Competition Graphs and Resource Graphs of Digraphs

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ABSTRACT. In this paper we study competition graphs of digraphs of restricted degree. We introduce the notion of restricted competition numbers of graphs. We complete the characterization of competition graphs of indegree at most 2 and their restricted competition numbers. We characterize interval (2,3)-graphs and give a recognition algorithm for interval (2,3)-digraphs. We characterize competition graphs and interval competition graphs of digraphs of outdegree at most 2. The relationship between restricted competition numbers and ordinary competition numbers are studied for several classes of graphs.

1. INTRODUCTION

Graphs and digraphs in this paper are always finite. A graph $G = (V, E)$ has vertex set V and edge set E. A digraph $D = (V, A)$ has vertex set V and arc set A. The vertex set, edge set and arc set of graphs and digraphs are also denoted $V(G)$, $E(G)$, $V(D)$, $A(D)$, respectively. An edge of two ends u, v is denoted (u, v). An arc from u to v is also denoted (u, v). Using terminology in food webs, we say that u *preys on* v and v is a *prey* of u. Multigraphs are allowed to have parallel edges and loops. For definitions of various graph theory terminology, book [35] is suggested.

Given an acyclic digraph $D = (V, A)$, the *competition graph* of D is a graph with vertex set V and there is an edge between $u, v \in V$ if there are arcs from u and v to the same vertex in D. We will denote the competition graph of digraph D by $C(D)$. The *resource graph* of D is the competition graph of the digraph obtained from D by reversing directions of all arcs of D. The resource graph of digraph D will be denoted by $R(D)$. The *competition number* of a graph is the minimum number of additional isolated vertices needed for a graph to be a competition graph of an acyclic digraph. The competition number of graph is denoted by $k(G)$.

This paper is organized as follows: In the rest of this section, we give a brief review of problems studied and introduce restricted competition graphs and restricted competition numbers. In the second section, we study competition graphs of acyclic digraphs of indegree at most 2. In the third section, we study competition graphs of acyclic digraphs of outdegree at most 2, which are resource graphs of digraphs studied in Section 2.

1.1. Background. Competition graphs were introduced in the study of the dimension of ecological niche spaces of food webs by Cohen [8]. Applications of competition graphs and their generalizations have also been studied in other fields such as channel assignment, communication over noisy channels, radio and television transmission, modeling of complex systems and network problems [16, 17, 18, 26, 27, 30, 31, 32]. In this paper, we focus on problems related to food webs. In his study of food webs, Cohen asked, given a knowledge of competition, what is the minimum dimension of a niche space such that the competition relation corresponds to niche overlap in that space. The *boxicity* of a graph is the minimum dimension of an Euclidean space such that the graph is the intersection graph of a collection of "boxes" in the space. In the terminology of "boxicity", defined by Roberts [34], Cohen's question is equivalent to the following: What is the boxicity of competition graphs?

Surprisingly, Cohen and his colleagues observed [8, 9, 10] that almost all competition graphs arising from actual ecosystems are of dimension 1, i.e., they are interval graphs. From a statistical point of view, Cohen [10] studied this problem by considering a food web as a random digraph. By making statistical models of food webs, he found that the probability that the corresponding competition graph of a food web is an interval graph is high. Sugihara [41] gave some ecological explanation on the high frequency of food webs having interval competition graphs and statistically showed that it could be accounted for by requiring the competition graph to be a triangulated graph. Further developments on the random digraph model can be found in Cohen and Newman [13], Cohen, Newman and Briand [14], Cohen, Briand, and Newman [12], and Newman and Cohen [28].

The observation that most food webs from the real world have interval competition graphs led Roberts [36] to ask whether it is an artifact that all competition graphs were of boxicity 1. On the contrary, Roberts [36] proved that by adding a sufficient number of additional isolated vertices, any graph can be made into a competition graph of some acyclic digraph. So the boxicity of competition graphs can be arbitrarily high. Then, interested also from the biological point of view, Roberts [35, 36] raised the problem of characterizing food webs that have interval competition graphs. Roberts [37] says that this problem remains the fundamental open problem in the subject.

It was observed by Steif [40] that there is no forbidden subgraph characterization of acyclic digraphs whose competition graphs are interval graphs. One approach proposed by Cohen [10] is to study the so called sink induced subgraphs of acyclic digraphs. Cohen proved that an acyclic digraph has an interval competition graph if and only if every sink induced subgraph does. Though Steif [40] proved that there exists a forbidden sink induced subgraph list for interval digraphs, no one has been able to present such a list.

On the other hand, Lundgren and Maybee [25] characterized digraphs whose competition graphs are interval graphs by means of edge clique covers of the competition graph. (Some further results related to that can be found in [26, 27, 32]). Though their result is interesting, it is mainly a transformation of the characterization of interval graphs given by Fulkerson and Gross [15] and it does not provide much information about the structure of the digraph. Therefore, the characterization of acyclic digraphs having interval competition graphs is still one of the most important open problems in the theory of competition graphs.

Roberts [36] obtained competition numbers for several classes of graphs, including triangulated graph and triangle-free graphs. Opsut [29] showed that computing the competition number is NP-complete. Several variations of competition number have been studied in literature.

Recently, when considering this problem on some more restricted classes of digraphs, progress was made by Hefner, Jones, Kim, Lundgren and Roberts in [19]. Empirical results of Cohen, Briand and Newman [6, 11, 13, 28, 12] suggest that digraphs built up from real food webs usually have very small degrees. The average is about two. Hefner, et al. studied competition graphs of acyclic digraphs with restricted indegree and outdegree. Part of our study here is a continuing effort in this direction. We focus on the following problems:

- Introducing restricted competition numbers;
- Characterizing competition graphs of acyclic digraphs of restricted degrees;
- Studying the relation between ordinary competition numbers and restricted competition numbers;
- Characterizing acyclic digraphs of restricted degrees which have interval competition graphs and resource graphs;
- Characterizing interval graphs which are competition graphs of degree restricted acyclic digraphs.

Remark: We focus our attention on *acyclic* digraphs because of the biological fact that most digraphs obtained from real world food webs are acyclic.

1.2. Restricted Competition Graph and Restricted Competition Number. We introduce restricted competition graphs and competition numbers here. Given an acyclic digraph D, it is an

(1). (i,j)-*digraph:* If every vertex of D has indegree $\leq i$ and outdegree $\leq j$;

(2). $(i,\bar{j})$-*digraph:* If every vertex of D has indegree $\leq i$ and outdegree either 0 or j;

(3). $(i,*)$-*digraph:* If every vertex of D has indegree at most i;

(4). $(\bar{i},j)$-*digraph:* If every vertex of D has indegree either 0 or i and outdegree $\leq j$;

(5). $(\bar{i},\bar{j})$-*digraph:* If every vertex of D has indegree either 0 or i and outdegree either 0 or j;

(6). $(\bar{i},*)$-*digraph:* If every vertex of D has indegree either 0 or i;

(7). $(*,j)$-*digraph:* If every vertex of D has outdegree at most j;

(8). $(*,\bar{j})$-*digraph:* If every vertex of D has outdegree either 0 or j.

Then a graph G is a *(u,v)-graph* or *(u,v)-competition graph* if there is some acyclic (u,v)-digraph D such that $C(D) = G$, where u, v take the form of $i, \bar{i}, *$ and $j, \bar{j}, *$, respectively.

(1), (2), (4) and (5) were introduced by Hefner, et al. in [19].

A graph G is an *interval (u,v)-graph* if it is an interval graph and there is an acyclic (u,v)-digraph D such that $C(D) = G$. An acyclic digraph is an *interval digraph* if its competition graph is an interval graph. An acyclic digraph is an *interval (u,v)-digraph* if it is both a (u,v)-digraph and an interval digraph.

The *(u,v)-competition number* of a graph G is the minimum number k of isolated vertices needed to add to G such that $G \cup I_k$ is a competition graph of some acyclic (u,v)-digraph, where u, v take the form of $i, \bar{i}, *$ and $j, \bar{j}, *$, respectively. When there is no such number, the (u,v)-competition number is ∞. We denote the (u,v)-competition number of a graph G by $k_{u,v}(G)$.

In general, we call (u,v)-competition numbers *restricted competition numbers*. It is clear that we can view the ordinary competition number as a $(*,*)$-competition number. Given graph $G = (V, E)$, an *edge clique covering* of G is a family of cliques such that every edge of G is in at least one of the clique in the family. Dutton and Brigham [7] characterized competition graphs by means of edge clique coverings.

Theorem 1.1 [Dutton and Brigham [7]]. *Suppose that $G = (V, E)$ and $|V| = n$. Then G is the competition graph of an acyclic digraph if and only if G has an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of vertices such that if $v_i \in C_k$, then $i > k$.*

(u,v)-graphs can be similarly characterized by means of edge clique coverings of the graph. Let $G = (V, E)$ be a graph of n vertices. Suppose $C_1, C_2, ..., C_n$ is an edge clique covering of G, and $v_1, v_2, ..., v_n$ is an ordering of V. We shall be interested in the following properties:

(a) if $v_i \in C_k$, then $i > k$;

(b) $|C_k| \le i$ for $k = 1, ..., n$;

(c) $|C_k| = i$ or $|C_k| = 0$ for $k = 1, ..., n$;

(d) $\forall v_i \in V(G)$, v_i is in at most j of the C_k 's;

(e) $\forall v_i \in V(G)$, v_i is in either none or j of the C_k's.

Theorem 1.2. *Suppose G is a graph of n vertices and (a), (b), (c), (d), (e) are given as above. Then*

(1) G is an (i, j)-graph if and only if there is an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of $V(G)$ such that $(a), (b)$ and (d) hold;

(2) G is an $(\bar{i}, j)$-graph if and only if there is an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of $V(G)$ such that $(a), (c)$ and (d) hold;

(3) G is an $(i, \bar{j})$-graph if and only if there is an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of $V(G)$ such that $(a), (b)$ and (e) hold;

(4) G is an $(\bar{i}, \bar{j})$-graph if and only if there is an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of $V(G)$ such that $(a), (c)$ and (e) hold.

(5) G is an $(i, *)$-graph if and only if there is an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of $V(G)$ such that (a) and (b) hold;

(6) G is an $(\bar{i}, *)$-graph if and only if there is an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of $V(G)$ such that (a) and (c) hold;

(7) G is a $(*, j)$-graph if and only if there is an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of $V(G)$ such that (a) and (d) hold;

(8) G is a $(*, \bar{j})$-graph if and only if there is an edge clique covering $C_1, ..., C_n$ and a labeling $v_1, ..., v_n$ of $V(G)$ such that (a) and (e) hold;

Proof. We give a detailed proof for (1). The others can be proved similarly.

Suppose G is an (i, j)-graph of n vertices and let D be an acyclic (i, j)-digraph D such that $C(D) = G$. Since D is acyclic, we can always label vertices of D as $v_1, v_2, ..., v_n$ such that $(v_x, v_y) \in A(D) \Rightarrow x > y$. Consider the cliques C_i, $i = 1, ..., n$, given by $C_k = \{v_x : (v_x, v_k) \in A(D)\}$. Clearly this is an edge clique covering. Then (a) holds. Since the indegree of any vertex of D is at most i, $|C_k| \leq i$ for any k showing (b). Also, since the outdegree of any vertex of D is at most j, a vertex can be in at most j of such C_k's and (d) follows.

On the other hand, suppose G has an edge clique covering $C = \{C_i\}$ satisfying (a), (b) and (d). Construct digraph D such that $V(D) = V(G)$ and $A(D)$ consists of all arcs (v_j, v_i) such that $v_j \in C_i, i = 1, ..., n-1$. By $(a), C_n = \emptyset$. Therefore $C(D) = G$. If $(v_j, v_i) \in A(D)$, then $j > i$, i.e., D is acyclic. By (b) and (d), D has indegree at most i, outdegree at most j. So (1) is proved. □

2. Competition Graphs of Acyclic Digraphs with Low Indegree

Food webs from the real world usually have very small indegrees and outdegrees. Especially, average indegrees or outdegrees are about 2. Here, we are interested on the restriction of indegrees only. An example of such a food web of small indegree is in the Figure 1.

2.1. $(2, v)$ and $(\overline{2}, v)$-Competition Graphs. A digraph $D = (V, A)$ is *irredundant* if it has no subgraph $P(2,2)$ in Figure 2. Competition graphs of acyclic digraphs of restricted degree were first studied by Hefner, et al. [19]. The following theorem summarizes their main results on digraphs of small degree.

Theorem 2.1 [Hefner, et al [19]]. *Let G be a graph and I be a collection of sufficiently many isolatex vertices. Then*

(1) $G \cup I$ is a $(2,2)$-graph if and only if each connected component of G is either an isolated vertex, or a path, or a cycle.

(2) $G \cup I$ is a $(\overline{2}, \overline{2})$-graph if and only if each connected component of G is either an isolated vertex, or an edge, or a cycle.

(3) *$G \cup I$ is an interval $(2,2)$-graph if and only if each component of G is either an isolated vertex, or a path, or a triangle.*

(4) *$G \cup I$ is an interval $(\overline{2},\overline{2})$-graph if and only if each component of G is either an isolated vertex, or an edge, or a triangle.*

(5) *A $(2,2)$-digraph D has an interval competition graph if and only if each $(2,2)$-irredundant subgraph of D of at least one arc induces one of the digraphs in Figure 3 as a subgraph.*

To complete the remaining cases of indegree at most 2, we characterize $(2,v)$-graphs and $(\overline{2},v)$-graphs for v in the form of $j, \overline{j}$ and $*$. Intervality will be discussed in the next subsection for the $(2,3)$ case. Most results of Theorem 2.1 follow as corollaries of our results here.

Denote the maximum degree of the vertices of G by $\Delta(G)$. A stable set of size k is denoted by I_k.

Lemma 2.2. *Suppose that $G = (V,E)$ is a $(2,j)$-graph. Then $\Delta(G) \leq j$.*

Proof. Suppose that $D = (V,A)$ is an acyclic $(2,j)$-digraph such that $C(D) = G$. Since the indegree of every vertex of D is at most 2, for any vertex v of G, its different neighbors in G, if there are any, correspond to different outgoing arcs in D started at v. So the outdegree of v in D is at least the number of neighbors of v in G, i.e., $j \geq d_D^+(v) \geq d_G(v)$. The lemma follows. □

Lemma 2.3. *If $\Delta(G) \leq j$, then G with sufficiently many additional isolated vertices is a $(\overline{2},j)$ -graph.*

Proof. Construct an acyclic digraph $D = (V(D), A(D))$ as follows. For each edge $e \in E(G)$, add a new vertex v_e. Let the vertex set of D be $V(D) = V(G) \cup \{v_e | e \in E\}$ and let the arc set of D be $A(D) = \{(v,v_e),(u,v_e) | e = (v,u) \in E\}$. Then D is acyclic and $C(D) = G \cup I_k$ where $k = |E(G)|$. The indegree of every vertex of D is either 2 or 0. If v is a vertex of nonzero outdegree in D, then $d_D^+(v) = d_G(v)$. Therefore D is a $(\overline{2},j)$-digraph. □

Lemma 2.4. *G with sufficiently many additional isolated vertices is a $(2, \bar{j})$-graph if and only if $\Delta(G) \leq j$.*

Proof. If $\Delta(G) \leq j$, then there is a $(2, j)$-digraph D such that $C(D) = G \cup I_k$ for some integer k by Lemma 2.3 since a $(\bar{2}, j)$-digraph is a $(2, j)$-digraph. For every vertex v of outdegree $d^+(v)$ in D such that $0 < d^+(v) < j$, add $j - d^+(v)$ new vertices to D and add arcs from v to those new vertices. Then we obtain a $(2, \bar{j})$-digraph. If D is acyclic, then the new digraph is also acyclic and its competition graph is G together with some isolated vertices. On the other hand, if G is a $(2, \bar{j})$-graph, it is also a $(2, j)$-graph, it follows from Lemma 2.2 that $\Delta(G) \leq j$. □

Theorem 2.5. *Every graph with sufficiently many additional isolated vertices is a $(2, *)$ -graph and also a $(\bar{2}, *)$-graph.*

Proof. This follows from Lemma 2.3 and that $(\bar{2}, j)$-graphs are $(2, j)$-graphs. □

Theorem 2.6. *Given a graph G, the following are equivalent:*

(1) G with sufficiently many additional isolated vertices is a $(2, j)$-graph;

(2) G with sufficiently many additional isolated vertices is a $(2, \bar{j})$-graph;

(3) G with sufficiently many additional isolated vertices is a $(\bar{2}, j)$-graph;

(4) $\Delta(G) \leq j$.

Proof. This follows from Lemmas 2.2, 2.3 , 2.4 and the observation that a $(\bar{2}, j)$-graph is also a $(2, j)$ -graph. □

If D is a $(2, j)$-digraph, then D is a (i, j)-digraph for any $i \geq 2$. Therefore, the following corollary follows from Theorem 2.6 (1).

Corollary 2.7. *For any graph G, $k_{i,*}(G) < \infty$ for all $i \geq 2$.*

The only remaining case is $(\overline{2}, \overline{j})$-graphs. Let N be the nonnegative integers. Let b_v be a positive integer assigned to a vertex $v \in V$ of a graph $G = (V, E)$. A function $B : E \to N$ is a b-*matching* if

$$0 \le \sum_{e|v\in e} B(e) \le b_v, \ \forall v \in V.$$

It is a *perfect* b-*matching* if

$$\sum_{e|v\in e} B(e) = b_v, \ \forall v \in V.$$

If $B(e) \neq 0$ for all $e \in E$, we say it is a *nowhere-zero* b-*matching*. If $b_v = j$ or 0 for all $v \in V$, we call it as $\overline{j}$-matching.

Theorem 2.8. *A graph $G = (V, E)$ with sufficiently many additional isolated vertices is a $(\overline{2}, \overline{j})$-graph if and only if it has a nowhere-zero perfect $\overline{j}$-matching.*

Proof. First suppose that there is a $(\overline{2}, \overline{j})$-digraph D such that $C(D) = G \cup I_k$. For any edge $e = (u, v) \in E$, let $B(e)$ equal the number of pairs of arcs from u and v preying on the same vertex in D, then $\sum_{e|v\in e} B(e) = j$ since D is a $(\overline{2}, \overline{j})$-digraph, i.e., G has a nowhere-zero perfect $\overline{j}$ -matching. Next, if G has a nowhere-zero perfect $\overline{j}$-matching $B(e)$, then for each edge $e = (u, v)$, add vertices $w_{e,i}, 1 \le i \le B(e)$, and two arcs from u, v to $w_{e,i}$. Since $B(e) > 0$ for any $e \in E$, if the degree of vertex v is not 0, in D there are exactly j arcs going out from v. Then we have a $(\overline{2}, \overline{j})$-digraph such that its competition graph is $G \cup I_k$ where $k = \sum_e B(e)$. □

A graph G is called *regularizable* if a regular multigraph can be obtained from G by adding edges parallel to some edges of G. The *regularizable index* of G is the minimum degree of all possible regular realization of G. Regularizable graphs were introduced and studied by Berge [1, 2, 3, 4]. Regularizable graphs were also a special notion of *magic graphs* studied by others. From Theorem 2.8, it is easy to see that, in fact, a graph G is a $(\overline{2}, \overline{j})$-graph if and only if each nontrivial connected component of G is regularizable with regularizable index at most j.

Given a graph $G = (V, E)$, a 2-matching is an assignment of nonnegative integers to edges of G $f : E \to \{0, 1, 2\}$ such that $\sum_{e|v\in e} f(e) = 2$ for all

$v \in V$. From observation that a $(\bar{2}, \bar{j})$-graph is j regularizable, following is other characterization of $(\bar{2}, \bar{j})$-graphs borrowed over from characterization of regularizable graphs of Berge [1].

Theorem 2.9. *A graph G is a $(\bar{2}, \bar{j})$-graph if and only if*

(i) every edge of G is in some 2-matching of G;

(ii) there are j 2-matchings covering all edges of G.

(1) of Theorem 2.1 follows immediately from Theorem 2.6. (2) of Theorem 2.1 can be easily derived from Theorem 2.8. Then (3) and (4) of Theorem 2.1 follows from (1), (2) and that intervals graphs are triangulated.

2.2. Interval $(2,3)$-Graphs. Hefner, et al. [19] made progress on characterizing interval digraphs. They characterized interval $(2,2)$-digraphs and gave an algorithm based on their characterization to recognize if a $(2,2)$-digraph has an interval competition graph. We make further progress on this problem by characterizing $(2,3)$-interval digraphs and giving a recognition algorithm.

Though Hefner, et al. [19] proved that there is a forbidden subgraph characterization of interval $(2,2)$-digraphs, there is no forbidden subgraph characterization of interval $(2,j)$-digraphs for general j, even for $j = 3$. An example is illustrated in Figure 4. In the digraph D, $D - e$ induces a subgraph whose competition graph has an induced cycle of length 4. But the competition graph of D is an interval graph.

Now we characterize interval $(2,3)$-graphs and interval $(2,3)$-digraphs. A maximal subgraph of a graph without cut vertices and having at least two vertices is called a *block*. We use $K_n - e$ to denote the graph obtained from K_n by deleting one edge.

Lemma 2.10. *If $\Delta(G) \leq 3$ and G is triangulated, then each block of G is either a K_2, or a K_3, or a $K_4 - e$, or a K_4.*

Proof. We may suppose that G is 2-connected and has at least two vertices. It is easy to check that if G has at most 4 vertices, then G is either a K_2, or a K_3, or a $K_4 - e$, or a K_4. Suppose that G has more than 4 vertices. There must be vertices u, v nonadjacent to each other. By 2-connectivity, there are two vertex disjoint paths P_1, P_2 from u to v. Without loss of generality, we may assume that P_1, P_2 are induced. Since u, v are nonadjacent, $\Delta(G) \leq 3$ and G is triangulated, P_1, P_2 form a cycle of length exactly four containing u, v and another two vertices x, y such that $(x, y) \in E$ but (x, y) is not on neither of P_1, P_2, i.e., u, v, x, y induces a $K_4 - e$. Since $\Delta(G) \leq 3$, x, y have no other neighbor in G. Let w be any other vertex of G. By 2-connectivity there must be a path P_1' from w to u and a path P_2' from w to v. Now by $\Delta(G) \leq 3$, G must has an induced cycle of length at least four containing u, v, x, w, contrary to the fact that G is triangulated. So G cannot have more that four vertices. □

Double star and *crab* are graphs shown in Figure 5. In a graph G, three vertices a, b, c form an *asteroidal triple* if there are simple paths P_1, P_2, P_3 such that P_1 is from a to b, P_2 is from b to c, P_3 is from c to a and c is not adjacent to any vertex on P_1, a is not adjacent to any vertex on P_2 and b is not adjacent to any vertex on P_3.

Theorem 2.11 [Lekkerkerker and Boland [24]]. *A graph G is an interval graph if and only if it is triangulated and has no asteroidal triples.*

Lemma 2.12. *Suppose that G plus sufficiently many additional isolated vertices is a triangulated $(2,3)$-graph. If G has a subgraph (not necessarily induced) which is a double star, then G has either an induced double star or an induced crab.*

Proof. Suppose that S is a double star of G but it is not induced. Let the vertices of S be $V(S) = \{x, u_1, u_2, u_3, v_1, v_2, v_3\}$ such that $d_S(x) = 3, d_S(u_i) = 2,\ i = 1,2,3$ and $d_S(v_i) = 1,\ i = 1,2,3$, and (u_i, v_i) are edges of S, $i = 1,2,3$. Since $G \cup I_k$ is a $(2,3)$-graph for some k, $\Delta(G) \leq 3$. There must be an edge between vertices of S other than those edges in the double star. Since G is triangulated, there must be an edge between two u_i, say u_1, u_2. Now we have a crab as a subgraph. If this crab is not induced, G has a 4-cycle without a chord, which is impossible. □

Theorem 2.13. *A graph G with sufficiently many additional isolated vertices is an interval $(2,3)$-graph if and only if $\Delta(G) \leq 3$, G is triangulated and has no induced double star or crab.*

Proof. Since double stars and crabs both have asteroidal triples, the necessity follows from Lemma 2.2 and Theorem 2.11.

If $\Delta(G) \leq 2$, since G is triangulated, each component of G is either an isolated vertex, a path or a triangle. G is clearly an interval graph. After adding isolated vertices to G, we still have an interval graph. Hence G together with sufficiently many isolated vertices is an interval $(2,3)$-graph by Theorem 2.6.

Suppose that $\Delta(G) = 3$ and G is not an interval graph. By Theorem 2.11, G has an asteroidal triple with vertices a, b, c, simple paths P_1, P_2, P_3. If P_1, P_2, P_3 only meet at vertices a, b, c, then G would have a cycle of at least 6 vertices, contrary to Lemma 2.10. Without loss of generality, suppose that P_1, P_2 meet at some vertices other than b. Starting travel from vertex b along P_1, let the last common vertex of P_1, P_2 be w. Let the next vertex on P_1 be u and let the next vertex on P_2 be v. See Figure 6. Let the vertex on P_1 before w be x. Then $\{x, u, v\} \cap \{a, b, c\} = \emptyset$ since $\{a, b, c\}$ is an asteroidal triple. Now G has a subgraph which is a double star. By Lemma 2.12, G has either an induced double star or an induced crab, contradicting our assumption. So there is no asteroidal triple in G. By Theorem 2.11, G is an interval graph.

Again, since adding isolated vertices to an interval graph produces an interval graph, by Theorem 2.6, G with sufficiently many additional isolated vertices is an interval $(2,3)$-graph. □

2.3. Interval $(2,3)$-digraphs. In this section, we characterize $(2,3)$-interval digraphs. We call a digraph *nontrivial* if it has at least one arc.

Lemma 2.14. *For a $(2,3)$-digraph D, $C(D)$ has a cycle (not necessarily induced) if and only if D has a nontrivial irredundant $(\overline{2},\overline{2})$-subgraph.*

Proof. If D has a nontrivial irredundant $(\overline{2},\overline{2})$-subgraph D^*, then by Theorem 2.6, $\Delta(C(D^*)) \leq 2$. Since D^* is irredundant, there is no vertex of degree 1 in $C(D^*)$. Each component of $C(D^*)$ is either an isolated vertex or a cycle. Since D^* has at least one arc, there is a component which is not an isolated vertex. It follows that $C(D)$ has a cycle since $C(D^*)$ is a subgraph of $C(D)$.

On the other hand, suppose $C(D)$ has a cycle with vertices $x_1, ..., x_k$ such that $(x_i, x_{i+1}) \in E(C(D))$. (In this paragraph, additions of subscripts are always modulo k.) Then for each edge (x_i, x_{i+1}), in the digraph D, there is a v_i such that $(x_i, v_i), (x_{i+1}, v_i) \in A$. Since the indegree of D is at most 2, each v_i is distinct. Hence arcs $(x_i, v_i), (x_{i+1}, v_i)$, $i = 1, ..., k$ together with vertices $x_1, ..., x_k, v_1, ..., v_k$ give a $(\overline{2},\overline{2})$-subgraph of at least one arc. The irredundancy follows from $k \geq 3$. □

Given digraph $D = (V, A)$, for any subset $S \subseteq V \cup A$, denote by $D - S$ the digraph obtained from D by deleting S. When $|S| = 1$, we simplify $D - S$ as $D - v$ for $v \in V$ or $D - a$ for $a \in A$.

Lemma 2.15. *Suppose $D = (V, A)$ is an acyclic $(2,3)$-digraph such that $C(D)$ has a triangle but $C(D - v)$ has no triangle for any $v \in V$. Then D is one of the digraphs in Figure 7. Conversely, if D is a digraph in Figure 7, then $C(D)$ has a triangle.*

Proof. Let $\{x_1, x_2, x_3\}$ be a triangle of $C(D)$. Let D' be the subgraph of D consisting of x_1, x_2, x_3 and common prey of each of the pairs of x_1 and x_2, x_1 and x_3, x_2 and x_3, as well as arcs from x_i's to these prey. These common prey are different because D is a $(2,3)$-digraph. Then $C(D')$ has a triangle. So $V(D) = V(D')$ by the minimality. Now D' contains either the vertices and arcs in D_1, or the vertices and arcs in D_9. Since D is an acyclic $(2,3)$-digraph, it is easy to check that D_1 through D_{10} are the only possibilities under the assumption that D is acyclic. The last part of the lemma is obvious. □

Given an irredundant $(\overline{2},\overline{2})$-subgraph $D^* = (V^*, A^*)$ of a digraph $D = (V, A)$, two vertices $u, v \in V^*$ are a *chord pair* of D^* if there is a w such that $(u, w), (v, w) \in A - A^*$ but u, v have no common prey in D^*. Let v be a vertex of D. The *competitors* S_v of v are vertices which have common prey with v, i.e., $S_v = \{z | (z, w), (v, w) \in A(D) \text{ for some } w\}$. The *out-competitor* *family* of v is the collection of $\{S_u - \{v\} | \ u$ is a competitor of $v\}$.

Given a family of subsets $\mathcal{S} = \{S_i\}$ of a set S, a subset $R \subseteq S$ is a *system of distinct representatives* of $\mathcal{S}$ if $|R \cap S_i| = 1$ for all i and $R \cap S_i \neq R \cap S_j$ for all $i \neq j$. A digraph $D = V, A)$ is *minimal* $(\overline{2}, \overline{2})$*-digraph* if $D - S$ is not a $(\overline{2}, \overline{2})$-digraph for any $S \subseteq V \cup A$.

Theorem 2.16. *An acyclic* $(2,3)$*-digraph* $D = (V, A)$ *is interval if and only if the following hold:*

(1) Every nontrivial minimal irredundant $(\overline{2}, \overline{2})$*-subgraph either has a chord pair or it has at most 6 vertices inducing a subgraph which has one of the digraphs in Figure 7 as an induced subgraph.*

(2) There is no outdegree 3 vertex such that its out-competitor family has a system of distinctive representative.

(3) There are no three vertices u_1, u_2, u_3 *such that every pair of them has a common prey and that there are* $v_i \in S_{u_i} - \{u_1, u_2, u_3\}$ *with all* v_i *distinct,* $i = 1, 2, 3$.

Proof. The necessity: Suppose that $C(D)$ is an interval graph.

If D^* is a minimal irredundant $(\overline{2}, \overline{2})$-subgraph of D (not necessarily induced), then by Lemma 2.14 and minimality, $C(D^*)$ is a cycle with some isolated vertices. Since $C(D)$ is triangulated, $C(D^*)$ either has a chord (u, v) in $C(D)$ for u, v nonadjacent in $C(D^*)$ or $C(D^*)$ is a triangle with at most three isolated vertices. If $C(D^*)$ has a chord in $C(D)$, since u, v are two nonadjacent vertices on $C(D^*)$, they cannot have a common prey in D^*. Therefore u, v is a chord pair of D^*. If $C(D^*)$ is a triangle with at most 3 isolated vertices, then D^* has at most 6 vertices. Let D' be the subgraph induced by D^*. Then $C(D')$ has an triangle. Now D' has vertex minimal subgraphs as in Lemma 2.15. Hence D' has one of the digraphs in Figure 7 as an induced subgraph. This gives (1).

Let v be a vertex of outdegree 3 such that u_1, u_2, u_3 are its competitors and that $v_i \in S_{u_i} - v$, $i = 1, 2, 3$, v_1, v_2, v_3 distinct. Now $C(D)$ would have a double star subgraph with vertices v, u_1, u_2, u_3, v_1, v_2, v_3. By Lemma

2.12, since $C(D)$ is a $(2,3)$-graph, $C(D)$ has either an induced double star or an induced crab, which contradicts the assumption that $C(D)$ is interval. Therefore (2) holds.

(3) must hold for otherwise u_1, u_2, u_3 and v_1, v_2, v_3 in (3), if any, give an induced crab in $C(D)$ since D is a $(2,3)$-digraph. Again, this contradicts the assumption that D is an interval graph.

The sufficiency: From (1), $C(D)$ is triangulated: Let $C = \{x_1, ..., x_k\}$ be a cycle of $C(D)$ such that $(x_i, x_{i+1}) \in E(C(D))$, $i = 1, ..., k$, where addition of subscripts is modulo k. Let v_i be the common prey for x_i and x_{i+1}. Then the x_i's and v_i's form a nontrivial irredundant $(\overline{2}, \overline{2})$-subgraph D^* of D. Since $C(D^* - x)$ does not contain any cycle for any arc or vertex x of D^*, D^* is a minimal irredundant $(\overline{2}, \overline{2})$-digraph. If there is a chord pair in D^*, then C has a chord. If D^* has no such chord pair, then D^* has at most 6 vertices which implies that $C(D^*)$ is a triangle.

$C(D)$ has no induced double star and crab. Suppose first $C(D)$ has an induced double star centered at v. Then it is easy to see that in D the out-competitor family of v has a system of distinct representative, violating (2). Next, if $C(D)$ has an induced crab, then it is easy to see that the three vertices in the triangle of the crab and the three vertices of degree 1 in the crab violate (3). So by Theorem 2.13, $C(D)$ is interval. □

Now we present an algorithm to recognize interval $(2,3)$-digraphs directly from the digraph. Rather than using the characterization of Theorem 2.16, the algorithm utilizes as a subroutine the algorithm given by Hefner, et al. [19] for recognizing interval $(2,2)$-digraphs. It enables us to avoid the difficulty of directly checking the long forbidden induced subgraph list given in Figure 7 and the existence of chord pairs.

The algorithm is implemented in two stages. At the end of the first stage, the algorithm either reports that the digraph is not an interval digraph, or it produces a $(2,2)$-digraph. We will show that the newly produced $(2,2)$-digraph is interval if and only if the original $(2,3)$-digraph is interval. In the second stage we use the algorithm given by Hefner, et al. [19] on the $(2,2)$-digraph produced in stage 1.

Algorithm 2.17. *Given a $(2,3)$-digraph $D = (V, A)$.*

Step 0 If D has $P(2,2)$ as a subgraph, then deleting one of the two pairs of arcs of $P(2,2)$ preying on the same vertex will not change the competition graph. Therefore we can first reduce the digraph into an irredundant $(2,3)$-digraph. Next, reduce the digraph D into a $(\overline{2},3)$-digraph by removing all arcs which go to vertices of indegree 1. Go to Step 1.

Step 1 If there is no vertex of outdegree 3, then go to Step 3. Otherwise take a vertex v of outdegree 3. Let a, b, c be the vertices which have common prey with v.

If every one of a, b, c has outdegree at least two, then go to Step 2. If one of a, b, c, say a, has out degree one, then:

1.1 If at least one of b, c has outdegree at most two, then remove the single arc going out from a and all arcs coming to the prey of a. Go to Step 1.

1.2 Now a has outdegree one and b, c both have outdegree 3. If b, c have a common prey but no common competitor, then stop and report that D is not interval.

1.3 If b, c have a common prey and also a common competitor, remove all arcs going out from b and all arcs coming to the prey of b. Go to Step 1.

1.4 If b, c have no common prey, then remove the single arc going out from a and all arcs coming to the prey of a. Go to Step 1.

Step 2 Now every one of a, b, c has outdegree at least 2. Check common prey of pairs a and b, b and c, a and c.

2.1 If every pair of a and b, a and c, b and c has a common prey, then delete all arcs with tail in $\{v, a, b, c, \}$. Go to Step 1.

2.2 If there are exactly two pairs of a and b, a and c, b and c having common prey, then delete all arcs e_1, e_2, e_3 going out from v and arcs sharing heads with e_1, e_2, e_3. Go to Step 1.

2.3 If there is exactly one pair, say a and b, of a and b, a and c, b and c having one common prey, then:

If a, b have a common prey but at least one of them, say a, has outdegree 2, then remove any arc going out from a and any arc going to prey of a. Go to Step 1.

If both a, b have outdegree 3, then

(a) if a, b have no common competitor, then stop. Report that D is not an interval digraph.

(b) if a, b have a common competitor, then remove any arcs going out from a and arcs going to prey of a. Go to Step 1.

2.4 If there is no pair of a and b, a and c, b and c having common prey, then stop. Report that D is not interval.

Step 3 We have an irredundant (2, 2)-digraph. We now use the algorithm giving by Hefner, et al for testing if an irredundant (2, 2)-digraph is interval or not. Make all vertices are unlabeled. Choose an unlabeled vertex v. Go to the following:

LABEL Construct an irredundant $(\overline{2}, \overline{2})$-subgraph D^* of D which contains v. If G^* gives a cycle of length larger than 4, then stop. Report D is not an interval graph. Otherwise label all vertices of G^* having nonzero outdegree. Then check if there is unlabeled vertices. If not, report that D is interval. Otherwise, repeat LABEL.

Now we prove that the above algorithm works.

Lemma 2.18. *Suppose that u is a vertex of G such that $N[u] = N[v]$ for some vertex $v \in V(G)$. Then G is interval if and only if $G - \{(u, w) | (u, w) \in E(G)\}$ is interval.*

Proof. $G - \{(u, w) | (u, w) \in E(G)\}$ is isomorphic to $(G - u) \cup I_1$ where I_1 is an isolated vertex. So if G is interval, $G - \{(u, w) | (u, w) \in E(G)\}$ is also interval. Now if $G - \{(u, w) | (u, w) \in E(G)\}$ is the intersection graph of a collection of intervals $\{I_w | w \in V(G - \{(u, w) \in E(G)\})\}$, then replacing interval I_u by $I_u^* = I_w$, we have that the intersection graph of $(I - I_u) \cup I_u^*$ is isomorphic to G. □

Lemma 2.19. *Let D be an irredundant $(\overline{2}, 3)$-digraph and $G = C(D)$. Then $d_G(v) = d_D^+(v)$ $\forall v \in V(D)$. Let v be a vertex of outdegree 3 in D and let a, b, c be neighbors of v in G such that $d_G(a) = 1$.*

(1) If one of b, c has degree at most 2, then G is interval if and only if $G - a$ is interval.

(2) If $d_G(b) = d_G(c) = 3$ and in D, b, c have a common prey only if they have a common competitor, then when b, c have common prey G is interval if and only $G - b$ is interval and when b, c have no common prey G is interval if and only if $G - a$ is interval.

Proof. The necessities are clear for both parts of the lemma. Now, for the sufficiency, suppose that $G - a$ is interval in (1) and (2). If at least one of b, c, say b, has degree 1 in G, and $G - a$ is interval but G is not, then there must be an asteroidal triple T in G such that $a \in T$. But T can contain only one of a, b. Since a, b have the same neighbor, $\{T - a\} \cup b$ is an asteroidal triple of $G - a$, which contradicts the assumption that $G - a$ is interval. Hence $d_G(b), d_G(c) \geq 2$. If b, c are nonadjacent, by $d_G(v) = 3$ and $G - a$ is triangulated, b, c are in different blocks of $G - a$, i.e., v is a cut vertex. Then no asteroidal triple of G can contain all a, b and c. Now it follows that if there is an asteroidal triple in G containing a, then there will be an asteroidal triple in $G - a$. Therefore b, c are adjacent. If $d_G(b) = d_G(c) = 2$, then a, b, c, v induces a connected component of G and (1) follows. If exactly one of $d_G(b), d_G(c)$ is 3, say, $d_G(b) = 3$, then b is a cut vertex and there is an asteroidal triple contains a if and only if there is an asteroidal triple contains v. Therefore (1) follows.

Now, to prove (2), suppose that both b and c have degree 3 in G and in D b, c have a common prey only if they have common competitor. Assume $G - b$ is interval and suppose b, c have a common prey. Then they have a common competitor d. Hence v, b, c, d induce a $K_4 - e$ in G such that $d_G(b) = d_G(c) = 3$. By $\Delta(G) \leq 3$, $N[b] = N[c]$. Then by Lemma 2.18, G is interval if and only if the graph obtained by removing all edges incident to b is interval. (The new graph is $G - b$ plus an isolated vertex.) Suppose b, c have no common prey, and suppose $G - a$ is interval. Let $G - a$ be the intersection graph of $I = \{I_u\}$ where I_u is an interval corresponding to $u \in V(G - a)$. Then $I_b \cap I_c = \emptyset$ by b not adjacent to c and $I_v \cap I_b \neq \emptyset$, $I_v \cap I_c \neq \emptyset$. Since in $G - a$ the degree of v is 2, there is at least one point of I_v not in any other intervals. Then we can choose a small interval I_a such that G is the intersection graph of $\{I_u\} \cup I_a$. □

Lemma 2.20. *Suppose $\{u, v, w\}$ induces a triangle in a $(\overline{2}, 3)$-graph such that $d(u) = 2$. Then G is interval if and only if $G - \{(u, v), (u, w)\}$ is interval.*

Proof. The necessity follows from that delete any vertex of an interval graph and then add an isolated vertex to the graph gives an interval graph. Now suppose that $G - \{(u,v),(u,w)\}$ is the intersection graph of open intervals $I = \{I_x | x \in V(G)\}$. If one of v, w, say v, is of degree 2, then $N[v] = N[u]$. The lemma follows by Lemma 2.18. Now suppose that both v, w are of degree 3. (By Theorem 2.6, their degree ≤ 3.)

Case 1: v, w have no common neighbor other than u. Since $I_w \cap I_v \neq \emptyset$, without loss of generality, we may suppose that $a = \max\{y \in I_w\} > b = \min\{y \in I_v\}$. Since v, w are of degree 2 in $G - \{(u,v),(u,w)\}$, we can choose $\epsilon > 0$ such that $I_u^* = (b+\epsilon, a-\epsilon)$ is an interval and G is isomorphic to the intersection graph of $(I - I_u) \cup I_u^*$.

Case 2: v, w have a common neighbor x other than u. Then $\{x, u, v, w\}$ induces a $K_4 - e$ and x is adjacent to at most one neighbor other than v, w. If x has no other neighbor, then $\{u, v, w, x\}$ induces a $K_4 - e$ as a connected component and the Lemma follows. Suppose x is adjacent to z other than v, w. Since z is not adjacent to v, w, I_z is totally on the left (right) side of I_v, I_w. Since v, w have the same neighborhood, we can let $I_v = I_w$ and extend I_v, I_w such that they are not contained in I_x. We can choose I_u intersecting I_v, I_w but not I_x. Then G is the intersection graph of $I \cup \{I_a\}$. □

Theorem 2.21. *Algorithm 2.17 correctly recognizes if an acyclic* $(2,3)$-*digraph is interval.*

Proof. The reduction in Step 0 certainly does not change the competition graph. After Step 0, the degree of a vertex in the competition graph equals its outdegree in the new digraph. Also, every time when some arcs are removed, the produced new digraph is still an irredundant $(\overline{2},3)$-digraph since whenever we remove an arc a, we also remove all arcs sharing heads with a.

If there is no vertex of outdegree 3, then the digraph is a $(2,2)$-digraph. So we can go to Step 3 to start the subroutine to check the intervality of a $(2,2)$-digraph. Now suppose after Step 0 the digraph still has a vertex of outdegree 3.

If the algorithm reports that the digraph is not interval in Step 1. It stops in Step 1.2. Let v, a, b, c be as described in the algorithm. In the competition graph, v, b, c all have degree 3, and b is adjacent to c. Then G has crab as a subgraph containing vertices v, a, b, c. Since $\Delta(G) \leq 3$, either this is an induced crab, or there is an induced cycle of length larger than 3. So the algorithm correctly reports that G is not interval.

Suppose the algorithm does not stop at Step 1. If there are no arcs removed, the digraph is not changed. Suppose some arcs are removed. If some arcs are removed in Step 1.1, then v has degree 3, and a has degree 1 in G. One of b, or c, say b, has degree at most 2. It follows from the first part of Lemma 2.19 that G is interval if and only if $G - a$ is interval. So removing arcs in Step 1.1, which is equivalent to deleting a than add in an isolated vertex, does not change the intervality of D.

If some arcs are removed in Step 1.3, then b and c have a common prey and a common competitor. By the second part of Lemma 2.19, D is interval if and only if the newly obtained digraph is interval since removing arcs in Step 1.3 is equivalent to removing b from G and adding an isolatd vertex.

If some arcs are removed in Step 1.4, then by the second part of Lemma 2.19, G is interval if and only if $G - a$ is, if and only if $G - a$ plus an isolated vertex is.

In Step 2.1, every pair of vertices a, b and c has a common prey. Therefore $\{v, a, b, c\}$ induces a K_4 subgraph of $C(D)$. $\{v, a, b, c\}$ must be a connected component of $C(D)$. $C(D)$ is interval if and only if $(C(D) - \{v, a, b, c, \}) \cup I_4$ is interval, where I_4 is a set of four isolated vertices. So $C(D)$ is interval if and only if the digraph obtained after Step 2.1 is interval.

In Step 2.2, vertices v, a, b, c induce a subgraph $K_4 - e$ in $C(D)$. Then one of a, b, or c, say a, will have the same neighbors as v by $\Delta(C(D)) \leq 3$. Arcs going out of v and arcs coming to prey of v are removed. This is equivalent to removing edges $\{(v, w) | (v, w) \in E\}$ from $C(D)$. Then by Lemma 2.18, $C(D)$ is interval if and only if $C(D) - \{(v, w) | (v, w) \in E(G)\}$ is interval. So $C(D)$ is interval if and only if the digraph obtained after Step 2.2 is interval.

In Step 2.3, v is a vertex of outdegree 3. If the algorithm stops, then among competitors a, b, and c of v, there is exactly one pair, say a and b, such that a, b have a common prey. If a has degree 2, then by Lemma 2.20 (1), $C(D)$ is interval if and only if $C(D)$ deleting a and adding an isolated vertex is. If both of a, b have degree 3 and a, b have no common competitor, then either v, a, b, c is in an induced crab of $C(D)$ or $C(D)$ has a cycle of length larger than 4. $C(D)$ is not interval. The algorithm will report this correctly. If a, b have a common common prey, then $N[a] = N[b]$ in $C(D)$. By Lemma 2.18, $C(D)$ is interval if and only if the newly obtained digraph is interval.

At Step 2.4, the degrees of a, b and c in $C(D)$ are at least 2, and these three vertices form a stable set. Either $C(D)$ has some induced cycle of length at least 4 or it has a double star. Hence D is not interval.

Step 3 utilizes the algorithm of Hefner et al. for $(2,2)$-digraphs. To entering Step 3, the digraph cannot have any vertex of outdegree 3. So it must be $(2,2)$-digraph. Therefore, the correctness of the algorithm is proved. □

2.4. $(2, j)$, $(2, *)$-Competition Numbers and Ordinary Competition Numbers. In this section, we initialize studying the relation between the ordinary competition number and the restricted competition numbers. We first compute the $(2, *)$-competition numbers.

Lemma 2.22. *For any graph $G = (V, E)$, $k_{2,*}(G) \geq |E| - |V| + 2$*

Proof. Let D be an acyclic $(2, *)$-digraph such that $G \cup I_m$ is the competition graph of D and $|V(D)| = |V(G)| + m$. Since indegree in D is at most 2, each edge of G corresponds to a distinct vertex of D. There is a labeling of vertices of D such that there is an arc from i to j only if $i < j$. The vertices with the smallest two labels have at most one incoming arc. It follows that $|V| + m \geq |E| + 2$. So $k_{2,*}(G) \geq |E| - |V| + 2$. □

Theorem 2.23. *For any connected graph $G = (V(G), E(G))$ with at least one edge, $k_{2,*}(G) = |E(G)| - |V(G)| + 2$.*

Proof. Roberts [36] showed that if H is a connected triangle-free graph, then the competition number of H is exactly $|E(H)| - |V(H)| + 2$. Now suppose G is a connected graph. We subdivide every edge (u,v) of G by adding a new vertex x_{uv} and replacing edge (u,v) by edges $(u,x_{uv}),(v,x_{uv})$. Then the new graph G^* is a connected triangle-free graph with $|V(G^*)| = |V(G)|+|E(G)|$ and $|E(G^*)| = 2|E(G)|$. There is an acyclic digraph D^* such that $G^* \cup I_{k(G^*)} = C(D^*)$ and $k(G^*) = |E(G^*)| - |V(G^*)| + 2 = |E(G)| - |V(G)| + 2$. It is clear that D^* is a $(2,*)$-digraph. Let s_u, s_v be prey of u, x_{uv} and x_{uv}, v, respectively, in D^*. For vertices s_u, s_v, assume without loss of generality that there is no directed path in D^* from s_u to s_v. Now for all u, v, delete from D^* vertices x_{uv} and arcs incident to x_{uv}. Let u, v prey on s_u and let vertices which prey on x_{uv} prey on s_v. Then the new digraph D is still an acyclic $(2,j)$-digraph. $C(D) = G \cup I_m$ and

$$\begin{aligned} |V(G)|+m &= |V(D^*)| - |E(G)| \\ &= |V(G^*)| + k(G^*) - |E(G)| \\ &= |V(G)| + |E(G)| + |E(G^*)| - |V(G^*)| + 2 - |E(G)| \\ &= |E(G)| + 2 \end{aligned}$$

i.e., $|V(D)| = |E(G)|+2$. Since $|V(D)| \geq |V(G)| + k_{2,*}(G)$, $|E(G)| - |V(G)| + 2 \geq k_{2,*}(G)$. The theorem follows from Lemma 2.22. □

Corollary 2.24. *For any connected graph G with at least one edge and for any integer $i \geq 2$, $k(G) \leq k_{i,*}(G) \leq k_{2,*}(G) = |E(G)| - |V(G)| + 2$.*

Theorem 2.25. *Suppose graph G has no isolated vertices and has $m+t$ components, t of which are trees. Then*

$$k_{2,*}(G) = \max\{1, |E(G)| - |V(G)| + 2\}.$$

Proof. Suppose $T_1, ..., T_t, G_1, ..., G_m$ are connected components of G such that $T_1, ..., T_t$ are all tree components. By Theorem 2.23, $k_{2,*}(G_i) = |E(G_i)| - |V(G_i)| + 2 \geq 2$, $i = 1, ..., m$ and $k_{2,*}(T_i) = |E(T_i)| - |V(T_i)| + 2 = 1$, $i = 1, ..., t$.

Suppose $D_1, ..., D_{t+m}$ are acyclic $(2,*)$-digraphs such that $C(D_i) = T_i \cup \{a_i\}$, $i = 1, ..., t$, and $C(D_{i+t}) = G_i \cup I_i$, $i = 1, ..., m$, where $a_i, i = 1, .., t$, are isolated vertices and $|I_i| = |E(G_i)| - |V(G_i)| + 2, i = 1, ..., m$. Without loss of generality, we may assume that D_i has no vertex of indegree one. Therefore by D_i is a $(2,*)$-digraph, in each D_i there are exactly two vertices of indegree zero. Denote them as x_1^i, x_2^i, $i = 1, ..., t+m$.

In digraph $D_1 \cup D_2 \cup ... \cup D_{m+t}$, first let arcs from vertex x of D_i to a_i, $i = 2, ..., t$, be changed to from x to x_1^{i-1}. Next order isolated vertices in $I_1 \cup ... \cup I_m$ as $b_1 \prec b_2 ... \prec b_l \prec ...$ such that if $b_l \in I_i, b_h \in I_j, i < j$, then $b_l \prec b_h$. According to the order of

$$x_1^t, x_2^1, x_2^2, ..., x_2^t, x_1^{t+1}, x_2^{t+1}, x_1^{t+2}, x_2^{t+1}, ..., x_1^{t+m-1}, x_2^{t+m-1},$$

change arc (x, b_1) of $D_{t+1} \cup ... \cup D_m$ to (x, x_1^t), (x, b_2) to (x, x_2^1), (x, b_3) to (x, x_2^2), ..., until either all (x, b_i)'s are replaced or all $x_j^i, j = 1, 2, i = 1, ..., t+m-1$, are used. Remove any a_i's and b_i's having no arcs coming in. If all b_i's are replaced, then $k_{2,*}(G) \leq 1$. If all x_j^i, except x_1^{t+m}, x_2^{t+m}, are used up, then by $|E(T_i)| - |V(T_i)| = -1$,

$$\begin{aligned} 1 \leq k_{2,*}(G) &\leq t + \textstyle\sum_{i=1}^m (|E(G_i)| - |V(G_i)| + 2) - 2(t+m-1) \\ &= \textstyle\sum_{i=1}^t (|E(T_i)| - |V(T_i)| + 2) + \sum_{i=1}^m (|E(G_i)| \\ &\quad - |V(G_i)| + 2) - 2(t+m-1) \\ &= |E(G)| - |V(G)| + 2(t+m) - 2(t+m-1) \\ &= |E(G)| - |V(G)| + 2. \end{aligned}$$

Since in any acyclic $(2,*)$-representation of G, the number of vertices having at least 2 incoming arcs are at least the number of edges of G and also there are at least two vertices having at most one incoming arc, so $k_{2,*}(G) \geq |E(G)| - |V(G)| + 2$. The theorem follows. □

As we proposed here to study the relationship between competition numbers and restricted competition numbers, we characterize when the interesting equality $k(G) = k_{2,*}(G)$ holds for a graph G. The answer strengthens the well-known result by Roberts. Another similar result will be presented in the next section on the relation between the $(*,2)$-competition numbers $k_{*,2}(G)$ and competition numbers $k(G)$.

Lemma 2.26. *If a connected graph G has a triangle, then $k(G) < |E(G)| - |V(G)| + 2$.*

Proof. We proceed by induction on $|V(G)|$. The lemma is trivially true for graphs of only two vertices. Suppose that $|V(G)| \geq 3$. Since G is connected, there is at least one vertex v such that $G - v$ is still connected.

Case 1. If $G - v$ has a triangle, then by induction,

$$k(G-v) < |E(G-v)| - |V(G-v)| + 2 = |E(G)| - d(v) - |V(G)| + 1 + 2.$$

There is an acyclic digraph D such that $C(D) = (G - v) \cup I_{k(G-v)}$. Since $G - v$ is connected and has at least one edge, $k(G - v) \geq 1$. Suppose $a \in I_{k(G-v)}$. Let D' be an acyclic digraph as follows:

$$V(D') = [V(D) \cup \{v\} \cup \{x_w | w \in N(v)\}] - \{a\};$$
$$A(D') = A(D-a) \cup \{(w, x_w), (v, x_w) | w \in N(v)\} \cup \{(y, v) | (y, a) \in A(D)\}.$$

Then $C(D') = G \cup I_m$, $|I_m| = k(G - v) + d(v) - 1$. Therefore $k(G) \leq k(G - v) + d(v) - 1 < |E(G)| - |V(G)| + 2$.

Case 2. If $G - v$ has no triangle, then v is in least one triangle, say $\{v, f, g\}$. Note that $k(G - v) = |E(G - v)| - |V(G - v)| + 2 = |E(G)| - d(v) - |V(G)| + 1 + 2$. There is an acyclic digraph D such that $C(D) = (G - v) \cup I_{k(G-v)}$. Since $G - v$ is connected, $k(G - v) \geq 1$. Suppose $a \in I_{k(G-v)}$. Let D' be an acyclic digraph as follows.

$$V(D') = [V(D) \cup \{v\} \cup \{x_w | w \in N(v),\ w \neq f\}] - \{a\};$$
$$A(D') = A(D - a) \cup \{(w, x_w), (v, x_w) | w \in N(v) \text{ and } w \neq f\}$$
$$\cup \{(y, v) | (y, a) \in A(D)\} \cup \{(f, x_g)\}.$$

Then $C(D') = G \cup I_m$, $|I_m| = k(G - v) + (d(v) - 1) - 1$. Hence $k(G) < k(G - v) + d(v) - 1 = |E(G)| - |V(G)| + 2$. □

Using Lemma 2.26, we can strengthen the following theorem by Roberts.

Theorem 2.27 [Roberts [36]]. *Let G be a connected triangle-free graph with at least one edge. Then*

$$k(G) = |E(G)| - |V(G)| + 2.$$

Theorem 2.28. *Let G be a connected graph with at least one edge. Then*

$$k(G) \leq k_{2,*}(G) = |E(G)| - |V(G)| + 2$$

and

$$k(G) = k_{2,*}(G) = |E(G)| - |V(G)| + 2$$

if and only if G is a triangle-free graph.

Since $k(G) \leq k_{i,*}(G)$, it is natural to ask, at least for connected graphs with at least one edge, if $k(G) = k_{i,*}(G)$ is equivalent to that G is K_{i+1}-free. The following example shows that this is not true in general. Since triangulated graphs have competition numbers at most 1, the graph in Figure 8 has $k(G) = 1$. But it is easy to see $k_{4,*}(G) = 1$ by the acyclic $(4,*)$-digraph D in Figure 8. Nevertheless G has an induced K_5.

3. Competition Graph of Small Outdegree

Suppose $D = (V, A)$ is an acyclic digraph. The *resource graph* $R(D)$ of D is a graph having the same vertex set as D and there is an edge between two vertices a, b if and only if there is a vertex x in D such that $(x, a), (x, b)$ are two arcs of D. Resource graph can also be viewed as the competition graph of the digraph obtained by reversing directions of arcs in D. Resource graphs are also known as common-enemy graphs studied by Scott and others [21, 39, 38]. Study of resource graphs of food webs can be found in [41] by Sugihara. Since to study resource graphs of digraphs of restricted indegree is equivalent to studying competition graphs of digraphs of restricted outdegree, in this section we study competition graphs of digraphs of small outdegrees.

3.1. Characterization of $(*,2)$- and $(*,\overline{2})$-Graphs.

Lemma 3.1. *If $G = (V, E)$ with sufficiently many additional isolated vertices is a $(*,2)$-graph, then for any $v \in V$, $N(v)$ can be covered by at most two cliques.*

Proof. Let D be an acyclic $(*,2)$-digraph such that $C(D) = G \cup I_k$. Suppose that $v \in V$ is a vertex of G. Each neighbor of v preys on one of the (at most) two prey of v. Since vertices having the same prey induces a clique, vertices of $N(v)$ of v can be covered by the (at most) two cliques. □

Remark Not every graph in which the neighborhood of every vertex $v \in V$ can be covered by at most two cliques can be made into a $(*,2)$-graph by adding isolated vertices. An example is in Figure 9 . To see that the graph G in Figure 9 is not a $(*,2)$-graph after adding isolated vertices, let us suppose to the contrary that D is an acyclic $(*,2)$-digraph such that $C(D) = G \cup I_k$ for some k. Notice that the only way to cover the neighborhood of vertex b by two cliques is to choose one clique as $\{a,b,d\}$ and another clique as $\{b,c,e\}$. Therefore in D there are $v_1 \neq v_2$ such that a,b,d prey on v_1 and b,c,e prey on v_2. Then vertices d,f cannot prey on v_2 , since f is not adjacent to d,b and d is not adjacent to c. It implies that e has at least another two prey other than v_2, contradicting the assumption that D is a $(*,2)$ -digraph.

Lemma 3.2. *Every line graph plus sufficiently many additional isolated vertices is a $(*,2)$-graph. The collection of all graphs obtained from line graphs by adding sufficiently many additional isolated vertices is a proper subset of $(*,2)$-graphs.*

Proof. Let $L(G)$ be the line graph of graph G. Construct a food web as follows: Let $V(D) = V(G) \cup E(G)$. There is an arc in D from some $e \in E(G)$ to some $v \in V(G)$ if and only if v is an end of e. It is clear that $C(D) = L(G) \cup I_{|V(G)|}$ and D is an acyclic $(*,2)$ -digraph. On the other hand, the graph in Figure 10 is a $(3,2)$ -graph after adding in some isolated vertices, but is not a line graph. □

We call vertices x and y *equivalent*, denoted by $x \sim y$, if $N[x] = N[y]$. Then the relation $\sim$ is an equivalence relation. A graph is called *reduced* if it has no pair of equivalent vertices. Suppose $G = (V,E)$ is a simple graph of vertex partition $V_1, ..., V_m$. A *contraction* G^* of G is a graph having vertex set $\{V_1, ..., V_m\}$ with V_i, V_j adjacent in G^* if and only if there are $x \in V_i, y \in V_j$ such that $(x,y) \in E$. When the V_i's are equivalent classes under $\sim$, G is also called a *multiplication* of G^*.

Lemma 3.3. *Let graph G be a simple graph with vertex set partitioned into equivalence classes $V_1, ..., V_m$ by the equivalence relation $\sim$. Let G^* be the contraction of G under this partitioning. Then $k_{*,j}(G) \leq k_{*,j}(G^*)$.*

Proof. Suppose D is an acyclic $(*, j)$-digraph such that $C(D) = G^* \cup I_m$. Let V_i be a vertex of G^*. Replace each V_i by all vertices v in V_i. (u, v) is an arc if and only if $u \in V_i, v \in V_j$, and (V_i, V_j) is an arc of D. Then the new digraph is still acyclic and has G as its competition graph. Since no additional isolated vertices not in $G^* \cup I_m$ have been added to D. So $k_{*,j}(G) \leq k_{*,j}(G^*)$. □

Lemma 3.4. *If G is a $(*, j)$-graph and G' is an induced subgraph of G, then G' plus sufficiently many isolated vertices is a $(*, j)$-graph. If G is a $(*, j)$-graph, then any multiplication of it is also a $(*, j)$-graph.*

Proof. Suppose G has a $(*, j)$-representation D. If $x \in V(G)$, remove all outgoing arcs of x from D, replace x by a new vertex $a \notin V(G)$ in D, and let all vertices which prey on x prey on a. We obtain a $(*, j)$-representation of $(G - x) \cup \{a\}$. On the other hand, for $y \in V(G)$, if y is an isolated vertex, it is clear that G plus some additional isolated vertices is still a $(*, j)$-graph. If y is not an isolated vertex, then adding w to D such that w preys on any vertex which is preyed by y gives an $(i + 1, j)$-representation of $G' = (V(G) \cup \{w\}, E(G) \cup (y, w) \cup \{(x, w) | (x, y) \in E(G)\})$. □

Theorem 3.5 [Krausz [23]]. *A graph G is a line graph of a simple graph if and only if there is an edge clique covering $C_1, ..., C_m$ such that every edge of G is in exactly one of the C_i's and every vertex is in at most two of the C_i's.*

Theorem 3.6 [Hemminger [20], Bermond and Meyer [5]]. *A graph is the line graph of a multigraph if and only if it does not contain the graphs in Figure 11 as induced subgraphs.*

Theorem 3.7. *The following are equivalent*

(1) G together with sufficiently many additional isolated vertices is a $(*, 2)$-graph;

(2) G together with sufficiently many additional isolated vertices is a $(*, \overline{2})$-graph;

(3) *G is the line graph of a multigraph;*

(4) *G does not contain any of the graphs in Figure 11 as induced subgraphs.*

Proof. (1) $\Rightarrow$ (2): Let D be an acyclic $(*,2)$-digraph such that $C(D) = G \cup I_k$ for integer $k \geq 0$. Then for any vertex v of outdegree 1, add an isolated vertex x_v and add an arc from v to x_v.

(2) $\Rightarrow$ (3): Let G be a $(*,\overline{2})$-graph. Contract all equivalent vertices in G to obtain a reduced graph G'. By Lemma 3.4, G' with sufficiently many additional isolated vertices is also a $(*,2)$-graph. Let D be a $(*,2)$-representation of G'. Let $S_1, ..., S_m$ be such that S_i contains all vertices preying on the same vertex v_i of D. Since D is a $(*,2)$-digraph and G' has no equivalent vertices, $|S_i \cap S_j| \leq 1 \;\; \forall i \neq j$. Every S_i induces a clique of G'. Now $S_1, ..., S_m$ give an edge clique covering of G' such that each edge is in exactly one clique and each vertex is in at most 2 such cliques. By Theorem 3.5, G' is a line graph of a simple graph. Hence G is a line graph of a multigraph.

(3) $\Rightarrow$ (1): Since line graphs of simple graphs are $(*,2)$-graphs by Lemma 3.2 when sufficiently many additional isolated vertices are added, it follows from Lemma 3.4 that line graphs of multigraphs are also $(*,2)$-graphs when sufficiently many additional isolated vertices are added.

(3) $\Leftrightarrow$ (4): This follows from the characterization of line graphs of multigraphs of Theorem 3.6 by Hemminger [20], Bermond and Meyer [5]. □

3.2. $(*,2)$-Competition Number. In this subsection we discuss the relationship between $(*,2)$ -competition number and the competition number. The $(*,2)$ competition number is also related to a conjecture proposed a decade ago by Opsut on competition numbers.

Conjecture 3.8 [Opsut [29]]. *If vertices in the neighborhood of every vertex of G can be covered by at most two cliques, then the competition number of G is at most 2.*

Opsut [29] proved that the competition number of a line graph is at most 2. By the characterization given in the last subsection, $(*, 2)$-graphs are exactly line graphs of multigraphs. Therefore it is natural to ask what is the competition number of an $(i, 2)$-graph.

Given graph $G = (V, E)$, for any $S \subseteq V$, let $\theta(S)$ be the minimum number of cliques needed to cover all vertices of S. For vertex $v \in V$, $N(v)$ is the neighborhood of v, i.e., all vertices adjacent to v. We say that $\theta^*(N(v)) \leq 2$ if $\theta(N(v)) \leq 2$ and there are two cliques C_1, C_2 covering $N(v)$ such that for all $w \in C_1$, $\theta(N(w) - C_1) \leq 1$. We say $\theta^*(N(v)) = 2$ if $\theta^*(N(v)) \leq 2$ and $\theta(N(v)) = 2$.

Lemma 3.9. *If $G \cup I_m$ is a $(*, 2)$-graph, then $\theta^*(N(v)) \leq 2$ for all $v \in V(G)$.*

Proof. Since $(*, 2)$-graphs are line graphs of multigraphs, suppose $G = L(H)$ where H is a multigraph. Suppose that $e = (u, v)$ is an edge of H. Let E_u and E_v be the sets of edges incident to u and v, respectively. In the line graph $L(H)$, E_u, E_v are two cliques covering $N_G(e)$. Suppose $w \in E_u$. Any neighbor of w which is not in $N_G(e)$ is an edge of H not incident to either u or v. So all such edges are incident to the same end of w other than u, i.e., $\theta(N_G(w) - E_u) \leq 1$. The argument is similar for $w \in E_v$. □

Theorem 3.10 [Kim and Roberts [22]]. *If G is such that $\theta^*(N(v)) \leq 2$ for all $v \in V(G)$, then $k(G) \leq 2$ and $k(G) = 2$ if and only if $\theta(N(v)) = 2$ for all $v \in V(G)$.*

By Theorem 3.10, we have the following.

Theorem 3.11. *If G with sufficiently many additional isolated vertices is a $(*, 2)$-graph, then the competition number of G is at most 2, and $k(G) = 2$ if and only if $\theta(N(v)) = 2$ for all $v \in V$.*

We have shown that for triangle-free graphs, $k_{2,*}(G) = k(G)$. Now we show that if G is a $(*, 2)$-graph, then $k_{*,2}(G) = k(G)$. Denote by $P(s)$ the set of vertices which prey on s for $s \in V(G)$.

Lemma 3.12. *Suppose G is a reduced $(*, 2)$-graph with a representation D . Let C be a clique of G corresponding to a $P(t)$ for some $t \in V(D)$. Then removing all edges in C produces another $(*, 2)$-graph G'.*

Proof. Note that $|P(t) \cap P(s)| \leq 1$ for any $s \neq t$, i.e., all edges in C are only produced by prey on t. For if there are $s \neq t, s, t \in V(D)$ such that $x, y \in C$ and $x, y \in P(s) \cap P(t)$, then $N[x] = N[y]$, contrary to the assumption that G is reduced. Therefore, removing edges in C corresponds to removing all incoming arcs of t in D, i.e., $(G - E(C)) \cup I_m = C(D - \{(r, t) | (r, t) \in A(D)\})$ for some m. So $G - E(C)$ is a $(*, 2)$-graph. □

Given a graph G, a *minimal (i, j)-representation* is an acyclic (i, j)-digraph D such $C(D) = G \cup I_m$ for some integer m but for any subgraph D' of D, $C(D') \neq G \cup I_{m'}$ for any m'.

Lemma 3.13. *Suppose G is a $(*, 2)$-graph. Suppose v is a simplicial vertex of G. Then there is a minimal $(*, 2)$-representation of G such that v has exactly one prey. Moreover, if $N(v)$ contains no equivalent vertices, then in any minimal $(*, 2)$-representation, v has exactly one prey.*

Proof. Let D be a minimal $(*, 2)$-representation of G. If v has only one prey in D, then we are done. Suppose v has two distinct prey x, y in D. Now add a new vertex z to D and let all arcs going to x, y instead to go to z. In the new digraph D', v has only one prey. D' is still acyclic. If $m \in P(x) \cup P(y)$ and $n \in P(x) \cup P(y)$, then $m, n \in N[v]$. Therefore m is adjacent to n and D' is a $(*, 2)$-representation of $G \cup z$. So there is a minimal representation of G such that v has only one prey.

Now suppose that v has no equivalent neighbors in G. Suppose that D is a minimal $(*, 2)$-representation of G and v has two different prey x, y in D. Since D is a minimal representation, $P(x) \not\subset P(y)$ and $P(y) \not\subset P(x)$. Hence there are two vertices $m \neq n$ such that $m \in P(x) - P(y)$; $n \in P(y) - P(x)$. Since m, n are both neighbors of v and v is simplicial, there must be some $z \in V$ such that $z \neq x, y$ and $\{m, n\} \subseteq P(z)$. Let $w \in N(n)$. Then either $w \in P(y)$ or $w \in P(z)$ since n has only z, y as prey. If $w \in P(z)$ then $w \in N(m)$; if $w \notin P(z)$, then $w \in P(y)$. Since v is simplicial, it follows that $w \in N(v)$, $w \in N(m)$. Therefore, $N[n] \subseteq N[m]$. By symmetry, $N[n] = N[m]$, contradicting the assumption that v has no equivalent neighbors. The lemma is proved. □

Lemma 3.14. *Suppose G is a reduced $(*, 2)$-graph. If v is a non-isolated simplicial vertex, then the graph obtained from G by removing edges in $N[v]$ and deleting v is also a $(*, 2)$-graph if sufficiently many isolated vertices are added.*

Proof. This follows from Lemmas 3.12 and 3.13. □

Lemma 3.15. *Suppose G is a reduced $(*, 2)$-graph. Suppose that $C(D) = G \cup I_k$. Then in the graph G^* obtained by removing edges in a clique C of G corresponding to a $P(s)$ of $s \in V(D)$, $N_{G^*}(v)$ has no equivalent vertices for any $v \in C$.*

Proof. That G^* is a $(*, 2)$-graph follows from Lemma 3.12. Now suppose that there are $u, w \in N_{G^*}(v)$ such that $N_{G^*}[u] = N_{G^*}[w]$ for some $v \in C$. It is clear that $u, w \notin C$ for otherwise u, w are not neighbors of v in G^*. Hence $N_{G^*}[u] = N_G[u]$ and $N_{G^*}[w] = N_G[w]$. This contradicts the assumption that G is reduced. □

Lemma 3.16. *If $\theta(N(v)) \geq k$ for all $v \in V$, then $k_{i,j}(G) \geq k$.*

Proof. Opsut [29] proved that if $\theta(N(v)) \geq k$ for all $v \in V$, then $k(G) \geq k$. Since for any fixed i, j, $k_{u,v}(G) \geq k(G)$, the lemma follows. □

Theorem 3.17. *If G plus sufficiently many additional isolated vertices is a $(*, 2)$-graph, then $k_{*,2}(G) \leq 2$, with equality if and only if $\theta(N(v)) = 2$ $\forall v \in V(G)$. Furthermore, there is a $(*, 2)$-representation of G such that every simplicial vertex has exactly one prey.*

Proof. The argument is by induction on $|E(G)| + |V(G)|$: It is trivial that the theorem i s true for graphs having at most 3 vertices. Suppose the theorem is true for graphs smaller than G.

Case 1. Suppose that G has an isolated vertex v. Apply induction, $k_{*,2}(G - v) \leq 2$. Then by using v as a prey, $k_{*,2}(G) \leq 1$.

Case 2. Suppose that G has a non-isolated simplicial vertex v. Let G^* be the reduced graph obtained from G by contracting all equivalent vertices. Still denote the equivalence class containing v as v. Since G^* plus sufficiently many isolated vertices is a reduced $(*,2)$-graph, by Lemma 3.14, removing edges in N_{G^*} and deleting v from G^* produces another graph G' with fewer edges. G' is a $(*,2)$-graph after sufficiently many isolated vertices are added. All vertices in $N_{G^*}(v)$ are simplicial in G'. By induction, $k_{*,2}(G') \leq 1$. By Lemma 3.15 and Lemma 3.13, every simplicial vertex of G' in $N_{G^*}(v)$ has exactly one prey in any minimal representation. Let D be a minimal $(*,2)$-representation of G' and let $C(D) = G' \cup \{a\}$. Now add v to G'. Let all vertices preying on a instead prey on v and let v and vertices in $N_{G^*}(v)$ prey on a. Then the $(*,2)$-competition number of G^* is at most 1. By Lemma 3.3, the $(*,2)$-competition number of G is at most 1.

Case 3. Suppose that G has no simplicial vertex or isolated vertex. Let G^* be the graph obtained from G by contracting equivalent vertices. G^* has no simplicial vertex or isolated vertex. Let D^* be a $(*,2)$-representation of $G \cup I_{k_{*,2}(G)}$. Let C be a clique of G^* corresponding to a $P(s)$ for some $s \in V(D^*)$. Let G' be the graph obtained from removing all edges in C from G^*. G' is a $(*,2)$-graph by Lemma 3.12. Every vertex of C in G' is a simplicial vertex because it has outdegree 2 in D^*. By induction $k_{*,2}(G') \leq 1$. By Lemmas 3.15 and 3.13, there is a digraph D' such that $C(D') = G' \cup \{a\}$, where $\{a\}$ is an isolated vertex, and that every vertex in C has at most one prey. We can add one more isolated vertex to D' such that all vertices in C prey on it. So $k_{*,2}(G^*) \leq 2$ which gives $k_{*,2}(G) \leq 2$.

On the other hand, by Lemma 3.16, if G has no simplicial or isolated vertex, then $k_{*,2}(G) \geq 2$. □

The following corollary is similar to $k_{2,*}(G) = k(G)$ for triangle-free graphs.

Corollary 3.18. *$k_{*,2}(G) = k(G)$ for $(*,2)$-graphs.*

Proof. This follows from Lemma 3.9 and Theorems 3.10 and 3.17. □

3.3. Interval $(*, 2)$-Graphs. We characterize interval $(*, 2)$-graphs and interval $(*, 2)$-digraphs in this section.

Lemma 3.19. *Suppose H is a graph such that $L(H)$ is triangulated. Then H has no cycle of length greater than 3. So each block of H is either an edge or a triangle.*

Proof. The lemmafollows from that any cycle, not necessarily an induced one, of H corresponds to an induced cycle in the line graph $L(H)$. □

A connected graph G is a *worm* of length m if G has a longest induced path of length $m + 1$, i.e., having $m + 1$ edges, such that every edge of H is either on P or incident to P and each block of G is either an edge or a triangle. The *end vertices* of a worm are vertices which can be the first and the second vertices of a longest path of the worm. An example of a worm is given in Figure 12. Dark vertices in Figure 12 are end vertices of the worm. A triangle of a regular worm or a worm is called an *end triangle* if it contains some end vertices.

Given graph $G = (V, E)$, we say that its maximal cliques can be consecutively order if there is a labeling of its all maximal cliques $C_1, ..., C_m$ such that for any $v \in V$ and $i < j$, if $v \in C_i \cap C_j$, then $v \in C_k$ for any $i \leq k \leq j$.

Theorem 3.20 [Fulkerson and Gross [15]]. *A graph G is an interval graph if and only if its maximal cliques can be consecutively ordered.*

Double star is the graph obtainable from a star $K_{1,m}$ by subdivide each edge by a vertex. It is easy to check that the line graph of a double star is not interval if $m \geq 3$. Since any induced subgraph of an interval graph is still interval, if the line graph $L(H)$ of H is interval, then H has no double star subgraph.

Theorem 3.21. *A line graph $L(H)$ of a simple graph H is interval if and only if each component of H is a worm.*

Proof. Necessity: Without loss of generality, we may assume that H is connected. Since $L(H)$ is interval, Lemma 3.19 implies that each block of H is either a triangle or an edge. If H has at most three vertices, clearly H is a worm. Suppose H has more that 3 vertices and suppose by induction that theorem is true for graphs having less vertices than H.

H must have a cut vertex v. Let $H_1, ..., H_m$, $m \geq 2$, be components of $H - v$. At most two of the H_i's are nontrivial, i.e., have at least one edge, for otherwise H would have a double star subgraph. Suppose that there is only one nontrivial component, say H_1. Then $H' = G(H_1 \cup v)$ is an induced subgraph of H. By induction, H' is a worm. Since there is at least one different vertex w in H_2 djacent to v, if v is not an end vertex of H', then H would have a double star as a subgraph. Therefore, v must be an end vertex of H'. Then H is also a worm since H' is the only nontrivial component of $H - v$.

If there are exactly two nontrivial components H_1, H_2, let $H_1' = G(H_1 \cup v)$ and $H_2' = G(H_2 \cup v)$. By induction, H_1', H_2' are worms. Similar to the case of one nontrivial component, v must be an end vertex of H_1' and H_2'. Furthermore, in at least one of H_1' and H_2', v is an end of a longest path there. For otherwise, $L(H)$ would have an asteroidal triple. Then it follows that H is a worm.

Sufficiency: It is easy to see that we can consecutively label maximal cliques of a worm $L(H)$. Then by Theorem 3.20, $L(H)$ is interval. □

Theorem 3.22 [Roberts [33]]. *Suppose G is a graph. Then the following are equivalent:*

(1) G is a unit interval graph.

(2) G is a proper interval graph.

(3) G is an interval graph and does not contain $K_{1,3}$ as induced subgraph.

(4) G is triangulated and does not contain any graph of Figure 13 as an induced subgraph.

The *underline graph* of a multigraph is the graph obtained from it by removing parallel edges. A multigraph is a *multi-worm* if its underline graph is a worm.

Theorem 3.23. *Given graph G, the following are equivalent:*

(1) G with sufficiently many additional isolated vertices is a unit interval $(*, 2)$-graph;

(2) G with sufficiently many additional isolated vertices is an interval $(*, 2)$-graph;

(3) Each component of G is the line graph of some multi-worm;

(4) G is triangulated and does not contain any graph in Figure 14 as an induced subgraph.

Proof. It is sufficient to consider only connected graphs.

(1) $\Rightarrow$ (2): This is trivial.

(2) $\Rightarrow$ (3): By Theorem 3.21, the line graph $L(H)$ of a simple graph is interval if and only if every component of H is a worm. A multigraph is interval if and only if the line graph of its underlying graph is interval. By Theorem 3.7, G plus sufficiently many additional isolated vertices is an interval $(*, 2)$-graph if and only if H is the line graph of a multi-worm.

(3) $\Rightarrow$ (4): Suppose that G is the line graph of a multi-worm. Then $\theta(N(v)) \leq 2\ \forall v \in V(G)$. G has no G_1 as induced subgraph. The characterization given by Hemminger [20], Bermond and Meyer [5] implies that G does not have G_4 as an induced subgraph. It is easy to check that G_2 and G_3 are not line graphs of multi-worms. Since any induced connected subgraph of the line graph of a worm is still line graph of a worm, so G_2 and G_3 cannot be induced subgraphs for line graphs of worms.

(4) $\Rightarrow$ (1): Suppose that G is a triangulated graph having no induced subgraphs as in Figure 14. Then by Roberts' Theorem 3.22, G is a unit interval graph. The graphs of Figure 11 all contain an induced $C_k, k \geq 4$, except F_1 and F_2. Since G is triangulated, F_3 through F_7 are not induced subgraphs of G. But F_1 and F_2 are G_1 and G_4 in Figure 14. Hence by

Theorem 3.7 G with sufficiently many additional isolated vertices is a $(*, 2)$-graph. □

4. Closing Remarks

(1) The fundamental problem of characterizing interval digraphs remains open.

(2) It is interesting to characterize competition graphs for which their restricted competition numbers equal their competition numbers.

(3) An interesting problem is to characterize acyclic digraphs whose competition graph and resource graph are both interval graphs.

(4) Another problem of interest from the point of view of both ecology and graph theory is to characterize digraphs which have triangulated competition graphs.

(5) Another simpler but interesting problem related to characterizing interval digraphs is to characterize digraphs whose every induced subgraph is interval. Similar questions can be asked for triangulated competition graphs.

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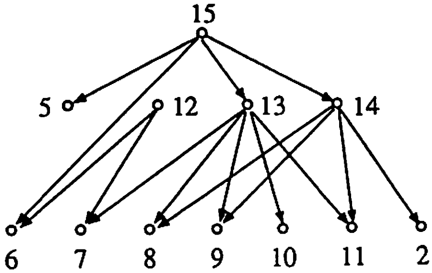
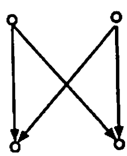


2. Attached Plants
5. Hyporhamphus
6. Mugil
7. Upogebia
8. Lamya
9. Solen
10. Arenicola
11. Hymenosoma
12. Johnius
13. Lithognathus
14. Rhabdosargus
15. Hypacanthus

Figure 1: Digraph for the Knysna esturary community (Day 1967, Sugihara 1982)



P(2,2)

Figure 2: Forbidden structure of irredundancy: $P(2,2)$

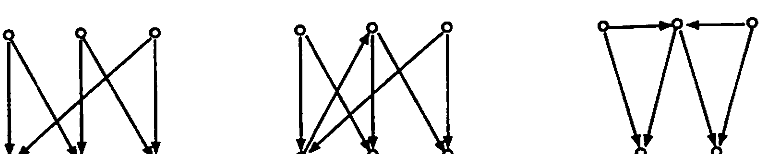


Figure 3: Digraphs in $(2,2)$-interval digraphs

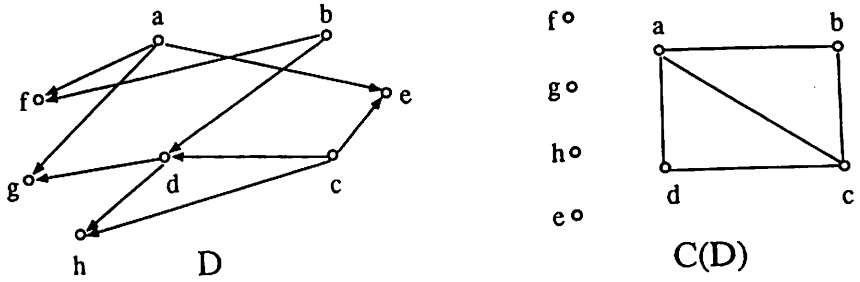


Figure 4: Interval (2, 3)-digraph containing induced non-interval subgraph

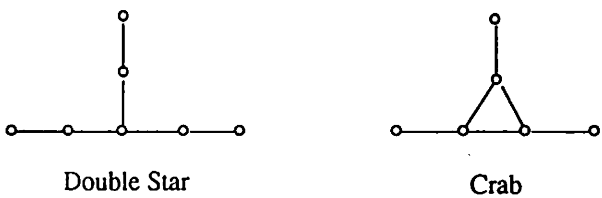


Figure 5: Double star and crab

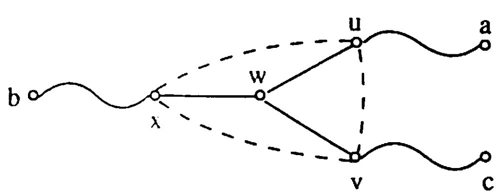


Figure 6: Proof of Theorem 2.13

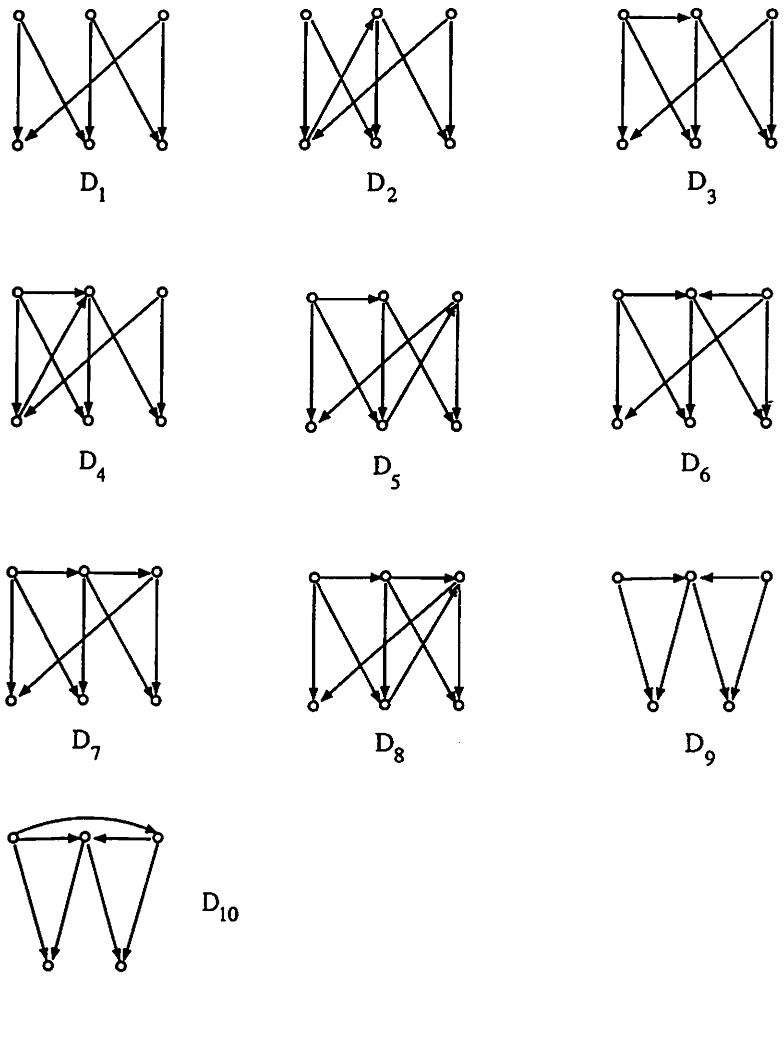


Figure 7: Subgraphs of (2, 3)-digraphs

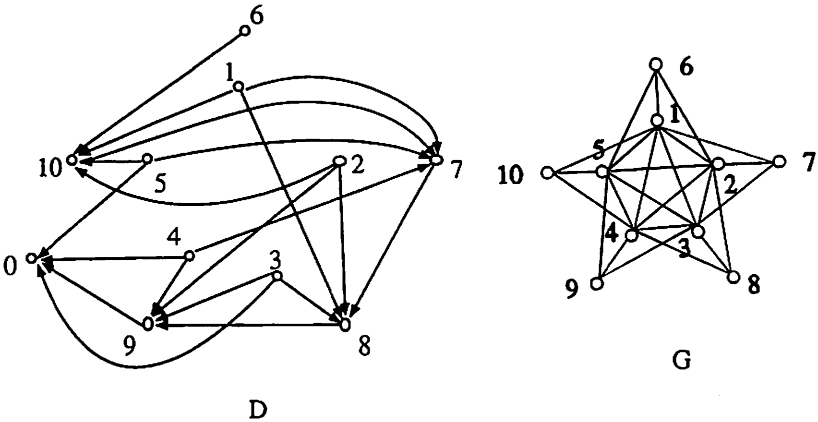


Figure 8

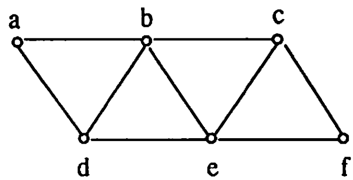


Figure 9

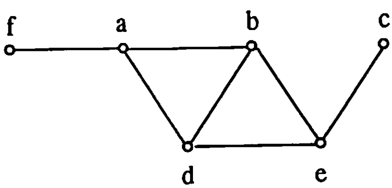


Figure 10

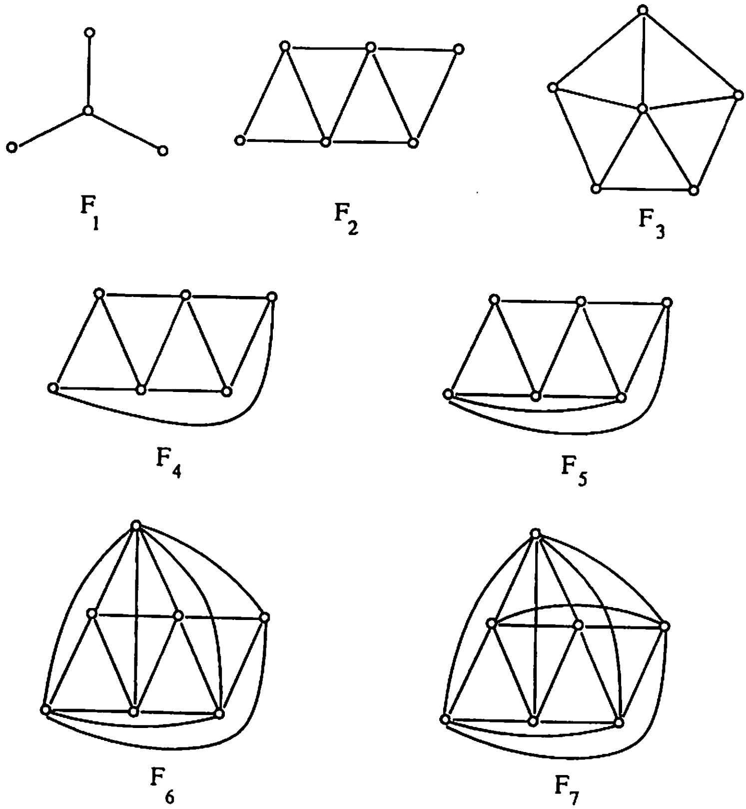


Figure 11

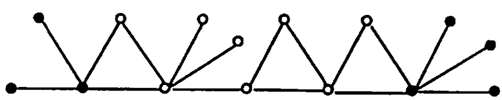


Figure 12

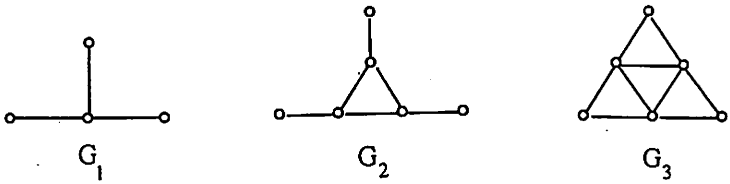


Figure 13

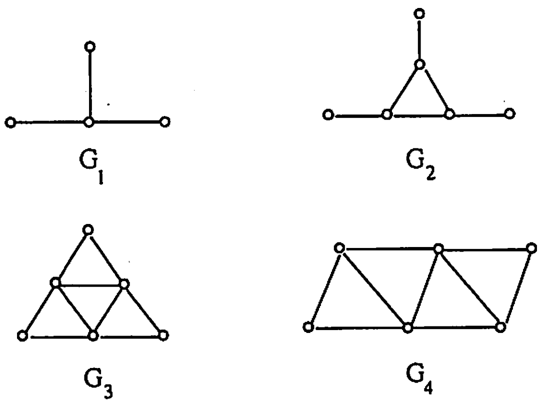


Figure 14