which holey transversals are used.

Construction 3.4 Suppose there is a frame SOLS of type $h^1 t^g$, where H is the size h hole, having k HS transversals with hole H such that all these transversals are disjoint. For $1 \leq j \leq k$, let v_j be non-negative integers. Let m be a positive integer, $m \neq 2$ or 6, and suppose there exist IMOLS($m + v_j, v_j$) for $1 \leq j \leq k$. Then there is a holey SOLS with g holes of size mt and one hole of size $mh + v$, where $v = \sum v_j$.

4 An example ISOLS(38, 12)

In this section we shall state some known results useful in obtaining our main result. We shall also give the first example in the class ISOLS($6m + 2, 2m$), namely an ISOLS(38, 12).

The following theorems provide the "ingredients" when applying the recursive constructions given in the previous two sections.

Theorem 4.1 [2] There exists an MOLS(v) for all values of v, $v \neq 2$ or 6.

Theorem 4.2 [12] There exists an IMOLS(v, n) for all values of v and n satisfying $v \geq 3n$ except that an IMOLS(6, 1) does not exist.

Theorem 4.3 [3] There exists an SOLS(v) for all values of v, $v \neq 2, 3$ or 6.

Theorem 4.4 [13, 14] There exists an ISOLS(v, n) for all values of v and n satisfying $v \geq 3n + 1$, except for $v = 6$ and $(v, n) = (8, 2)$ and perhaps excepting $(v, n) = (6m + 2, 2m)$, $m \geq 2$.

We are now in a position to give our first example.

Lemma 4.5 There exists an ISOLS(38, 12).

Proof Start with a frame SOLS of type 1^4 which comes from an SOLS(4) in Theorem 4.3. Apply Construction 3.2 with an IMOLS(9, 3) which comes from Theorem 4.2. We obtain an I-frame SOLS of type $(9, 3)^4$. Further apply Construction 2.2 using an ISOLS(11; 3, 2) shown in Table 4.1. Since an ISOLS(11, 3) exists from Theorem 4.4, we obtain the required ISOLS(38, 12).

1	7	5	2	10	9	4	11	3	6	8
6	2	8	7	3	10	1	5	11	4	9
9	4	3	10	8	1	11	2	6	5	7
5	11	9	4	7	2	6	10	1	8	3
7	6	11	3	5	8	2	4	10	9	1
11	8	4	9	1	6	10	3	5	7	2
3	10	6	1	11	5				2	4
4	1	10	6	2	11				3	5
10	5	2	11	4	3				1	6
8	9	7	5	6	4	3	1	2		
2	3	1	8	9	7	5	6	4		

Table 4.1 ISOLS(11;3,2)

A frame SOLS of type $k^1 1^{n-k}$ which has a holey symmetric transversal with size k hole (so $n-k$ is even) will be referred to as an HSTFSOLS(n, k). If $k = 0$, we simply write an HSTFSOLS(n, 0) as an STFSOLS(n).

Lemma 4.6 If there exists an HSTFSOLS(n, k), then there exists an ISOLS($3n+2, n$), provided an ISOLS($3k+2, k$) exists.

Proof First, apply Construction 3.4 to the given HSTFSOLS(n, k) and take $m = 3$ and $v = 1$. We obtain an I-frame SOLS of type $(3k+1, k)^1(3,1)^{n-k}$. Next, apply Construction 2.2 with the requisite ISOLS(4; 1, 1) from Theorem 4.3 and the known ISOLS($3k+2, k$). We obtain the required ISOLS($3n+2, n$).

In view of Lemma 4.6, the existence of an ISOLS($6m+2, 2m$) can be obtained from the existence of an HSTFSOLS($2m$, 12) since an ISOLS(38, 12) is already known from Lemma 4.5.

5 Existence of HSTFSOLS($2m$, 12)

In this section we shall show the existence of an HSTFSOLS($2m$, 12) for any $m \geq 50$. The following known result is useful.

Theorem 5.1 [7], [17] For all even n, $n \notin E = \{2, 6, 10, 14, 46, 54, 58, 62, 66, 70\}$, there exists an SOLS($n$) with a symmetric orthogonal mate which has a constant element on the main diagonal.

Since each element of the orthogonal mate in Theorem 5.1 determines a symmetric transversal in the SOLS(n) when n is even and $n \notin E$, there exists a frame SOLS of type 1^n having $n-1$ disjoint symmetric transversals. With this in mind we restate Theorem 2.5 of [14] as follows.

Lemma 5.2 (a) If q is an odd prime power, $q \geq 5$, then there exists a frame SOLS of type 1^q, having $q-1$ disjoint transversals which occur as $(q-1)/2$ symmetric pairs. (b) If q is an even integer and $q \notin E$, then there exists a frame SOLS of type 1^q having $q-1$ disjoint symmetric transversals. In particular, there exists an $STFSOLS(q)$.

Lemma 5.3 Suppose q is an even integer, $q \notin E$, or $q \geq 5$ is an odd prime power. For $1 \leq i \leq q-1$ let m and u_i be integers, $0 \leq 2u_i \leq m$, $m \neq 6$. If there exists an $STFSOLS(m)$, then there exists an HSTFSOLS($qm+u, u$), u is even and $u = \sum u_i$. Further, if there exists an $STFSOLS(u)$, then there exists an HSTFSOLS($qm+u, m$).

Proof First, applying Lemma 5.2 we obtain a frame SOLS of type 1^q having $q-1$ disjoint symmetric transversals when q is even, or $(q-1)/2$ symmetric pairs of transversals. Next, apply Construction 3.3 with the required IMOLS($m+u_i, u_i$) from Theorem 4.2, we obtain a frame SOLS of type $m^q u^1$, where we can take $u_1 = u_2, u_3 = u_4$ etc if q is odd since u is even. Then the conclusion follows from the Filling in Holes construction.

We now prove the existence of an HSTFSOLS($2n$, 12).

Lemma 5.4 There exists an HSTFSOLS($2n$, 12) for any integer $n \geq 50$.

Proof For $n \geq 180$ we may write $2n = 12q + u$ such that $q \equiv 0 \pmod 4$ and $q \geq 24$ and such that u is even and $72 \leq u \leq 120$: then $q, u \notin E$. We can apply Lemma 5.3 with m = 12 and suitably chosen u_i such that $0 \leq u_i \leq 6$ and $u = \sum u_i$. Since both STFSOLS(12) and STFSOLS(u) exist, we obtain an HSTFSOLS($2n$, 12) for $2n \geq 360$.

For $100 \leq 2n \leq 358$ we again apply Lemma 5.3 to obtain an HSTFSOLS($2n$, 12) shown in Table 5.1. The required ingredients are all known from Theorem 5.1, Theorem 4.1 and Theorem 4.2.

6 Concluding remarks

Combining Lemmas 5.4, 4.6 and 4.5 we have the main result of this paper.

Theorem 6.1 An ISOLS($6m+2, 2m$) exists for any $m \geq 50$.

We conjecture that an ISOLS($6m+2, 2m$) also exists for $2 \leq m \leq 49$. Some of them can be constructed using methods similar to the above so that only numbers $2m$ for which the existence of an ISOLS($6m+2, 2m$)

Table 5.1

q	m	u	$2n = qm + u$
7	12	16 - 26	100 - 110
8	12	16 - 26	112 - 122
9	12	16 - 38	124 - 146
11	12	16 - 38	148 - 170
13	12	16 - 44	172 - 200
5	38	12	202
13	12	48 - 50	204 - 206
16	12	16 - 38	208 - 230
18	12	16 - 38	232 - 254
20	12	16 - 38	256 - 278
22	12	16 - 38	280 - 302
24	12	16 - 38	304 - 326
26	12	16 - 38	328 - 350
28	12	16 - 22	352 - 358

is still unknown are $2m \in \{4, 6, 8, 10, 14, 16, 18, 20, 22, 26, 28, 32, 34, 46\}$. It is easy to see that Lemma 4.5 with SOLS(4) replaced by SOLS(k) for k even and $8 \leq k \leq 32$ gives designs for 2m $\in$ {24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96}. To resolve the case 2m = 38, one observes that the ISOLS(38, 12) is in fact an HSTFSOLS(38, 12). Applying Lemma 5.3 with q = 7, m = 4 and u = 12 gives an HSTFSOLS(40, 12). An HSTFSOLS(44, 12) can be constructed similarly by taking q = 4, m = 8 and u = 12. If one takes m = 12 and q $\in$ {4,5,7 } in Lemma 5.3, one may obtain an HSTFSOLS($2m$, 12) for 2m $\in$ {52,56,64,68,76,80,82,88, 92}. The remaining eight numbers 2m can be done by a more complicated argument. In [8], an ISOLS(12, 2) with 12 disjoint symmetric transversals is given, of caurse two of them are HS transversals. Apply Inflation Construction with ISOLS($m + u_i, u_i$) for $1 \leq i \leq 12$, where $u = \sum u_i$ and $u_1 = u_2 = 1$ if m = 4, or $u_1 = 2$ if m = 5,7. If one fills in the size u hole with an HSTFSOLS(u, 2) and another hole of size 2m + 2 with an HSTFSOLS($2m + 2$, 2), then one obtains an HSTFSOLS($12m + u$, 12). Taking m = 4 and u = 2,10 gives an HSTFSOLS($2n$, 12) for 2n = 50,58. Taking m = 5 and u = 2, 10, 14 solves the casees 2n = 62,70,74 and taking m = 7 and u = 2, 10, 14 solves the cases 2n = 86, 94, 98. The ISOLS(10, 2) and ISOLS(14, 2) given by the starter-adder type construction in [14,Table 2.1] are indeed an HSTFSOLS(10, 2) and an HSTFSOLS(14, 2) respectively. The requisite HSTFSOLS(12, 2) and HSTFSOLS(16, 2) come from the same source. This takes care of all cases for $2m \leq 98$ leaving 14 unknown cases. Further, combining this with Theorems 4.4 and 6.1 we have the best knowm result on the existence of an ISOLS(v, n).

Theorem 6.2 There exists an ISOLS(v, n) for all values of v and n satisfying $v \geq 3n+1$, except for $v = 6$ and $(v, n) = (8, 2)$ and perhaps excepting $(v, n) = (6m+2, 2m)$, $2m \in \{4, 6, 8, 10, 14, 16, 18, 20, 22, 26, 28, 32, 34, 46\}$.

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References

[1] T. Beth, D. Jungnickel and H. Lenz, Design Theory(Cambridge University Press, 1986).
[2] R.C. Bose, S.S. Shrikhande and E.T. Parker, Further results on the construction of mutually orthogonal latin squares and the falsity of Euler's conjecture, Canadian J. Math. 12(1960) 189–203.
[3] R.B. Brayton, D. Coppersmith and A.J. Hoffman, Self-orthogonal latin squares of all orders $n \neq 2, 3, 6$, (Coll. Internationale sulle Teorie Combinatorie, Roma, 1973, Atti dei convegni Lincei, No. 17, Tomo II, 1976), 509–517.
[4] A.E. Brouwer and G.H.J. van Rees, More mutually orthogonal latin squares, Discrete Math. 39(1982) 263–281.
[5] D.J.Crampin and A.J.W. Hilton, On the spectra of certain type of latin square, J. Combinatorial Theory 19 (1975) 84–94.
[6] D.A. Drake and H. Lenz, Orthogonal latin squares with orthogonal subsquares, Arch. Math. 34(1980) 565–576.
[7] B. Du, A few more resolvable spouse-avoiding mixed-doubles round

robin tournaments, Ars Combinatoria, to appear.
[8] B. Du, On the existence of incomplete transversal designs with block size five, Discrete Math., to appear.
[9] A. Hedayat, A generalization of sum composition: self-orthogonal latin square design with sub self-orthogonal latin square designs, J. Combinatorial Theory Ser. A 24(1978) 202–210.
[10] K. Heinrich, Self-orthogonal latin squares with self-orthogonal subsquares, Ars Combinatoria 3(1977) 251–266.
[11] K. Heinrich, Latin squares with and without subsquares of prescribed type, Ann. Discrete Math. 46(1991).
[12] K. Heinrich and L. Zhu, Existence of orthogonal latin squares with aligned subsquares, Discrete Math. 59(1986) 69–78.
[13] K. Heinrich and L. Zhu, Incomplete self-orthogonal latin squares, J. Austral. Math. Soc. Ser. A 42(1987) 365–384.
[14] K. Heinrich, L. Wu and L. Zhu, Incomplete self-orthogonal latin squares ISOLS($6m + 6, 2m$) exist for all m, Discrete Math. 87(1991) 281–290.
[15] E.T. Parker, Orthogonal latin squares, Proc. Nat. Acad. Sci. U.S.A. 45(1959) 859–862.
[16] D.R. Stinson, A general construction for group divisible designs, Discrete Math. 33(1981) 89–94.
[17] S.M.P. Wang, On self-orthogonal latin squares and partial transversals of latin squares, Ph.D. Thesis, Ohio State University, Columbus Ohio, 1978.