The Irregularity Strength of $m \times n$ Grids for $m, n \geq 18$

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Abstract: Given a graph G with weighting $w : E(G) \to Z^+$, the *strength* of $G(w)$ is the maximum weight on any edge. The *weight* of a vertex in $G(w)$ is the sum of the weights of all its incident edges. The network $G(w)$ is *irregular* if the vertex weights are distinct. The *irregularity strength* of G is the minimum strength of the graph under all irregular weightings. We determine the irregularity strength of the $m \times n$ grid for all $m, n \geq 18$.

1 Introduction

A network $G(w)$ consists of the graph G together with an assignment $w : E(G) \to Z^+$. The *strength* s of $G(w)$ is defined by $s(G(w)) = max\{w(e) : e \in E(G)\}$. For each vertex $v \in V(G)$, define the *weight* $w(v)$ of v in $G(w)$ as the sum of the weights of all edges incident on v. The network $G(w)$ is *irregular* if for all distinct $u, v \in V(G)$, $w(u) \neq w(v)$; there exists an irregular weighting on G only if it contains no K_2 component and at most one isolated vertex.

The *irregularity strength* $I(G)$ is defined to be $min\{s(G(w)) : G(w)$ is irregular$\}$. Thus the irregularity strength of a graph G is the minimum integer t such that G has an irregular weighting with maximum weight t. We call an irregular weighting of G *minimal* if its strength equals $I(G)$.

The study of $I(G)$ was proposed in [1]. In [7] the following lower bound was obtained:

Theorem 1.1 *Let d_k be the number of vertices of degree k in $V(G)$, then*

$$I(G) \geq \lambda(G) = \left\lceil max \left\{ \left(\left(\sum_{k=i}^{j} d_k \right) + i - 1 \right) / j : i \leq j \right\} \right\rceil$$

We call an irregular weighting w of graph G a λ-*weighting* of G if $s(G(w)) = \lambda(G)$.

ARS COMBINATORIA 40(1995), pp. 143-151

There are not many graphs for which the irregularity strength is known. In [1] it was shown that $I(K_n) = 3$ and $I(K_{2n,2n}) = 3$; $I(P_n)$ was also determined. That $I(K_{2n+1,2n+1}) = 4$ was proven in [6]. Work has also been done on dense graphs, cycles, and the disjoint unions of paths ([4, 8]). The irregularity strengths of wheels, k-cubes and $2 \times n$ grids has also been determined [3]. In each of these cases it was found that $I(G) = \lambda(G)$ or $\lambda(G) + 1$. Results on irregularity strengths of graphs are surveyed in [9].

In this paper, we determine the irregularity strength of the $m \times n$ grid $X_{m,n}$ for all $m, n \geq 18$. It is easy to derive the following result from Theorem 1.1.

Proposition 1.2 *If $m, n \geq 3$, and $\{m, n\} \neq \{3, 5\}$, then $\lambda(X_{m,n}) = \lceil (mn+1)/4 \rceil$.*

However, it was shown in [2] that for certain values of m and n, $I(X_{m,n}) > \lambda(X_{m,n})$. We state this result in the following proposition.

Proposition 1.3 *When $m, n \geq 3$, $\{m, n\} \neq \{3, 5\}$, and $mn \equiv 3 \pmod 4$, then $I(X_{m,n}) > \lambda(X_{m,n})$.*

The authors of [2] provided a set of constructions which, given a minimal irregular weighting of a grid (with the proper ingredients), yields minimal irregular weightings for an infinite class of grids. In this paper, we describe a direct construction for minimal irregular weightings of all $X_{16,16+k}$, $k \equiv 0 \pmod 4$. These weightings have the necessary ingredients for the constructions of [2], and the overlapping classes of grids which can be generated cover the set of all $X_{m,n}$ for $m, n \geq 18$. Thus, we will show that, with the exception of the cases specified in Proposition 1.3, $I(X_{m,n}) = \lambda(X_{m,n})$ when $m, n \geq 18$. In the exceptional cases $I(X_{m,n}) = \lambda(X_{m,n}) + 1$.

The next section provides the construction for minimal irregular weightings of $X_{16,16+k}$, $k \equiv 0 \pmod 4$. In Section 3 we show that the weightings from the construction satisfy the constraints in [2], and we prove our main result.

2 λ-weightings of $X_{16,16+k}$, $k \equiv 0 \pmod 4$

We now present the construction which yields λ-weightings of $X_{16,16+k}$ for $k \equiv 0 \pmod 4$.

Theorem 2.1 *There exists a λ-weighting of $X_{16,16+k}$ when $k \equiv 0 \pmod 4$.*

Proof: From Proposition 1.2 it follows that $\lambda(X_{16,16+k}) = 65 + 4k$ when $k \equiv 0 \pmod 4$. Since the minimum vertex degree in a grid is 2, the smallest vertex weight in a grid, under any irregular weighting, is at least 2. Since the maximum vertex degree of a grid is 4, then in a λ-weighting of $X_{16,16+k}$, where $k \equiv 0 \pmod 4$, the maximum vertex weight possible is $260 + 16k$. However, to satisfy the constraints that will be described in Section 3, the three highest vertex weights will not be used. Thus, since $|V(X_{16,16+k})| = 256 + 16k$, each vertex weight from 2 to $257 + 16k$ will be used in the weighting we develop.

We begin by partitioning $X_{16,16+k}$ into 8 concentric cycles $C_i, 1 \leq i \leq 8$, and the sets of edges that connect adjacent cycles; as an exception, we treat the edges bridging the vertices in C_8 as belonging to $E(C_8)$. The vertices in $C_i, 1 \leq i \leq 8$, are all those vertices that are distance $i - 1$ from the border of the grid. For example, C_1 consists of the vertices and edges around the border of the grid. The vertices and edges in C_8 are those that make up the $2 \times (2 + k)$ grid in the center of the graph. The $\{i, i + 1\}$-connectors $(1 \leq i \leq 7)$ are the edges that connect vertices in C_i to vertices in C_{i+1}.

Under the weighting w that we describe in this construction, edge weights are assigned such that if $u \in V(C_i)$, and $v \in V(C_j)$ where $i < j$, then $w(u) < w(v)$. Further, all edges in a given set of connectors have the same weight; thus, if distinct edges e and e' are both $\{i, i + 1\}$-connectors for some i, then $w(e) = w(e')$.

Figure 1 shows the vertex weights for $C_i, 1 \leq i \leq 7$. In cycle i, vertices on the two sides have the $36 - 4i$ smallest weights; of these weights, those equivalent to 2 (mod 4) appear on the vertices on the top half of the left side, those equivalent to 3 (mod 4) appear on the vertices on the bottom half of the left side, those equivalent to 0 (mod 4) appear on the vertices on the top half of the right side, and those equivalent to 1 (mod 4) appear on the vertices on the bottom half of the right side. The next $k + (16 - 2i)$ vertex weights in C_i appear on the vertices along the top of the cycle (exclusive of the vertices in the corners of the cycle); of these weights, the even weights progress from the left side toward the center, and the odd weights progress from the right side toward the center. In a manner similar to the vertices along the top, the vertices with the $k + (16 - 2i)$ greatest vertex weights appear along the bottom of the cycle. Note that Figure 1 shows the weights for all vertices in the outer 7 cycles of $X_{16,16}$ under weighting w. The grid is split down the middle and shows that the vertex weights increase in the same pattern for the k vertices in the middle of the top of each cycle, and the k vertices in the middle of the bottom of each cycle. In each cycle, the vertex with the smallest weight is on the top left corner, and the vertex with the greatest weight is in the center of the bottom.

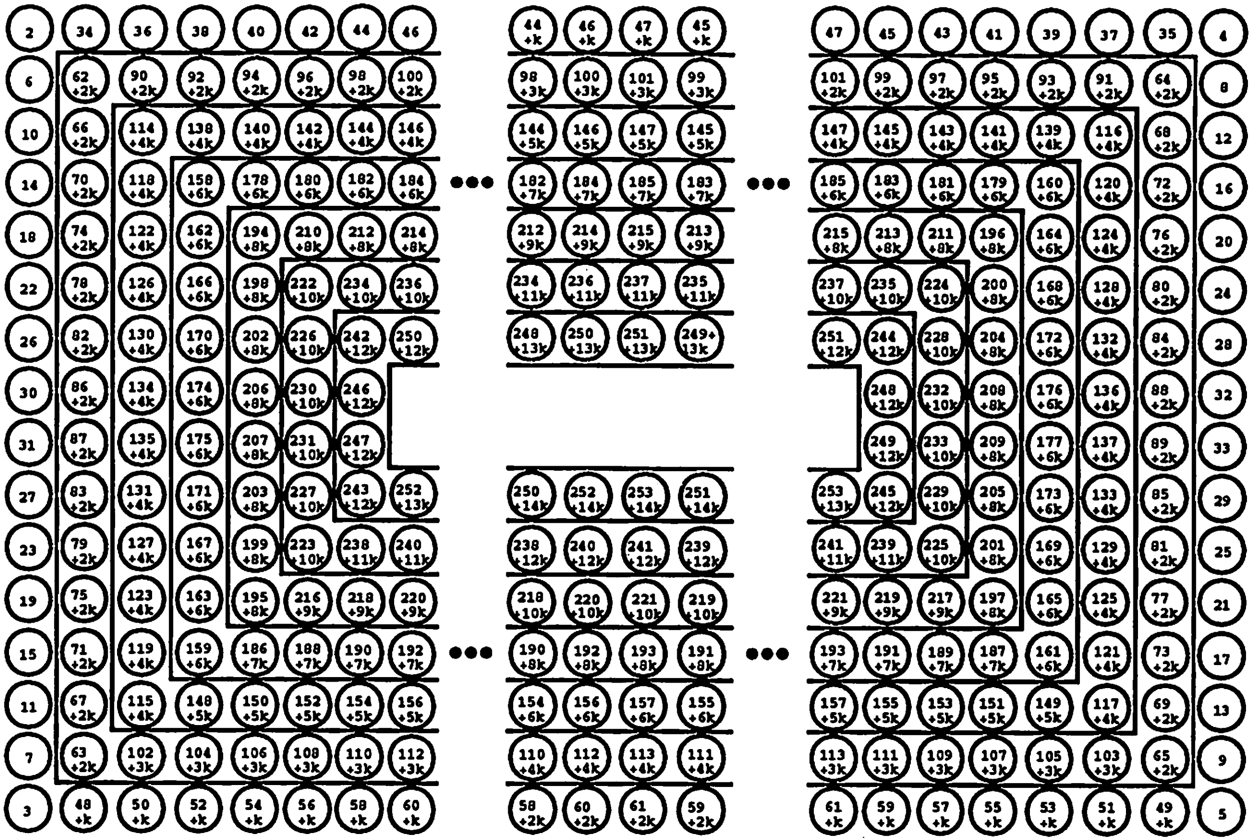

Figure 1: Vertex weights for cycles 1 through 7

Figure 2 shows the edge weights under w for $E(C_i)$ and the $\{i, i+1\}$-connectors, for $1 \leq i \leq 7$. All such edge weights are shown for the weighting of $X_{16,16}$ under w; the figure is split open in the middle to show how the pattern of increasing edge weights continues to the middle of the top (and bottom) of each cycle. Thus, for example, note that along the top of C_1 in $X_{16,16+k}$, the i^{th} edge from the left has weight $i + 27$ when i is even and $i \leq (16+k)/2$. When $k = 0$, the center edge along the top of C_1, which is the eighth edge from the left, has weight 35. When $k > 0$, the center edge along the top of C_1 is an additional $k/2$ edges from the left, and the weight on that edge is $k/2$ greater than the center edge along the top of $X_{16,16}$.

Figure 3 shows the vertex weights for C_8. Note the following difference between C_8 and the other 7 cycles. In C_1 through C_7, the vertex weights that would appear in an irregular minimal weighting of $X_{16,16}$ were on the right and left sides of the figure; the $2k$ additional vertices in each cycle were shown in the middle of the figure. In Figure 3, the four vertices from cycle 8 of $X_{16,16}$ are shown at the center of the figure; the $2k$ additional vertices extend out to either side, with the greatest vertex weights at the extreme left and right ends of C_8.

In Figure 4 we give the weights for $E(C_8)$, the weight that is assigned to all $\{7,8\}$-connectors $(= 65 + 4k)$, and the weights on all edges that are "inside" of C_8; note that all edges of this last type have weight $62 + 4k$, with the exception of one edge at the center which has weight $60 + 4k$.

This completes the construction. It is straightforward to see that the weighting is irregular, and has strength $\lambda(X_{16,16+k}) = 65 + 4k$, when $k \equiv 0 \pmod 4$. ∎

3 The irregularity strength of $X_{m,n}$ for $m, n \geq 18$

The authors of [2] showed that whenever there exists a properly constrained λ-weighting of a grid $G = X_{m,n}$, then there exist irregular weightings of strength λ or $\lambda+1$ for an infinite class of grids. A weighting w of $X_{m,n}$ that satisfies the constraints necessary to begin the recursive constructions in [2] is called *Type 1*. In order to understand the definition of Type 1, we first develop some terminology. The edges in C_1 of $X_{m,n}$ are called *border* edges. The border edges are further described as *left, right, top*, and *bottom* edges. When m and n are both even, and if the border edges along one side are labeled $1, 2, 3, \ldots$ starting from a corner vertex, then the edges with odd labels are termed *heavy edges*.

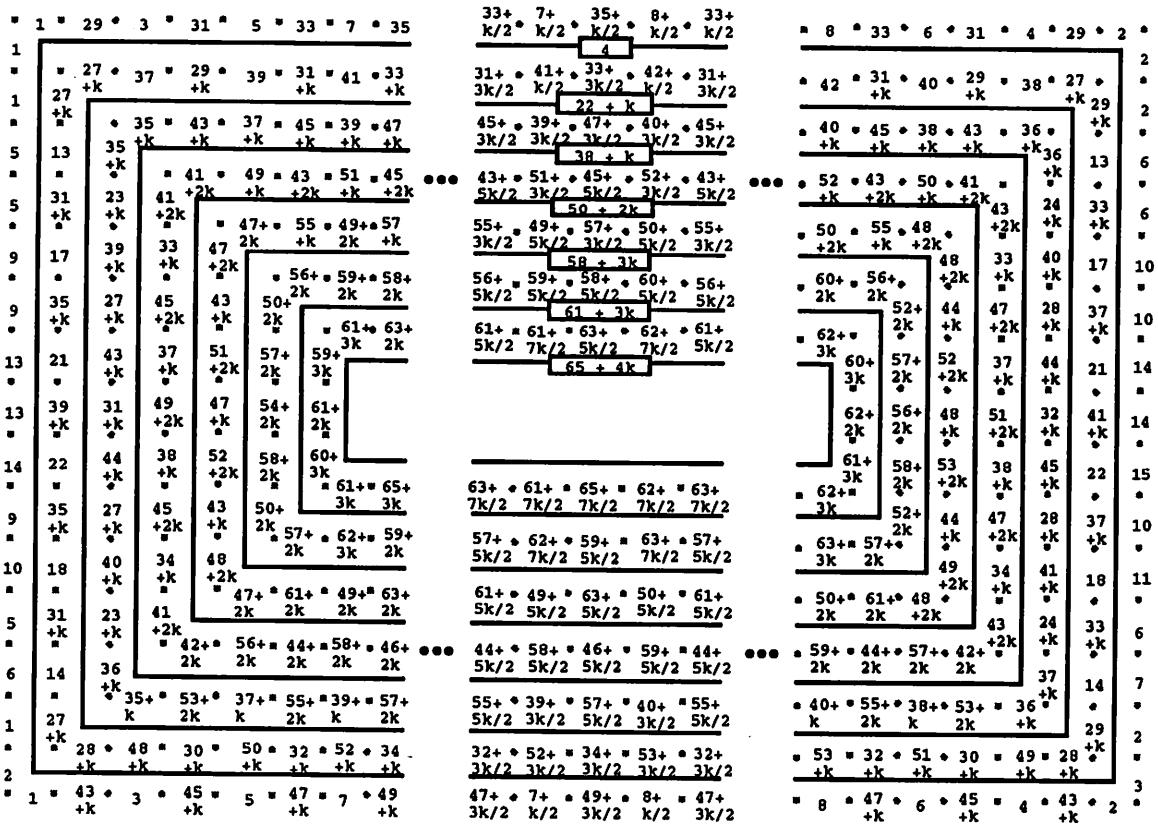

Figure 2: Labelings of the edges in cycles 1 through 7, and of the connecting edges between the cycles

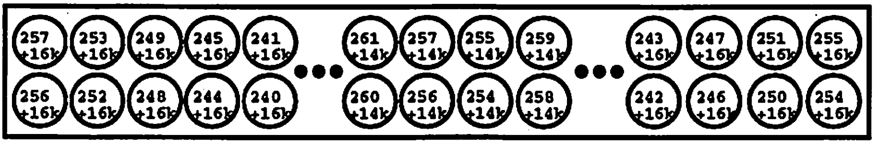

Figure 3: Vertex weights for cycle 8

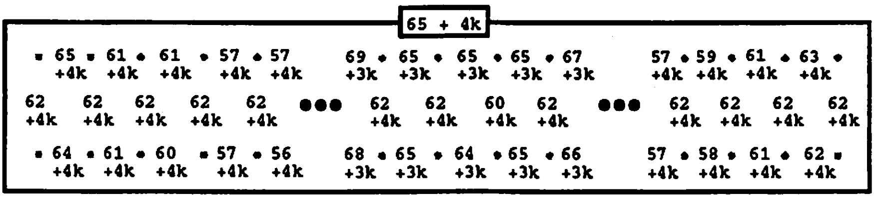

Figure 4: Weights on all edges incident to vertices in cycle 8

The definition of Type 1 given in [2] applied to all $X_{m,n}$ when m and n were even. We constrain the definition of Type 1 weightings to the grids $X_{16,16+k}$ when $k \equiv 0 \pmod 4$; these are the grids we examined in the previous section.

Definition. An irregular weighting w of $X_{16,16+k}$, $k \equiv 0 \pmod 4$ is *Type 1* if for every edge e, $w(e) \le 65 + 4k$ $(=\lambda(X_{16,16+k}))$ and furthermore,

1. If e is a top or bottom heavy edge, then $w(e) \le 39 + 7k/2$;
2. If e is a left heavy edge, then $w(e) \le 55 + 9k/2$;
3. If e is a right heavy edge, then $w(e) \le 55 + 7k/2$; and
4. There is a one-factor f in $X_{16,16+k}$ which has the property that for all $e \in f$, $w(e)$ is less than or equal to the previous constraints minus three. Thus,
   (a) If e is internal or a non-heavy border edge, then $w(e) \le 62 + 4k$;
   (b) If e is a top or bottom heavy edge, then $w(e) \le 36 + 7k/2$;
   (c) If e is a left heavy edge, then $w(e) \le 52 + 9k/2$;
   (d) If e is a heavy right edge, then $w(e) \le 52 + 7k/2$.

Theorem 3.1 *There is a Type 1 weighting of $X_{16,16+k}$ for $k \equiv 0 \pmod 4$.*

Proof: One can check that the first three constraints for Type 1 are met by the construction in Theorem 2.1. We now show the existence of a 1-factor f which meets the tighter constraints of part 4 in the above definition. First we label the vertices in $V(X_{16,16+k}), k \equiv 0 \pmod 4$ as $v_{i,j}$, $1 \leq i \leq 16$ and $1 \leq j \leq 16+k$. The vertex $v_{1,1}$ is in the upper left corner of the grid, and $v_{16,16+k}$ is the vertex in the lower right corner. We now identify the 1-factor f which has the appropriate constraints.

$$\begin{aligned} f = \ & \{(v_{i,j}, v_{i+1,j}) | i \text{ is even}, 2 \leq i \leq 14, 1 \leq j \leq 16+k\} \bigcup \\ & \{(v_{i,j}, v_{i,j+1}) | i \in \{1, 16\}, j \text{ is odd}, 1 \leq j \leq 15+k\} \end{aligned}$$

The first set of edges in f includes all edges connecting a vertex in an even numbered row to its neighbor in the row below it. The second set of edges includes the alternate edges (starting with the first) in the first and last rows. ∎

Beginning with any Type 1 weighting of $X_{m,n}$ where $m, n \geq 16$, the constructions in [2] yield irregular weightings for all $X_{m',n'}$ where $m' = m+a$, $n' = n+b$, $a, b \geq 0$, and $2\lfloor (a+2)/4 \rfloor \leq b \leq 4\lfloor a/2 \rfloor$. The set of pairs (m', n') satisfying these constraints is called the *feasible region* of m and n: $\mathcal{F}(m,n)$. This is the infinite region in the first quadrant of the plane that is approximately bounded by the two lines through (m,n) with slopes 1/2 and 2. The weightings of $X_{m'n'}$ have strength $\lambda(X_{m'n'})$ when $m'n' \not\equiv 3 \pmod 4$, and strength $\lambda(X_{m'n'})+1$ when $m'n' \equiv 3 \pmod 4$. Thus, from the definition of Type 1, the set of constructions, and Proposition 1.3, [2] provides the following result.

Theorem 3.2 *If there is a Type 1 weighting of $X(m,n)$, where $m, n \geq 16$, then for all m' and n' such that $(m', n') \in \mathcal{F}(m,n)$,*

$$I(X_{m',n'}) = \begin{cases} \lambda(X_{m',n'}) & \text{if } m'n' \not\equiv 3 \pmod 4 \\ \lambda(X_{m',n'})+1 & \text{if } m'n' \equiv 3 \pmod 4 \end{cases}$$

We can now present the principal result of this paper.

Theorem 3.3 *For all $m, n \geq 18$,*

$$I(X_{m,n}) = \begin{cases} \lambda(X_{m,n}) & \text{if } mn \not\equiv 3 \pmod 4 \\ \lambda(X_{m,n})+1 & \text{if } mn \equiv 3 \pmod 4 \end{cases}$$

Proof: Assume that $m \leq n$. Note that $(m,n) \in \mathcal{F}(16, 16+k)$, where

$$k = \begin{cases} 4\lfloor (n-m)/4 \rfloor & \text{if } m \neq 19 \\ 4\lfloor (n-m+1)/4 \rfloor & \text{if } m = 19 \end{cases}$$

Since $k \geq 0$ and $k \equiv 0 \pmod 4$, then by Theorem 3.1 there exists a Type 1 weighting of $X_{16,16+k}$. Thus, by Theorem 3.2, there exists an irregular weighting of $X_{m,n}$ with the strength specified in the theorem. When $m > n$, a minimal irregular weighting of $X_{m,n}$ is produced by taking the transpose of a minimal irregular weighting of $X_{n,m}$. ∎

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