Designs Having the Parameters of $PG_{i+1}(d+1,q)$

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Abstract. A construction is given which uses $PG_i(d,q)$ and q copies of $AG_i(d,q)$ to construct designs having the parameters of $PG_{i+1}(d+1,q)$, where q is a prime power and $i \leq d-1$.

Introduction

The main tools used in this paper are the projective and affine geometry designs $PG_i(d,q)$ and $AG_i(d,q)$. Definitions, notation and results used in this paper, can be found in Dembowski [2] or Hughes and Piper [3].

In [1] Assmus and van Lint use a construction due to R. J. Wilson which takes a copy of $PG(2,4)$ and 4 copies of $AG(2,4)$ to construct 2-$(85,21,5)$ designs, that is designs having the parameters of $PG_2(3,4)$. The construction works in such a way that a specific collection of five points form an oval in the resulting design. Further they mention that a copy of $PG(2,3)$ together with 3 copies of $AG(2,3)$ can be used to construct 2-$(40,13,4)$ designs, that is designs having the parameters of $PG_2(3,3)$.

In [4] Jungnickel uses a similar construction which takes a copy of $AG_d(d+1,q)$ and a design having the parameters of $PG_{d-1}(d,q)$ to construct designs having the parameters of $PG_d(d+1,q)$. The resulting designs contain a copy of $AG_d(d+1,q)$ whose full automorphism group is known — Jungnickel uses this to derive a lower bound for the number of non-isomorphic designs having the parameters of $PG_d(d+1,q)$.

A general construction for designs with the parameters of $PG_{i+1}(d+1,q)$, using $PG_i(d,q)$ and $AG_i(d,q)$ as building blocks, might then give useful information in the classification, enumeration and the structure of such designs.

The following construction uses a copy of a $PG_i(d,q)$ together with q copies of $AG_i(d,q)$ to construct designs with the parameters of $PG_{i+1}(d+1,q)$.

Construction

Denote the design $PG_i(d,q)$ by $\mathcal{D}_1$, then $\mathcal{D}_1$ has the following parameters

$$v_1 = \frac{q^{d+1}-1}{q-1} \qquad k_1 = \frac{q^{i+1}-1}{q-1} \qquad b_1 = \phi(d,i,q)$$

$$r_1 = \phi(d-1,i-1,q) \qquad \lambda_1 = \phi(d-2,i-2,q)$$

where

$$\phi(d,i,q) = \frac{(q^{d+1}-1)(q^d-1)\ldots(q^{d-i+1}-1)}{(q^{i+1}-1)(q^i-1)\ldots(q-1)}$$

Let $\mathcal{D}_2,\mathcal{D}_3,\ldots,\mathcal{D}_{q+1}$ denote the q copies of $AG_i(d,q)$ then each of these designs has the following parameters

$$\begin{array}{lll} v_2 = q^d & k_2 = q^i & b_2 = q^{d-i}\phi(d-1,i-1,q) \\ r_2 = r_1 & \lambda_2 = \lambda_1 & \end{array}$$

Further, let Π denote the design to be constructed, then Π has the following parameters

$$v_3 = \frac{q^{d+2}-1}{q-1} \qquad k_3 = \frac{q^{i+2}-1}{q-1}$$

$$b_3 = \phi(d+1,i+1,q) \quad r_3 = \phi(d,i,q) \quad \lambda_3 = \phi(d-1,i-1,q)$$

The point set $\mathcal{P}$ of Π consists of the union of the point sets of $\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_{q+1}$, hence

$$|\mathcal{P}| = \frac{q^{d+1}-1}{q-1} + q(q^d) = \frac{q^{d+2}-1}{q-1}$$

as required. The block set $\mathcal{B}$ of Π is constructed in three parts.

Part 1

The initial $q^{2d-2i}\phi(d-1,i-1,q)$ blocks of Π are formed by joining together a block from each of $\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_{q+1}$, and have $k_3 = \frac{q^{i+1}-1}{q-1} + q(q^i) = \frac{q^{i+2}-1}{q-1}$ points incident with them. To construct these blocks, note that each of the designs $\mathcal{D}_2,\mathcal{D}_3,\ldots,\mathcal{D}_{q+1}$ is resolvable into $\frac{b_2k_2}{v_2} = \phi(d-1,i-1,q)$ parallel classes, each of which contains $\frac{k_2}{v_2} = q^{d-i}$ blocks.

Select a parallel class from each $\mathcal{D}_2,\mathcal{D}_3,\ldots,\mathcal{D}_{q+1}$ and q^{d-i} blocks from $\mathcal{D}_1$. Form blocks of Π by adjoining a block from each of the chosen parallel classes with a block from $\mathcal{D}_1$ in such a way that each block-pair from $\mathcal{D}_2,\mathcal{D}_3,\ldots,\mathcal{D}_{q+1}$ or $\mathcal{D}_1$ appear together exactly once in a block Π. For a given choice of parallel class, a total of q^{2d-2i} of Π can be formed in this way. To see this, count the total number of possible block-pairs. Subtract from this the number of block pairs from a given design $\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_{q+1}$ then divide by the number of block pairs required to form a block of Π.

$$\text{Number of blocks of } \Pi = \frac{\binom{q^{d-i+1}+q^{d-i}}{2} - (q+1)\binom{q^{d-i}}{2}}{\binom{q+1}{2}} = q^{2d-2i}$$

Repeating this process for each of the $\phi(d-1,i-1,q)$ parallel classes of the affine designs and using a different set of q^{d-i} blocks of $\mathcal{D}_1$ for each choice of parallel class yields a total of $q^{2d-2i}\phi(d-1,i-1,q)$ blocks of Π.

Part 2

Now construct a further $\phi(d, i+1, q)$ blocks of Π by constructing a design, $\mathcal{D}_{q+2}$, on the points $\mathcal{D}_1$. $\mathcal{D}_{q+2}$ has the following parameters

$$v_4 = \frac{q^{d+1}-1}{q-1} = v_1 \qquad k_4 = \frac{q^{i+2}-1}{q-1} = k_1$$

$$b_4 = \phi(d, i+1, q) \quad r_4 = \phi(d-1, i, q) \quad \lambda_4 = \phi(d-2, i-1, q)$$

Part 3

The remaining $q^{d-i}\phi(d-1, i, q)$ blocks of Π are constructed as follows. Construct a design having the parameters of $AG_{i+1}(d, q)$ on each of the point sets of $\mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_{q+1}$ denote these designs by $\mathcal{D}_{q+3}, \mathcal{D}_{q+4}, \ldots, \mathcal{D}_{2q+2}$, each of these designs has the following parameters

$$v_5 = q^d \quad k_5 = q^{i+1} \quad b_5 = q^{d-i-1}\phi(d-1, i, q) \quad r_5 = r_4 \quad \lambda_5 = \lambda_4$$

Each of the designs $\mathcal{D}_{q+3}, \mathcal{D}_{q+4}, \ldots, \mathcal{D}_{2q+2}$, is resolvable into $\phi(d-1, i, q)$ parallel classes, each of which contains q^{d-i-1} blocks. A bijection from the remaining $\phi(d-1, i, q)$ blocks of $\mathcal{D}_1$ onto the parallel classes of $\mathcal{D}_{q+3}, \mathcal{D}_{q+4}, \ldots, \mathcal{D}_{2q+2}$ gives blocks of the required size. The total number of blocks of Π which have been constructed is

$$q^{2d-2i}\phi(d-1, i-1, q) + \phi(d, i+1, q) + q^{d-i}\phi(d-1, i, q)$$
$$= \phi(d+1, i+1, q) = b_3, \text{ as required.}$$

Further each block has $\frac{q^{i+2}-1}{q-1} = k_3$ points incident with it.

If Π is a design with the parameters of $PG_{i+1}(d+1, q)$ then each point should be incident with $r_3 = (d, i, q)$ blocks and every pair of points should appear $\lambda_3 = \phi(d-1, i-1, q)$ times.

For the replication number, r, we consider two cases:

1. The points of $\mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_{q+1}$. Each point of these designs appears on $\phi(d-1, i-1, q) = r_2$ blocks of the design. Further each of the blocks appear as part of q^{d-i} blocks of Π in the first part of the construction. The points of $\mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_{q+1}$ also appear on $\phi(d-1, i, q)$ blocks of Π in the last part of the construction.

$$r_3 = q^{d-i}\phi(d-1, i-1, q) + \phi(d-1, i, q) = \phi(d, i, q) \text{ as required.}$$

2. The points of $\mathcal{D}_1$. Each of these points appears on $r_1 = \phi(d-1, i-1, q)$ blocks of $\mathcal{D}_1$. Each of these blocks appears as part of q^{d-i} blocks of Π in either the first or last part of the construction. Further each point of $\mathcal{D}_1$ appears on $r_4 = \phi(d-1, i, q)$ blocks of Π when the design $\mathcal{D}_{q+2}$ is constructed.

$$r_3 = q^{d-i}\phi(d-1, i-1, q) + \phi(d-1, i, q) = \phi(d, i, q) \text{ as required.}$$

For λ we need to consider four cases.

1. A pair of points from the same affine geometry. Each pair of points appear on $\lambda_2 = \phi(d-2, i-2, q)$ blocks of one of $\mathcal{D}_2, \ldots, \mathcal{D}_{q+1}$. Each of these blocks appears as part of q^{d-i} blocks of Π. Further each pair of points appear together on $\lambda_5 = \phi(d-2, i-1, q)$ blocks of one of $\mathcal{D}_{q+3}, \mathcal{D}_{q+4}, \ldots, \mathcal{D}_{2q+2}$.

$$\lambda_3 = q^{d-i}\phi(d-2, i-2, q) + \phi(d-2, i-1, q)$$
$$= \phi(d-1, i-1, q) \text{ as required.}$$

2. A pair of points from distinct $\mathcal{D}_2, \ldots, \mathcal{D}_{q+1}$. Each pair of points appear once per choice of parallel class, in the first part of the construction. Further such pairs do not occur again.

$$\lambda_3 = \phi(d-1, i-1, q) \text{ as required.}$$

3. A pair of points from $\mathcal{D}_1$. Such pairs appear on $\phi(d-2, i-2, q) = \lambda_1$ blocks of $\mathcal{D}_1$. Each of these blocks appear as part of q^{d-i} blocks of Π. Further each pair of points from $\mathcal{D}_1$ appears $\lambda_4 = \phi(d-2, i-1, q)$ time in the design $\mathcal{D}_{q+2}$.

$$\lambda_3 = q^{d-i}\phi(d-2, i-2, q) + \phi(d-2, i-1, q)$$
$$= \phi(d-1, i-1, q) \text{ as required.}$$

4. A pair of points, one from an affine geometry, the other from the projective geometry. In the first part of the construction each block, and therefore each point of the affine geometries appears with q^{d-i} distinct blocks of $\mathcal{D}_1$ for each choice of parallel class.

   In the designs $\mathcal{D}_{q+3}, \mathcal{D}_{q+4}, \ldots, \mathcal{D}_{2q+2}$ each point $\mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_{q+1}$ appears once with each of the remaining blocks of $\mathcal{D}_1$. Hence each pair appears

$$r_1 = \lambda_3 = \phi(d-1, i-1, q) \text{ as required.}$$

Hence Π is indeed a design having the parameters of $PG_{i+1}(d+1, q)$.

References

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