REGULAR TRIPLES WITH RESPECT TO A HYPEROVAL

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Abstract. In a previous paper, [6], we associated with every hyperoval of a projective plane of even order a Hadamard 2−design and investigated when this design has lines with three points. We study further this problem using the concept of regular triple and prove the existence of lines with three points in Hadamard designs associated with translation hyperovals. In the general case, the existence of a secant line of regular triples implies that the order of the projective plane is a power of two.

1. Introduction

Let Π_q be a projective plane of order q. A *hyperoval* of Π_q is a subset Ω of $q+2$ points no three of which are collinear. It is well known that q must be even and that every line of Π_q meets Ω in 0 or 2 points. The lines meeting Ω in 0 (resp., 2) points are said to be *exterior* (resp., *secant*). For a reference on hyperovals see [2], [11].

It is possible to associate with Ω a Hadamard 2−design, $\mathcal{H}$, in the following way. We denote the point-set of Π_q by the same symbol Π_q.

For every point P of Π_q not belonging to Ω, let

$$E(P) = \{Q \notin \Omega \mid Q \neq P \text{ and the line } PQ \text{ is exterior}\}$$
$$S(P) = \{Q \notin \Omega \mid Q \neq P \text{ and the line } PQ \text{ is secant}\} \cup \{P\}.$$

The incidence structure, $\mathcal{D}$, whose points are those of $\Pi_q \setminus \Omega$ and whose blocks are the sets $E(P)$, for every $P \notin \Omega$, is a symmetric $2-(q^2-1, q^2/2, q^2/4)$−design, which is the complementary design of a Hadamard 2−design, $\mathcal{H}$, whose blocks are the sets $S(P)$, see [6].

In [6] we give some necessary and sufficient conditions for a line of $\mathcal{H}$ to have three points, which is the maximum number of points on every line of $\mathcal{H}$.

In this paper we investigate further this problem introducing the concept of *regular triple*, that is a non ordered triple $\{X, Y, Z\}$ of distinct points of $\Pi_q \setminus \Omega$ such that $E(X) \cap E(Y) \cap E(Z) = \emptyset$. A regular triple exists if and only if there is a line of $\mathcal{H}$ with three points.

In Section 2 we give necessary and sufficient conditions for the existence of regular triples (Theorems 1 and 2). Furthermore, we will prove that if there exists a secant line of regular triples, then it is possible to construct a Steiner triple system on $q-1$ points, which is the 2−design of points and lines of a projective geometry over $GF(2)$, the Galois field of order 2 (Theorem 3). Thus, in this case, q is a power of two.

In Section 3, we will give a further necessary and sufficient condition for the existence of regular triples. This condition is satisfied by every translation hyperoval, as it is proved in Section 4.

2. Regular triples

Let Π_q be a projective plane of order q and Ω a hyperoval of Π_q. We will denote by $\mathcal{H}$ the Hadamard 2–design associated with Ω.

DEFINITION. *A non ordered triple $\{X, Y, Z\}$ of distinct points of $\Pi_q \setminus \Omega$ is said to be regular with respect to Ω if $E(X) \cap E(Y) \cap E(Z) = \emptyset$.*

The existence of a regular triple is equivalent to the existence of a line of $\mathcal{H}$ with three points (see [6, Proposition 4.1]).

It is obvious that given X and Y there exists at most one Z such that $\{X, Y, Z\}$ is regular, since every line of $\mathcal{H}$ has at most three points.

For every X and Y not in Ω, we put:

$$\mathsf{W} = E(X) \cap E(Y), \ \mathsf{X} = E(X) \setminus \mathsf{W}, \ \mathsf{Y} = E(Y) \setminus \mathsf{W}, \ \mathsf{A} = S(X) \cap S(Y).$$

A simple count proves that $|\mathsf{W}| = |\mathsf{X}| = |\mathsf{Y}| = \frac{q^2}{4}$ and $|\mathsf{A}| = \frac{q^2}{4} - 1$.

The following two theorems are proved in [6].

THEOREM A. *For every X, $Y \notin \Omega$ there exists Z such that $\{X, Y, Z\}$ is regular if and only if $q = 4$.*

THEOREM B. *If $\{X, Y, Z\}$ is regular and $q > 4$, then X, Y and Z are on a same secant line.*

From now on we suppose $q > 4$ and fix a secant line, ℓ.

If X and Y are distinct points on ℓ, X, $Y \notin \Omega$, we denote by Z_1, $Z_2, \ldots,$ Z_{q-3} the other points of $\ell \setminus (\ell \cap \Omega)$ and define:

$$\lambda_k = | E(X) \cap E(Y) \cap E(Z_k) | = | \mathsf{W} \cap E(Z_k) | \ , \ k = 1, 2, \ldots, q-3.$$

Therefore, $\{X, Y, Z_k\}$ is regular if and only if $\lambda_k = 0$.

We have the following identity:

$$\sum_{k=1}^{q-3} \lambda_k = \frac{q}{2}\left(\frac{q^2}{4} - q\right) = \frac{q^2}{8}(q-4) \tag{1}$$

which is obtained counting in two different ways the pairs (P, Z_k), $P \in \mathsf{W}$, such that the line PZ_k is exterior, observing that for every $P \in \mathsf{W}$ there are exactly $(q/2) - 2$ points Z_k such that the line PZ_k is exterior.

THEOREM 1. *The triple $\{X, Y, Z_k\}$ is regular if and only if $q \equiv 0 \pmod 4$ and $|E(P) \cap \mathbf{W}| = \frac{q^2}{8}$, for every $P \notin \Omega$, $P \neq X, Y, Z_k$.*

Proof: If $\{X, Y, Z_k\}$ is regular, then $E(Z_k) = \mathbf{X} \cup \mathbf{Y}$ and for every $P \neq X, Y, Z_k$, $|E(P) \cap \mathbf{W}| = h$, $0 < h < q^2/4$. Therefore

$$|E(P) \cap \mathbf{X}| = |E(P) \cap \mathbf{Y}| = (q^2/4) - h.$$

As $|E(P) \cap E(Z_k)| = q^2/4$, so $q^2/4 = 2\left((q^2/4) - h\right)$ follows. Hence $q \equiv 0 \pmod 4$ and $h = q^2/8$.

Conversely, let $q \equiv 0 \pmod 4$ and $|E(P) \cap \mathbf{W}| = q^2/8$, for every $P \notin \Omega$, $P \neq X, Y, Z_k$. Equality (1) may be written as

$$\left(\sum_{j=1, j \neq k}^{q-3} \lambda_j\right) + \lambda_k = \frac{q^2}{8}(q-4).$$

Since $\lambda_j = q^2/8$, $j = 1, \ldots, q-3, j \neq k$, then $\lambda_k = 0$, that is, the triple $\{X, Y, Z_k\}$ is regular. □

The next theorem is a refinement of the previous one.

THEOREM 2. *The triple $\{X, Y, Z_k\}$ is regular if and only if $q \equiv 0 \pmod 4$ and every line neither through X, Y nor Z_k has $q/4$ points on $\mathbf{W}$.*

Proof: If $q \equiv 0 \pmod 4$ and every line neither on X, Y nor Z_k has $q/4$ points on $\mathbf{W}$, then $\lambda_j = q^2/8$, for every $j = 1, \ldots, q-3$, $j \neq k$ and $\lambda_k = 0$. Thus $\{X, Y, Z_k\}$ is regular.

To prove the converse, let a_k be the number of pairs $(P, Q) \in \mathbf{W} \times \mathbf{W}$, such that the line PQZ_k is exterior, $k = 1, \ldots, q-3$, and let α_{ki} be the number of points that the i-th exterior line through Z_k has on $\mathbf{W}$. We have:

$$\lambda_k = \sum_{i=1}^{q/2} \alpha_{ki} \qquad \text{and} \qquad a_k = \sum_{i=1}^{q/2} \alpha_{ki}^2.$$

If n_{kj} is the number of exterior lines on Z_k which have j points on $\mathbf{W}$, $j = 0, 1, \ldots, q/2$, then

$$\sum_{j=0}^{q/2} n_{kj} = \frac{q}{2}, \qquad \sum_{j=0}^{q/2} j n_{kj} = \lambda_k, \qquad \sum_{j=0}^{q/2} j^2 n_{kj} = a_k.$$

From the quadratic form

$$\sum_{j=0}^{q/2} (x-j)^2 n_{kj} = \frac{q}{2}x^2 - 2\lambda_k x + a_k \geq 0$$

we deduce $a_k \geq \frac{2\lambda_k^2}{q}$, for every $k = 1, 2, \ldots, q-3$.

Assume that $\{X, Y, Z_{q-3}\}$ is regular. Then $q \equiv 0 \pmod 4$, $\lambda_k = q^2/8$, $\lambda_{q-3} = 0$ and $a_k \geq q^3/32$, for every $k = 1, 2 \ldots, q-4$. We have the following equality:

$$\sum_{k=1}^{q-4} a_k = \frac{q^3}{32}(q-4).$$

In fact,

$$\sum_{k=1}^{q-4} a_k = \left(\sum_{k=1}^{q-4} a_k - \sum_{k=1}^{q-4} \sum_{i=1}^{q/2} \alpha_{ki} \right) + \sum_{k=1}^{q-4} \sum_{i=1}^{q/2} \alpha_{ki}$$

and

$$\sum_{k=1}^{q-4} a_k - \sum_{k=1}^{q-4} \sum_{i=1}^{q/2} \alpha_{ki}$$

equals the number χ of pairs $(P, Q) \in \mathbf{W} \times \mathbf{W}$, such that $P \neq Q$ and the line PQ is an exterior line neither on X nor Y. Now,

$$\chi = \left(\frac{q^2}{8} - (q-2) \right) \frac{q^2}{4},$$

since, for every $Q \in \mathbf{W}$, $E(Q)$ has $q^2/8$ points in common with $\mathbf{W}$ (Theorem 1), of which $2(q/2 - 1) = q - 2$ belong to the exterior lines QX and QY. As

$$\sum_{k=1}^{q-4} \sum_{i=1}^{q/2} \alpha_{ki} = \frac{q^2}{8}(q-4)$$

so

$$\sum_{k=1}^{q-4} a_k = \left(\frac{q^2}{8} - (q-2) \right) \frac{q^2}{4} + \frac{q^2}{8}(q-4) = \frac{q^3}{32}(q-4).$$

Therefore,

$$a_k = \frac{q^3}{32}, \; k = 1, \ldots, q-4 \qquad \text{and} \qquad \sum_{i=1}^{q/2} \left(\alpha_{ki} - \frac{q}{4} \right)^2 = 0.$$

Hence $\alpha_{ki} = q/4$, for every $k = 1, \ldots, q-4$ and $i = 1, \ldots, q/2$.

A similar argument applies to secant lines.

In fact, define, for every $k = 1, \ldots, q-3$,

$$\mu_k = |S(Z_k) \cap \mathbf{W}|.$$

Then $\mu_k = (q^2/4) - \lambda_k$.

Let $\{O, N\} = \ell \cap \Omega$. If β_{ki} is the number of points that the i–th secant line other than ℓ on Z_k has on W and b_k is the number of pairs $(P, Q) \in W \times W$, such that the line PQZ_k is secant, proceding as before, we have

$$b_k \geq \frac{2\mu_k^2}{q}, \quad k = 1, \dots, q-3.$$

Let γ_i (resp., δ_i) be the number of points that the i–th line other than ℓ through O (resp., N) has on W. Then

(i)
$$\sum_{i=1}^{q} \gamma_i = \sum_{i=1}^{q} \delta_i = \frac{q^2}{4}$$

and

(ii)
$$\sum_{i=1}^{q} \gamma_i^2 \geq \frac{q^3}{16}, \quad \sum_{i=1}^{q} \delta_i^2 \geq \frac{q^3}{16}.$$

Let $\{X, Y, Z_{q-3}\}$ be regular. The number

$$\sum_{k=1}^{q-4} b_k - \sum_{k=1}^{q-4} \sum_{i=1}^{q/2} \beta_{ki} = \sum_{k=1}^{q-4} b_k - \frac{q^2}{8}(q-4)$$

equals the number Ψ of pairs $(P, Q) \in W \times W$ such that $P \neq Q$ and the line PQ is a secant line neither on O nor N nor Z_{q-3}. As $|S(Q) \cap W| = q^2/8$, for every $Q \in W$, so

$$\Psi = \sum_{Q \in W} \Big(|S(Q) \cap W| - (x_Q - 1) - (\tfrac{q}{2} - 1) \Big) = \frac{q^4}{32} + \frac{q^2}{2} - \frac{q^3}{8} - \sum_{Q \in W} x_Q$$

where $x_Q = |QO \cap W| + |QN \cap W|$.

It is easy to prove that

$$\sum_{Q \in W} x_Q = \sum_{i=1}^{q} (\gamma_i^2 + \delta_i^2).$$

Therefore, using (ii)

$$\sum_{k=1}^{q-4} b_k = \frac{q^4}{32} - \sum_{Q \in W} x_Q \leq \frac{q^3}{32}(q-4).$$

As $b_k \geq q^3/32,\ k = 1, \ldots, q-4$, so $b_k = q^3/32,\ k = 1, \ldots, q-4$.
It follows

$$\sum_{i=1}^{q/2} \left(\beta_{ki} - \frac{q}{4}\right)^2 = 0.$$

Hence $\beta_{ki} = q/4$, for every $i = 1, \ldots, q/2$ and $k = 1, \ldots, q-4$.
It remains to calculate γ_i and δ_i, for every $i = 1, \ldots, q$. Firstly

(iii) $$\sum_{i=1}^{q} (\gamma_i^2 + \delta_i^2) = \frac{q^3}{8}.$$

Let $Q \in \boldsymbol{W}$. One of the secant lines through Q is on Z_{q-3}, since the triple $\{X, Y, Z_{q-3}\}$ is regular. Thus

$$|S(Q) \cap \boldsymbol{W}| = 1 + (\frac{q}{4} - 1)(\frac{q}{2} - 2) + \frac{q}{2} - 1 + \delta_i - 1 + \gamma_j - 1 = \frac{q^2}{8}.$$

Hence $\delta_i + \gamma_j = q/2$, for every $i, j = 1, \ldots, q$. It follows

$$\sum_{i=1}^{q} \delta_i^2 = \sum_{j=1}^{q} (\frac{q}{2} - \gamma_j)^2 = \sum_{j=1}^{q} \gamma_j^2.$$

From this equality and equalities (i) and (iii), $\delta_i = \gamma_j = q/4$, for every i and j. □

Remark. Theorem 2 implies that the set $\boldsymbol{W}$ is of type $(0, q/4, q/2)$. This has suggested to investigate sets of such a type in a projective plane of order $q \equiv 0 \pmod 4$ (see [8]), which seem to be related with regular triples.

It is interesting to see what happens when there is a secant line ℓ of regular triples, that is, for every two distinct points X and Y of $\ell \setminus (\ell \cap \Omega)$ there exists a point Z which makes regular the triple $\{X, Y, Z\}$. Such a secant line will said to be *strongly regular*. In this case, as it is easily proved, the incidence structure $\mathcal{S}$, whose points are those of $\ell \setminus (\ell \cap \Omega)$ and whose blocks are the regular triples, is a Steiner triple system on $q-1$ points.

THEOREM 3. *Let Ω be a hyperoval of Π_q, admitting a strongly regular secant line. Then the Steiner triple system $\mathcal{S}$ is the 2–design of points and lines of a projective geometry over $GF(2)$. In particular, the order q of Π_q is a power of two.*

Proof: It sufficies to verify the Veblen–Young axiom: if X, Y and Z are three non collinear points of $\mathcal{S}$, then if D is on the block XY and E is on the block YZ, the block DE meets the block XY in its third point.

For this purpose, for every X and Y in $\mathcal{S}$, we denote by (XY) the third point of $\mathcal{S}$, which makes regular the triple $\{X, Y, (XY)\}$ and, for every point P of Π_q, by W^P, X^P, Y^P, A^P the sets $\mathsf{W} \cap E(P)$, $\mathsf{X} \cap E(P)$, $\mathsf{Y} \cap E(P)$, $\mathsf{A} \cap E(P)$, respectively.

Let X, Y, Z be any three distinct points of $\mathcal{S}$ such that $Z \neq (XY)$, i.e. the triple $\{X, Y, Z\}$ is not regular. To verify the Veblen–Young axiom, we must prove that the triple $\{(XY), (XZ), (YZ)\}$ is regular.

Now $E((XY)) = \mathsf{X} \cup \mathsf{Y}$, since $\{X, Y, (XY)\}$ is regular. Hence $E(X) = \mathsf{W} \cup \mathsf{X}$ and $E(Y) = \mathsf{W} \cup \mathsf{Y}$. Then

$$\text{(i)} \qquad \begin{cases} E((XZ)) = \mathsf{W}^{(XZ)} \cup \mathsf{X}^{(XZ)} \cup \mathsf{Y}^{(XZ)} \cup \mathsf{A}^{(XZ)} \\ E((YZ)) = \mathsf{W}^{(YZ)} \cup \mathsf{X}^{(YZ)} \cup \mathsf{Y}^{(YZ)} \cup \mathsf{A}^{(YZ)}. \end{cases}$$

Therefore

$$(*) \quad E((XY)) \cap E((XZ)) \cap E((YZ)) = (\mathsf{X}^{(XZ)} \cap \mathsf{X}^{(YZ)}) \cup (\mathsf{Y}^{(XZ)} \cap \mathsf{Y}^{(YZ)}).$$

(Observe that $\mathsf{W}, \mathsf{X}, \mathsf{Y}, \mathsf{A}$ are disjoint).

By way of contradiction, assume that (*) is not empty. Then, by Theorem 1

$$\text{(ii)} \qquad |\mathsf{X}^{(XZ)} \cap \mathsf{X}^{(YZ)}| + |\mathsf{Y}^{(XZ)} \cap \mathsf{Y}^{(YZ)}| = \frac{q^2}{8}.$$

Equalities (i) and $E(X) \cap E(Z) = \mathsf{W}^Z \cup \mathsf{X}^Z$ imply

$$E((XZ)) \cap E(X) \cap E(Z) = (\mathsf{W}^{(XZ)} \cap \mathsf{W}^Z) \cup (\mathsf{X}^{(XZ)} \cap \mathsf{X}^Z)$$

which is the emptyset, since $\{X, Z, (XZ)\}$ is regular. Thus

$$\text{(iii)} \qquad \mathsf{W}^{(XZ)} \cap \mathsf{W}^Z = \emptyset, \quad \mathsf{X}^{(XZ)} \cap \mathsf{X}^Z = \emptyset$$

and

$$\text{(iv)} \qquad \mathsf{W}^{(XZ)} \cup \mathsf{W}^Z = \mathsf{W}, \quad \mathsf{X}^{(XZ)} \cup \mathsf{X}^Z = \mathsf{X},$$

since

$$|\mathsf{W}^{(XZ)}| = |\mathsf{W}^Z| = |\mathsf{X}^{(XZ)}| = |\mathsf{X}^Z| = \frac{q^2}{8}, \text{ by Theorem 1.}$$

As $\{Y, Z, (YZ)\}$ is regular, so $E((YZ)) \cap E(Y) \cap E(Z) = \emptyset$. Hence

$$(\mathsf{W}^{(YZ)} \cap \mathsf{W}^Z) \cup (\mathsf{Y}^{(YZ)} \cap \mathsf{Y}^Z) = \emptyset.$$

Therefore, $\mathsf{Y}^{(YZ)} \cap \mathsf{Y}^Z = \emptyset$ and $\mathsf{W}^{(YZ)} \cap \mathsf{W}^Z = \emptyset$, which implies, together with (iii) and (iv)

$$\text{(v)} \qquad \mathsf{W}^{(XZ)} = \mathsf{W}^{(YZ)}.$$

Then $E((XZ)) \cap E((YZ))$ equals

$$\mathsf{W}^{(XZ)} \cup (\mathsf{X}^{(XZ)} \cap \mathsf{X}^{(YZ)}) \cup (\mathsf{Y}^{(XZ)} \cap \mathsf{Y}^{(YZ)}) \cup (\mathsf{A}^{(XZ)} \cap \mathsf{A}^{(YZ)}),$$

because of (v).

Thus $\mathsf{A}^{(XZ)} \cap \mathsf{A}^{(YZ)} = \emptyset$, since

$$| E((XZ)) \cap E((YZ)) | = \frac{q^2}{4}$$

and

$$| \mathsf{W}^{(XZ)} | = | \mathsf{X}^{(XZ)} \cap \mathsf{X}^{(YZ)} | + | \mathsf{Y}^{(XZ)} \cap \mathsf{Y}^{(YZ)} | = \frac{q^2}{8}.$$

Finally

$$E((XZ)) \cap E((YZ)) \cap E(Z) = (\mathsf{Y}^{(XZ)} \cap \mathsf{Y}^{(YZ)} \cap \mathsf{Y}^Z) \cup (\mathsf{A}^{(XZ)} \cap \mathsf{A}^{(YZ)} \cap \mathsf{A}^Z)$$

because of (iii).

As

$$\mathsf{Y}^{(YZ)} \cap \mathsf{Y}^Z = \mathsf{A}^{(XZ)} \cap \mathsf{A}^{(YZ)} = \emptyset$$

so

$$E((XZ)) \cap E((YZ)) \cap E(Z) = \emptyset$$

that is, the triple $\{(XZ), (YZ), Z\}$ is regular. But also $\{X, Z, (XZ)\}$ is regular. Therefore $X = (YZ)$, which implies that $\{X, Y, Z\}$ is regular, a contradiction. □

3. A necessary and sufficient condition for the existence of regular triples

In this Section we will give a further condition which guarantees the existence of regular triples and which is fulfilled by any translation hyperoval.

THEOREM 4. *Let Ω be a hyperoval of a projective plane Π_q of order $q \equiv 0 \pmod 4$ and s a secant line of Ω. Let $\{M, N\} = \Omega \cap s$ and, if X and Y are two distinct points on $s \setminus \{M, N\}$, let $\mathsf{W} = E(X) \cap E(Y)$, $\mathsf{A} = S(X) \cap S(Y)$ and $\{Z_1, \ldots, Z_{q-3}\}$ be the set of remaining points on s. Assume that:*

(1) *every exterior (resp., secant other than s) line on Z_k, $k = 1, \ldots, q-3$, has α_k (resp., β_k) points on W;*
(2) *every line other than s on M or N has $q/4$ points on W.*

Then there exists a point $Z \in \{Z_1, \ldots, Z_{q-3}\}$, such that the triple $\{X, Y, Z\}$ is regular.

Proof: By hypothesis

$$\lambda_k = |\mathbf{W} \cap E(Z_k)| = \frac{q}{2}\alpha_k, \; k = 1, \ldots, q-3.$$

Then equality (1) of Section 2 becomes

$$(2) \qquad \sum_{k=1}^{q-3} \alpha_k = \frac{q}{4}(q-4).$$

Furthermore

$$(3) \qquad \beta_k = \frac{q}{2} - \alpha_k$$

since $\mathbf{W} = (E(Z_k) \cap \mathbf{W}) \cup (S(Z_k) \cap \mathbf{W})$.

We split the proof in several steps.

Step 1. We prove that

$$(4) \qquad |E(P) \cap \mathbf{W}| = \frac{q^2}{8}$$

for every $P \notin \Omega \cup s$.

Consider the set of exterior lines on P. There are several cases to treat, according as the lines PX or PY are exterior or secant.

Assume that the lines PX, $PZ_1, \ldots, PZ_{\frac{q}{2}-1}$ are exterior. Then $P \notin \mathbf{W}$ and the lines PY and PZ_j, $j = q/2, \ldots, q-3$, are secant. Therefore PX has $q/2$ points on $\mathbf{W}$ and, by hypothesis, the line PZ_i, $i = 1, \ldots, (q/2)-1$, has α_i points on $\mathbf{W}$, the line PZ_j, $j = q/2, \ldots, q-3$, has $\beta_j = (q/2) - \alpha_j$ points on $\mathbf{W}$ and each of the lines PM and PN has $q/4$ points on $\mathbf{W}$.

Since $(E(P) \cap \mathbf{W}) \cup (S(P) \cap \mathbf{W}) = \mathbf{W}$, we obtain

$$(5) \qquad \frac{q}{2} + \sum_{i=1}^{(q/2)-1} \alpha_i + \sum_{j=q/2}^{q-3} \left(\frac{q}{2} - \alpha_j\right) + \frac{q}{2} = \frac{q^2}{4}.$$

Hence

$$|E(P) \cap \mathbf{W}| = \frac{q}{2} + \sum_{i=1}^{(q/2)-1} \alpha_i = \frac{q^2}{8}.$$

The other cases are similarly treated.

Step 2.

$$(6)\qquad \sum_{k=1}^{q-3} \alpha_k^2 = \frac{q^2}{16}(q-4).$$

The number of pairs $(P,Q) \in \mathsf{W} \times (\mathsf{A} \setminus s)$ is

$$\frac{q^2}{4}\left(\frac{q^2}{4} - q\right).$$

This number may be obtained in another way, adding the number Φ of pairs $(P,Q) \in \mathsf{W} \times (\mathsf{A} \setminus s)$, such that the line PQ is exterior, to the number Ψ of pairs $(R,S) \in \mathsf{W} \times (\mathsf{A} \setminus s)$, such that the line RS is secant.

Since

$$\Phi = \frac{q}{2} \sum_{k=1}^{q-3} \alpha_k^2$$

and

$$\Psi = \frac{q}{2} \sum_{k=1}^{q-3} \left(\frac{q}{2} - \alpha_k\right)\left(\frac{q}{2} - \alpha_k - 2\right) + \frac{q^2}{2}\left(\frac{q}{4} - 1\right) = \frac{q}{2} \sum_{k=1}^{q-3} \alpha_k^2$$

equality (6) follows.

Step 3. Let P be a point of $\mathsf{A} \setminus s$. We denote by $\mathcal{I}_P$ the set of $q/2$ indices for the points $Z_{i_P} \in \{Z_1, \dots, Z_{q-3}\}$, such that PZ_{i_P} is an exterior line and by $\overline{\mathcal{I}}_P$ the complementary set of $(q/2) - 3$ indices for the points Z_{j_P}, such that the line PZ_{j_P} is secant. Then equalities (4) and (6) may be written, respectively,

$$(7)\qquad \begin{cases} \displaystyle\sum_{i_P \in \mathcal{I}_P} \alpha_{i_P} = \frac{q^2}{8} \\[2ex] \displaystyle\sum_{i_P \in \mathcal{I}_P} \alpha_{i_P}^2 + \sum_{j_P \in \overline{\mathcal{I}}_P} \alpha_{j_P}^2 = \frac{q^2}{16}(q-4). \end{cases}$$

Now we count the ordered pairs $(U,V) \in \mathsf{W} \times \mathsf{W}$, such that the line UV is on P. Consider the $q+1$ lines on P: two are on X and Y, respectively, and have no point on W; $q/2$ are exterior lines and each of them has α_{i_P} points on W; thus they give

$$\sum_{i_P \in \mathcal{I}_P} \alpha_{i_P}^2$$

pairs of $\mathbf{W} \times \mathbf{W}$. The other lines on P are secant lines: two of them are on M and N, respectively, and give $2q^2/16 = q^2/8$ pairs; the other secant lines give

$$\sum_{j_P \in \overline{\mathcal{I}}_P} \left(\frac{q}{2} - \alpha_{i_P}\right)^2$$

pairs. Thus the total number of pairs is

$$\sum_{i_P \in \mathcal{I}_P} \alpha_{i_P}^2 + \sum_{j_P \in \overline{\mathcal{I}}_P} \left(\frac{q}{2} - \alpha_{i_P}\right)^2 + \frac{q^2}{8} = \frac{q^2(q+2)}{16}. \tag{8}$$

If P varies in $\mathbf{A} \setminus s$, we obtain the number of collinear triples $(P, Q, R) \in (\mathbf{A} \setminus s) \times \mathbf{W} \times \mathbf{W}$. This number is

$$\sum_{P \in (\mathbf{A} \setminus s)} \frac{q^2(q+2)}{16} = \frac{q^3}{64}(q+2)(q-4). \tag{9}$$

Using the first of equalities (7), an easy calculation shows that

$$\sum_{P \in (\mathbf{A} \setminus s)} \sum_{i_P \in \mathcal{I}_P} \left(\frac{q}{2} - \alpha_{i_P}\right)^2 = \sum_{P \in (\mathbf{A} \setminus s)} \sum_{i_P \in \mathcal{I}_P} \alpha_{i_P}^2. \tag{10}$$

Equality (10) means that the number of ordered triples $(P, Q, R) \in (\mathbf{A} \setminus s) \times \mathbf{W} \times \mathbf{W}$, such that PQR is a secant line, minus the number of triples collinear with M or N, equals the number of triples $(P, U, V) \in (\mathbf{A} \setminus s) \times \mathbf{W} \times \mathbf{W}$, such that PUV is an exterior line. Since the number of ordered triples (P, S, T) collinear with M or N is

$$\frac{q^2}{16}\left(\frac{q^2}{4} - q\right),$$

by equality (10) we obtain

$$\sum_{P \in (\mathbf{A} \setminus s)} \sum_{i_P \in \mathcal{I}_P} \alpha_{i_P}^2 = \frac{q^3}{32}\left(\frac{q^2}{4} - q\right). \tag{11}$$

Step 4. We prove that for every $P \in (\mathbf{A} \setminus s)$

$$\sum_{i_P \in \mathcal{I}_P} \alpha_{i_P}^2 \geq \frac{q^3}{32}. \tag{12}$$

Let n_i be the number of α_{i_P} which are equal to i, $i = 2, \ldots, (q/2) - 2$. Then

$$\sum_{i=2}^{(q/2)-2} n_i = \frac{q}{2}, \qquad \sum_{i=2}^{(q/2)-2} i n_i = \frac{q^2}{8}, \qquad \sum_{i=2}^{(q/2)-2} i^2 n_i = \sum_{i_P \in \mathcal{I}_P} \alpha_{i_P}^2.$$

From the quadratic form

$$\sum_{i=2}^{(q/2)-2}(x-i)^2 n_i = \frac{q}{2}x^2 - 2\frac{q^2}{8}x + \sum_{i_P\in\mathcal{I}_P}\alpha_{i_P}^2 \geq 0$$

inequality (12) follows.

Now, we can finish the proof of the theorem. Equality (11) and inequality (12) imply

$$\sum_{i_P\in\mathcal{I}_P}\alpha_{i_P}^2 = \frac{q^3}{32}$$

for every $P \in (\mathcal{A}\setminus s)$. Hence

$$\sum_{i_P\in\mathcal{I}_P}\left(\frac{q}{4}-\alpha_{i_P}\right)^2 = 0.$$

Whence,

$$\alpha_{i_P} = \frac{q}{4}, \text{ for every } i_P \in \mathcal{I}_P.$$

Since P is arbitrary, $\alpha_k = q/4$ for $q-4$ indices of $\{1,\dots,q-3\}$. Then, by Theorem 2, there is a point $Z \in \{Z_1,\dots,Z_{q-3}\}$ such that the triple $\{X,Y,Z\}$ is regular. □

Remark. Obviously, when $\{X,Y,Z\}$ is regular, hypothesis (1) and (2) of Theorem 4 are satisfied, as follows from Theorem 2.

4. Translation hyperovals

DEFINITION. *A hyperoval Ω of a projective plane Π_q is said to be a translation hyperoval, with respect to a line s, if for any two points $A, B \in \Omega\setminus s$ there exists an elation, that is a translation of the affine plane $\Pi_q\setminus s$, which maps A to B and stabilizes Ω.*

It is easily seen that all these elations form a group G, that s is a secant line, which is the common axis of all the elations of G and that G is sharply transitive on $\Omega\setminus(s\cap\Omega)$. Then G is an elementary Abelian 2–group.

If X and Y are two distinct points of $s\setminus\{M,N\}$, where $\{M,N\} = s\cap\Omega$, let $\mathbf{W} = E(X)\cap E(Y)$ and Z_k, $k = 1,2,\dots,q-3$, the remaining points of s.

THEOREM 5. *Let Ω be a translation hyperoval of Π_q with respect to the line s and with translation group G. Then*

(1) *G stabilizes $\mathbf{W}$;*

(2) *every exterior (resp., secant other than s) line on Z_k meets W in a constant number of points, α_k (resp., $(q/2) - \alpha_k$);*
(3) *every line other than s on M or N meets W in $q/4$ points;*
(4) *G has $q/4$ orbits on W, each consisting of q points and each orbit, together with $\{M, N\}$, is a translation hyperoval with respect to the line s.*

Proof: $W = E(X) \cap E(Y)$, X and Y are fixed by G and G transforms exterior lines in exterior lines. Thus G stabilizes W. Furthermore, G is transitive on the exterior (resp., secant but s) lines through Z_k, $k = 1, \ldots, q-3$, and on the lines other than s on M or N. Since $|W| = q^2/4$ and $W = (W \cap E(Z_k)) \cup (W \cap S(Z_k))$, for every $k = 1, \ldots, q-3$, (2) and (3) follow.

It is easy to show that G has $q/4$ orbits on W and that each orbit has q points. Let ℓ be any line other than s on Z_k. If ℓ is exterior (resp., secant), then ℓ^g is an exterior (resp., secant) line through Z_k. Thus if $P \in \ell \cap W$, then $P^g \in \ell^g \cap W$, for every $g \in G$. Since the number of exterior (resp., secant but s) lines on Z_k is $q/2$, every line through Z_k has 0 or 2 points on each orbit of G on W. To end the proof of (4), it remains to observe that every line but s on M or N meets each G–orbit on W in only one point, since G is sharply transitive on the lines other than s through M or N. □

By Theorems 4 and 5 we have the following

COROLLARY. *Let Ω be a translation hyperoval with respect to the line s. Then for every pair of distinct points X and Y in $s \setminus \{M, N\}$, there exists a point Z such that the triple $\{X, Y, Z\}$ is regular. In particular, the secant line s is strongly regular.*

Some known examples of translation hyperoval are the following.

A first class of tanslation hyperovals is that of complete conics in the Desarguesian projective plane $PG(2,q)$, where $q = 2^h$. Every complete conic, i.e. a conic plus its nucleus, is a translation hyperoval with respect to every tangent line of the conic. Therefore a complete conic is an example of hyperoval for which there are $q+1$ secant lines of regular triples (they are the tangent lines to the conic). This class of translation hyperovals has been studied in [8], using direct constructive methods.

In $PG(2,q)$, $q = 2^h$, all the translation hyperovals has been determined ([3], [10], [12]). They are, with respect to a suitable reference frame, the sets of points $\{(1, t, t^{2^g}), t \in GF(q)\} \cup \{(0,1,0), (0,0,1)\}$, where $(g, h) = 1$, $1 < g \leq h-1$ and $h \geq 3$.

In non-Desarguesian projective planes of even order, there are many new translation hyperovals (see [1], [4], [5], [6]). Therefore it seems to be interesting to investigate how one can distinguish between the associated projective geometry over $GF(2)$, when there are two different translation hyperovals with respect to the same line s. Some partial results are in [7], where it is proved the following

THEOREM. *Let Ω and $\overline{\Omega}$ be two hyperovals of Π_q with the same strongly regular secant line s. Then Ω and $\overline{\Omega}$ have the same regular triples if and only if $S_P^{\Omega} = S_Q^{\Omega}$, for every $P, Q \in \overline{\Omega}$.*

In the statement, S_P^{Ω} is the set $\{X \in s \mid \text{the line } PX \text{ is } \Omega\text{--secant}\}$.

In the case of translation hyperovals of $PG(2,q)$, this theorem implies that if Ω and $\overline{\Omega}$ are two translation hyperovals with respect to the same line s, then there exists an elation g of axis s and center on $\Omega \cap s$ such that $\Omega^g = \overline{\Omega}$. In particular then Ω and $\overline{\Omega}$ are equivalent under the action of the automorphism group of $PG(2,q)$. As a consequence, for example, a complete conic and a translation hyperoval, which is not a complete conic, cannot have the same regular triples.

Finally, equivalent translation hyperovals with respect to the same line may have different regular triples, as example 4.1 in [9] shows.

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