A Counter Example to a Conjecture of Gallai

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ABSTRACT. Let G be a graph and $t(G)$ be the number of triangles in G. Define $\mathcal{G}_n$ to be the set of all graphs on n vertices that do not contain a wheel and $t_n = \max\{t(G) : G \in \mathcal{G}_n\}$. T. Gallai conjectured that $t_n \leq \lfloor \frac{n^2}{8} \rfloor$ In this note we describe a graph on n vertices that contains no wheel and has at least $\frac{n^2+n}{8} - 3$ triangles.

In this note, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of a graph G. A graph W is a wheel if $V(W) = V(C) \cup \{v\}$ where C is a cycle and v is adjacent to every vertex of C. Let $\mathcal{G}_n$ be the set of all graphs on n vertices that do not contain a wheel. We define $t(G)$ to be the number of triangles, i.e. K_3's, in G and

$$t_n = \max\{t(G) : G \in \mathcal{G}_n\}.$$

In [1], P. Erdös posed a conjecture of T. Gallai:

Conjecture: $t_n \leq \frac{n^2}{8}$.

The main aim of this note is to describe a class of graphs that shows

$$t_n \geq \frac{n^2+n}{8} - 3.$$

The smallest counter example to Gallai's conjecture is the graph G_7 in Figure 1, where $V(G_7) = \{0, 1, \ldots, 6\}$ and $E(G_7) = \{(i, i+k) : 0 \leq i \leq 6, k = 1, 2\}$, and the addition is taken modulo 7. It can be shown that G_7 is the only graph in $\mathcal{G}_7$ that contains 7 triangles.

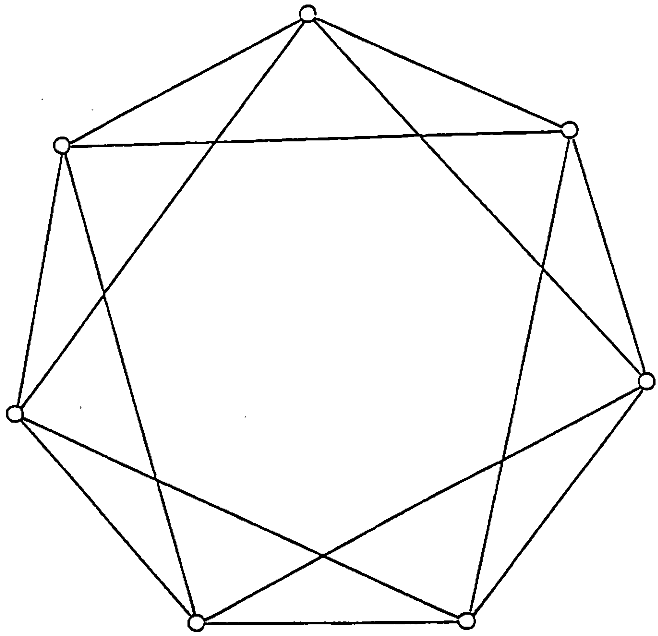


Figure 1

For larger counter examples, we construct graph G_n that contains many copies of G_7. First we assume that n is odd, then for some integer r and $-2 \leq k \leq 1$, $n = 8r + 2k + 1$, let

$$\begin{aligned} V(G) &= \{a, b_i, c_i, d_i, e_i, f_j, g_j \colon 1 \leq i \leq r, 1 \leq j \leq 2r + k\} \text{ and} \\ E(G) &= \{(a, b_i), (a, c_i), (a, d_i), (a, e_i), (b_i, c_i), (c_i, d_i), (d_i, e_i), (f_j, g_j), (b_i, f_j), \\ &\quad (b_i, g_j), (c_i, f_j), (d_i, g_j), (e_i, f_j), (e_i, g_j), \colon 1 \leq i \leq r, 1 \leq j \leq 2r + k\}. \end{aligned}$$

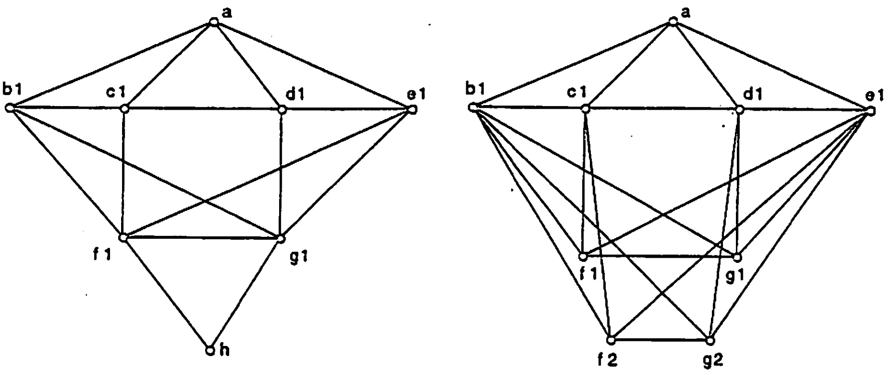
Similarly, if n is even, then $n = 8r + 2k + 2$ for some integer r and $-2 \leq k \leq 1$ and we define

$$\begin{aligned} V(G) &= \{a, b_i, c_i, d_i, e_i, f_j, g_j, h \colon 1 \leq i \leq r, 1 \leq j \leq 2r + k\} \text{ and} \\ E(G) &= \{(a, b_j), (a, c_j), (a, d_j), (a, e_j), (b_j, c_j), (c_j, d_j), (d_i, e_i), (f_j, g_j), \\ &\quad (b_i, f_j), (b_i, g_j), (c_i, f_j), (d_i, g_j), (e_i, f_j), (e_i, g_j), (h, f_j), \\ &\quad (h, g_j) \colon 1 \leq i \leq r, 1 \leq j \leq 2r + k\}. \end{aligned}$$

G_8 and G_9 are shown in Figure 2.

To show that G_n has the desired properties, we first state a simple observation as a lemma:

Lemma 1. *A graph G contains no wheel if and only if for every vertex v in G, there is no cycle in $N(v)$, where $N(v)$ is the subgraph of G induced by the neighbors of v.* □



G_8 G_9

Figure 2

For a graph G and a vertex v of G, we define $t(v)$ to be the number of triangles in G that contain v. It is easy to see that $t(v) = |E(N(v))|$ and $t(G) = \frac{1}{3}\sum_{v\in G} t(v)$.

Theorem 2. *G_n contains no wheel and for $-3 \leq p \leq 4$, and $r \geq 1$, if $n = 8r + p$, then $t(G_n) = \frac{n^2+n+c_p}{8}$, where $c_{-3} = c_2 = -6$, $c_{-2} = -18$, $c_{-1} = 0$, $c_0 = -8$, $c_1 = -2$, $c_3 = c_4 = -12$.*

Proof: For each vertex of G_n, we will show that $N(v)$ does not contain a cycle and count the number of edges in $N(v)$. By symmetry, we only need to do this for $v \in \{a, b_1, d_1, f_1\}$, if n is odd; and for $v \in \{a, b_1, d_1, f_1, h\}$ if n is even.

Suppose n is odd and $n = 8r+2k+1$, $-2 \leq k \leq 1$. We list the information describing the neighborhoods of specified vertices in the following table, where i is any integer such that $1 \leq i \leq r$ and j is any integer such that $1 \leq j \leq 2r + k$.

v	$V(N(v))$	$E(N(v))$	$t(v)$
a	b_i, c_i, d_i, e_i	$(b_i, c_i), (c_i, d_i), (d_i, e_i)$	$3r$
b_1	a, c_1, f_j, g_j	$(a, c_1), (c_1, f_j), (f_j, g_j)$	$4r + 2k + 1$
d_1	a, c_1, e_1, g_j	$(a, C_1), (a, e_1), (e_1, g_j)$	$2r + k + 2$
f_1	g_1, b_i, c_i, e_i	$(b_i, g_1), (b_i, c_i), (e_i, g_1)$	$3r$

It is easy to see that $N(v)$ does not contain any cycles for every $v \in \{a, b_1, d_1, f_1\}$. Therefore, G_n does not contain a wheel according to Lemma

1 and

$$t(G_n) = \frac{1}{3}[3r + 2r(4r + 2k + 1) + 2r(2r + k + 2) + 2(2r + k)3r]$$
$$= \frac{1}{3}[24r^2 + 12rk + 9r] = 8r^2 + 4rk + 3r = \frac{1}{8}(n^2 + n - 4k^2 - 6k - 2).$$

This gives the values of $t(G_n)$ when n is odd.

The case when n is even can be dealt with in exactly the same way and the details are left to the reader. □

Theorem 2 gives $t_n \geq \frac{n^2+n}{8} - 3$ for all values of n. We believe that $t_n = \frac{n^2}{8} + \mathcal{O}(n)$ but at this time, only the trivial upper bound $t_n \leq \frac{n^2-n}{6}$ is known.

Theorem 3. *If G is a graph on n vertices and contains no wheel, then $t(G) \leq \frac{n^2-n}{6}$.*

Proof: Let the vertex set of G be $\{v_1, v_2, \ldots, v_n\}$. Since G contains no wheel, $N(v_i)$ is a forest for all i.

Thus we have

$$3t(G) = \sum_{i=1}^{n} \text{number of triangles in } G \text{ that contain } v_i$$
$$\leq \sum_{i=1}^{n} [d(v_i) - 1]$$
$$= 2|E(G)| - n.$$

For every graph G in $\mathcal{G}_n$ from [1] it is known that $|E(G)| \leq \frac{n^2+n}{4}$, giving

$$t(G) \leq \frac{1}{3}\left(\frac{n^2+n}{2} - n\right) = \frac{n^2-n}{6}.$$

□

References

[1] P. Erdös, Problems and results in combinatorial analysis and graph theory, *Discrete Mathematics* **72**(1988), 81–92.