ON THE FACE VECTORS OF ARRANGEMENTS OF CURVES

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ABSTRACT. We consider the realizations of a sequence $(p_3^*, p_5^*, p_6^*, \dots)$ of nonnegative integers satisfying the equation $\sum_{k\geq 3}(k-4)p_k + 8 = 0$ as an arrangement of simple curves defined by B. Grünbaum [4]. In this paper, we show that an Eberhard-type theorem for a digon-free arrangement of simple curves is not valid in general, while some sequences are realizable as a digon-free arrangement of simple curves.

1. Introduction.

B. Grünbaum extended the notion of "an arrangement of lines" to "an arrangement of curves" in his book [4], and obtained many results. According to his definition, an arrangement of simple curves is a family of simple curves such that every two simple curves meet at precisely two points.

As we know, two curves can meet at a point in two different ways. That is, they either cross or osculate each other. A point is called an **intersection point** (a **kissing point** or an **osculation point**, respectively) if two curves cross (osculate, respectively) at this point. However, in this paper, we shall assume that every meeting point of curves is an intersection point unless stated otherwise. The terms "curves" and "arrangement" stand for "simple closed curves" and "arrangement of simple closed curves", respectively, and the terms "points" and "vertices" will be used interchangeably throughout this paper.

B. Grünbaum's definition allows that three or more curves may intersect at the same point, but we assume that only two curves intersect at the same point to keep the arrangements as 4-valent graphs. Thus, in this paper, **an arrangement of curves** $\mathcal{A} = \{C_1, \cdots, C_n\}$ in the Euclidean plane $\mathbb{E}^2$ is a finite family of n simple closed curves with the properties:

(1) Every pair of curves has 2 points in common.
(2) Exactly two curves meet at each point (or vertex).

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The graph of an arrangement of curves $\mathcal{A}$, denoted by $G_{\mathcal{A}}$, is a plane 4-valent graph whose vertices are intersection points and whose edges are the segments of curves between each pair of adjacent points (perhaps, with multiple edges). Let G be a 4-valent plane graph and let $p_k(G)$ (or p_k) denote the number of faces with k sides in G for all positive integer k. We call a sequence $(p_1, p_2, p_3, p_4, p_5, \dots)$ a **p-vector** (or **aface vector**) of G.

Let $\mathcal{A}$ is an arrangement, then $p_1(G_{\mathcal{A}}) = 0$ because of the definition of the arrangement. An arrangement $\mathcal{A}$ is called a **digon-free arrangement** if $p_2(G_{\mathcal{A}}) = 0$. In this case, we have

$$(*) \qquad p_3 = 8 + \sum_{k \geq 4} (k-4) \cdot p_k$$

from the well-known Euler's formula. Notice that the coefficient of p_4 is zero in the equation $(*)$.

A sequence $(p_3^*, p_5^*, p_6^*, \dots)$ of nonnegative integers satisfying the equation $(*)$ is said to be **realizable as a digon-free arrangement** if there is a nonnegative integer p_4^* and an arrangement $\mathcal{A}$ such that $p_k(G_{\mathcal{A}}) = p_k^*$ for all k, $3 \leq k \leq n$, and $p_k(G_{\mathcal{A}}) = 0$ for all $k > n$.

Let $\mathcal{A}$ be an arrangement of n simple curves. Then, each curve in $\mathcal{A}$ contains the same number of vertices (or edges) of $G_{\mathcal{A}}$. Let r be the number of edges (or vertices) in a simple curve, then $r = 2(n-1)$.

Now, let us state our main results which concern about an Eberhard-type question for an arrangement of curves raised by B. Grünbaum [4] (for the Eberhard Problem, see [2], and for the analogues of Eberhard's Theorem, see [3], [6], [8], [9] and [10]).

Theorem A. *A sequence $(p_3^*, p_5^*, p_6^*, \dots)$ of nonnegative integers such that $p_k^* = 1$ for an integer $k > 4$, $p_3^* = k + 4$, and $p_i^* = 0$ for all $i \neq 3, 4$, and k is not realizable as a digon-free arrangement.*

Theorem B. *A sequence $(p_3^*, p_5^*, p_6^*, \dots)$ of nonnegative integers satisfying the equation $(*)$ where p_k^* is even for all k is realizable as a digon-free arrangement.*

An arrangement of 4 curves in Figure 1 shows that the condition "digon-free" in Theorem A is necessary. An easy modification shows the necessity of the condition for all $k \geq 5$.

2. The proof of Theorem A.

For the proof of Theorem A, we need several lemmas and theorems.

Theorem 2.1 (W. Meyer [7]). *Let $\mathcal{A}$ be a digon-free arrangement of n simple curves, and let k be the largest integer such that $p_k(G_{\mathcal{A}}) \neq 0$. Then, $k \leq 2n - 4$ for all $n \geq 4$.*

Corollary 2.2. *Every face of the graph of a digon-free arrangement of 4 curves is either a triangle or a 4-gon.*

Proof. By Theorem 2.1, the largest face of the graph of any digon-free arrangement of 4 simple curves is a 4-gon. In fact, there are 8 triangles and six 4-gons in such an arrangement.

Lemma 2.3. *For a given sequence $(p_3^*, p_5^*, p_6^*, \ldots)$ of nonnegative integers such that $p_3^* = 8$ and $p_i^* = 0$ for all $i > 3$, there is only one digon-free arrangement of 3 curves (up to isomorphism) that realizes the given sequence if we do not allow the addition of 4-gons (observe that the above sequence satisfies the equation $(*)$).*

Proof. Let $\mathcal{A}$ be a digon-free arrangement of n curves. Clearly, $G_{\mathcal{A}}$ is a plane 4-valent graph. Thus,

$$2 \cdot e = 4 \cdot v, \text{ i.e.,} \quad e = 2 \cdot v,$$

where e and v are the number of edges and the number of vertices of $G_{\mathcal{A}}$, respectively. Moreover, $e = n \cdot r$ because every curve has r edges. Combining these equalities with Euler formula, we have the fact that the number of faces $f(G_{\mathcal{A}})$ in the graph $G_{\mathcal{A}}$ is $2 + \frac{1}{2}(n \cdot r)$. Since we have only 8 triangles, $nr = 12$. Hence, $n = 2, r = 6$ or $n = 3, r = 4$ because n cannot be larger than r. However, only the case "$n = 3, r = 4$" satisfies the equality $r = 2(n-1)$. The arrangement of 3 curves is unique (up to isomorphism) because there is only one arrangement of 2 curves that has 4 digons as its faces, and the third curve must cut all 4 edges in the arrangement of 2 curves to make it a digon-free arrangement of 3 curves. This completes the proof.

Lemma 2.4. *Let G be a plane graph and let F be a face of G. Then there is a plane graph G', isomorphic to G, such that the face of G' corresponding to F is the infinite face.*

Proof. Since G is plane, it can be embedded on the sphere (see J.A. Bondy [1], p138). Rotate the sphere in order to place the north pole inside the region corresponding to F. Now, embed the graph on the sphere into the plane by using the stereographic projection.

Now, we are ready to prove Theorem A which shows that an Eberhard-type theorem is not valid for the digon-free arrangements of curves.

We will use induction on m, the number of curves. Let G be the graph of a digon-free arrangement of m curves. If $m = 3$, then all the faces of G are triangles, by Lemma 2.3. If $m = 4$, then G has eight triangles and six 4-gons, by Corollary 2.1. Thus, our theorem is true for the case $m = 3$ and $m = 4$.

Assume that our theorem is true for all digon-free arrangements of n curves where $n < m$. That is, no digon-free arrangement of $n(< m)$ curves has only one k-gon ($k > 4$), $k + 4$ triangles and possibly some 4-gons.

Now, suppose that there exists a graph of a digon-free arrangement of m curves that has one k-gon ($k \geq 5$), $k + 4$ triangles, and some 4-gons as its faces. By Theorem 2.4, there exists a graph G', isomorphic to G, whose infinite face is the k-gon and all other faces are triangles and 4-gons (Figure 2 (a)).

Claim 1: No two sides of the infinite face of G' are on the same curve.

Otherwise, there is a curve that contains at least two edges of the infinite face. Suppose that C_1 is such a curve that contains at least two sides of the infinite face. Let e_1 be one of the edges of the infinite face contained in the curve C_1. Assign the labels $e_2, e_3, \ldots, e_k$ to all the other edges of the infinite face counterclockwise (Figure 2 (b)). Let e_i and $e_j, 2 < i \leq j < k$, be two edges of the curve C_1 such that no edges between e_1 and e_i, and no edges between e_j and e_1 can be an edge of the curve C_1 (note that i and j may be equal). Since two curves intersect each other only at two points, two edges e_p and e_q where $2 \leq p < i \leq j < q \leq k$, cannot lie on the same curve (otherwise, such a curve would intersect C_1 at least 4 times). Let C_2(C_3, respectively) be a curve that contains the edge e_2(e_k, respectively). Then, the two curves C_2 and C_3 must meet each other inside the curve C_1 (Figure 2 (b)). Since the two curves C_2 and C_3 create a digon, there must be a curve that cuts through such a digon to make this arrangement digon-free, and this curve must cut through the segments of the curve C_1 from v_1 and v_2 as well as from v_3 and v_4 to avoid making t-gons, where $t > 4$ (dotted line in Figure 2 (b)). Then, the curve C_1 and this curve have at least 4 intersection points in common. This violates the first condition of the arrangement. In other words, every edge of the infinite face must be contained in a different curve; that is, $m \geq k > 4$.

Claim 2: In G', there is no configuration of three triangles as in Figure 3.

Assume the contrary, i.e., assume that there is such a configuration of three triangles as in Figure 3. Since $m \geq k > 4$ and $r = 2(m-1)$, r cannot be less than 8. Thus, the face F in Figure 3 that is adjacent to two triangles is neither a triangle nor a 4-gon. In fact, if the face F were a triangle, then $m = 3, r = 4 < 8$. If the face F were a 4-gon, then the curve C is not simple. Hence, the face F must be the infinite face of G'. This implies that the infinite face contains two edges e and e' which are on the same curve (Figure 3), and this contradicts to the Claim 1. Thus, Claim 2 is true.

Claim 3: There is at least one curve C such that the arrangement of $m - 1$ curves after removing the curve C from G' is also digon-free.

Suppose that removing any curve in this arrangement creates a digon. This means that every curve cuts through at least one digon. By Claim 2, we can find at least k edge-disjoint digons, one for each curve. Thus, we can assign two triangles to each curve, so the number of triangles in this digon-free arrangement is not less than $2k$. On the other hand, the number of triangles $p_3 = k + 4$. Therefore, $k + 4 > 2k$. But, this is absurd because $k \geq 5$. Thus, Claim 3 is true.

Now, let's complete the proof. By Claim 3, we can find a curve C such that the arrangement of $m - 1$ curves, obtained from the arrangement of m curves by removing the curve C, is still a digon-free arrangement. Then, the infinite face becomes a k-gon or a $(k - 1)$-gon, depending on the face F' (see Figure 4 (a) and (b)). Hence, we have a digon-free arrangement of $m - 1$ curves with only one k-gon or one $(k - 1)$-gon, as well as some triangles and some 4-gons as its faces. This violates our induction hypothesis. Therefore, there is no digon-free arrangement of m curves that realizes the given sequence.

3. The proof of Theorem B.

For the proof, let us introduce an operation called "**adding a curve**" for the arrangement of curves. The first step of the operation is to "draw a parallel curve". Let $\mathcal{A}$ be an arrangement of n curves. Select a curve C in $\mathcal{A}$, and draw a new curve C' inside the curve C such that C' is parallel to the curve C and no vertex is between two curves C' and C. Clearly, we can apply this operation to the curve C from outside. Observe that this process only creates exactly $2(n - 1)$ 4-gons since there are $2(n - 1)$ edges on the curve C. The second step of this operation is to move a part of the new curve C' over the curve C to create two intersection points of C' and C. Figure 5 illustrates the application of the operation "adding a curve", to the digon-free arrangement of 3 curves. Hence, if we apply this operation to an (digon-free, resp.) arrangement of n curves, then the result is also an (digon-free, resp.) arrangement of $n + 1$ curves. This yields the following fact.

Lemma 3.1. *Let $\mathcal{A}$ be an (digon-free, resp.) arrangement of n curves, then the result of the application of the operation "adding a curve" is an (digon-free, resp.) arrangement of $n + 1$ curves.*

Now, let's consider the change of the face structure due to the operation, "adding a curve". As we see in Figure 5, the number of sides of the four faces are increased by one while four triangles and some 4-gons are created. Let $F_i, i = 1, 2, 3, 4$ in Figure 6 (a) be the four faces to which the second step of the "adding a curve" is applied. Figure 6 (b) shows the change of the face structure after the application of the operation where C_1 is a new curve. We are also able to apply the same operation to the curve

C_1 again, which increases the number of sides of the faces, F_1 and F_2, by one once more. Therefore, if the faces, $F_i, i = 1, 2$, are k-gons, and $F_i, i = 3, 4$, are triangles, then we can change the faces $F_i, i = 1, 2$ to $(k+m)$-gons by applying the operation "adding a curve" m times consecutively. Observe that only several triangles and some 4-gons are created due to this application. Moreover, in the final configuration, we can locate the two 4-gons and two triangles (see Figure 6 (c)). Thus, we are able to change two 4-gons to another two k-gons by repeating the same process.

To construct a digon-free arrangement which realizes the given sequence in Theorem B, we will use the operation "adding a curve" as follows. The base configuration of our construction is the digon-free arrangement of 3 curves. Let k be the largest integer such that $p_k \neq 0$ in the given sequence. Select a curve and apply the operation "adding a curve" consecutively to construct two k-gons. After making two k-gons, we can locate the configuration of two triangles and two 4-gons as in Figure 6 (c). Hence we can start the application of the operation "adding a curve" again to change these two 4-gons to another two k-gons. Using this method, we are able to have a digon-free arrangement of curves that contains p_k k-gons, some triangles, and some 4-gons. For another i-gons such that $p_i \neq 0$, repeat the above process. Finally, we can have a digon-free arrangement realizing the given sequence. This completes the proof.

4. Other results.

In the arrangement of curves, two curves have exactly two common points. However, J. Malkevitch generalized the definition of an arrangement of curves by allowing that every pair of curves can have more than 2 common points (the common point is either an intersection point or an osculation point).

A generalized arrangement of curves $\mathcal{A} = \{C_1, \cdots, C_n\}$ in the Euclidean plane $\mathbb{E}^2$ is a finite family of n simple closed curves with the properties:

(1) Every pair of curves has exactly t intersection points (t is even) and k kissing points in common,
(2) Exactly two curves meet at each point (or vertex),

Clearly, every curve has the same number of edges. Let r be the number of edges in a curve. Then, $r = (n-1)(t+k)$. We will denote a generalized arrangement of curves by an $(\mathbf{n}, \mathbf{r}, \mathbf{t}, \mathbf{k})$- arrangement of curves. If there is no digons in the graph of an (n, r, t, k)-arrangement, then we call it a **digon-free** $(\mathbf{n}, \mathbf{r}, \mathbf{t}, \mathbf{k})$**-arrangement**.

Among the several results on the generalized arrangements, we'd like to state some of them without proof. For the proof and other results, see D. Y. Jeong [5]. One of them shows the relationship between the p-vectors and the values n, r, t, and k of the arrangement.

Theorem 4.1. *Suppose that $(p_3, p_4, p_5, p_6, \dots)$ be the p-vector of a digon-free (n, r, t, k)-arrangement. Then,*

$$\sum_{i \geq 4} (i-3) \cdot p_i = \frac{n(n-1)(t+k)}{2} - 6$$

We also have a theorem that is similar to Theorem B.

Theorem 4.2. *A sequence $(p_3^*, p_5^*, p_6^*, \dots)$ of nonnegative integers satisfying the equation $(*)$, where p_k^* is even for all $k \equiv 0 \pmod 3$ and $p_k^* = 0$ otherwise, is realizable as a digon-free $(3, r, t, 0)$-arrangement. In this case, $t = 2 + \sum_{i \geq 2} (i-1) p_{3i}$ and $r = 2t$.*

Finally, we may extend Theorem A to the generalized arrangement as follows:

Conjecture. *For a given sequence $(p_3^*, p_5^*, p_6^*, \dots)$ of nonnegative integers satisfying the equation $(*)$ where $p_k = 1$ for an integer $k > 4, p_i = 0$ for all $i \neq k$ and 3, and $p_3 = k + 4$, there is no digon-free $(n, r, t, 0)$-arrangement with $t > 2$, which realizes the given sequence.*

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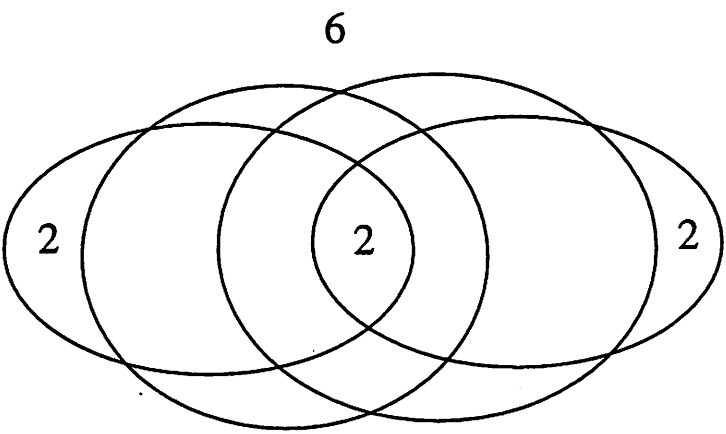
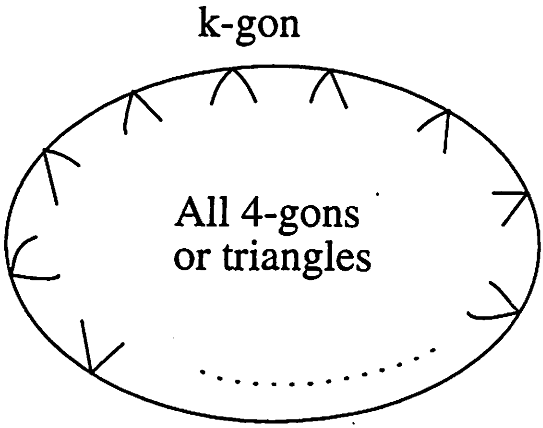
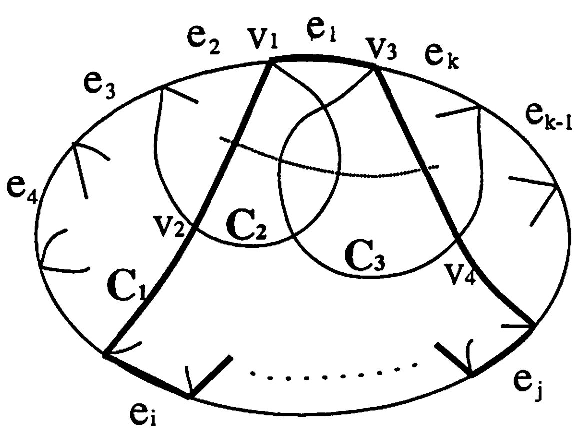


Figure 1



k-gon
All 4-gons
or triangles

(a)



e2
v1
e1
v3
ek
e3
ek-1
e4
v2
C2
C3
C1
v4
ei
ej

(b)

Figure 2

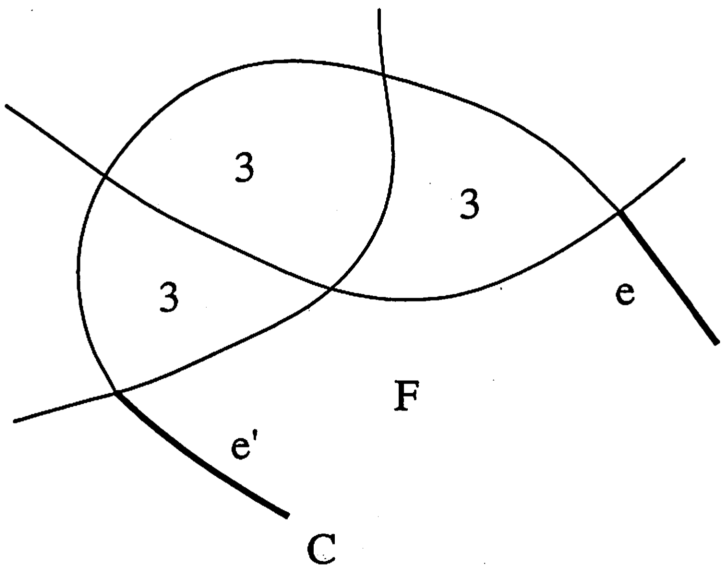
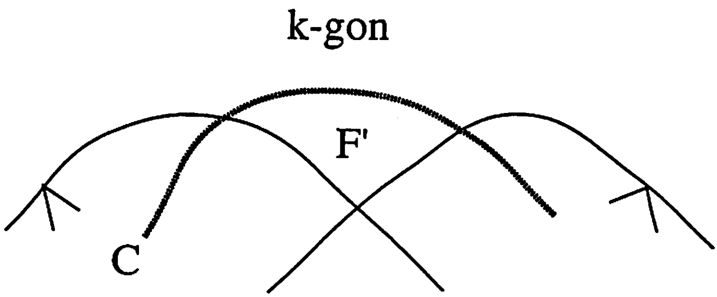
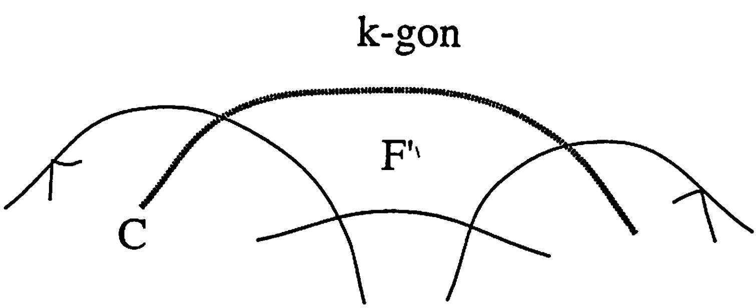


Figure 3



(a) k-gon to (k-1)-gon



(b) k-gon to k-gon

Figure 4

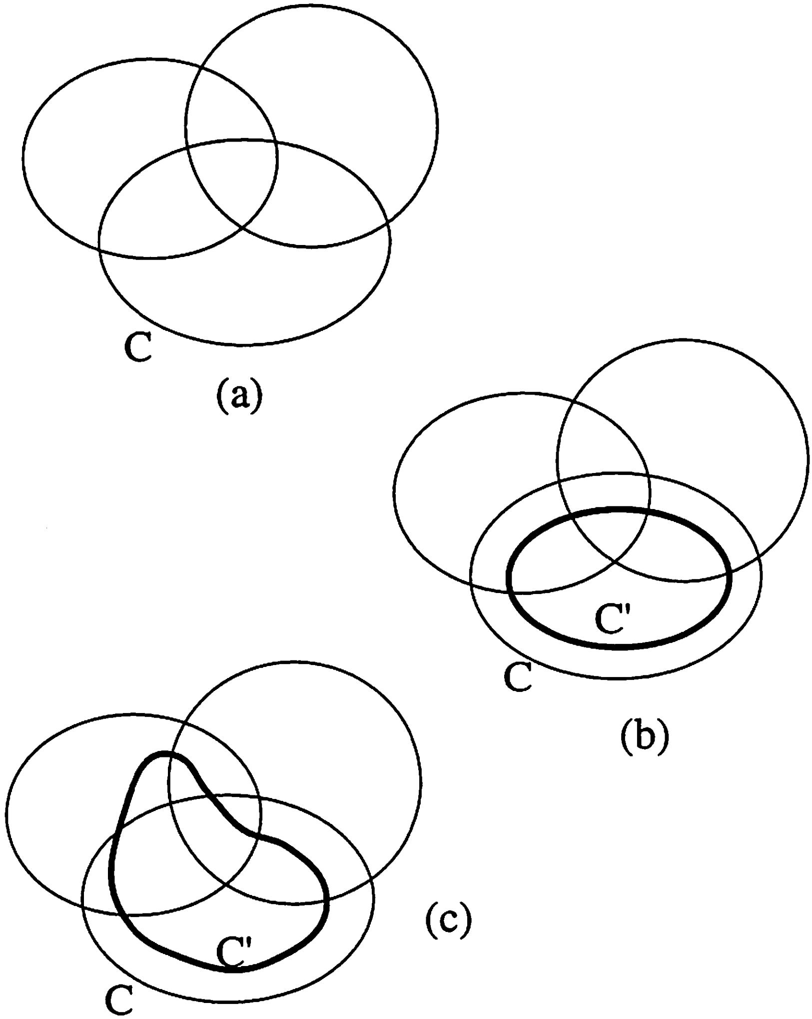


Figure 5

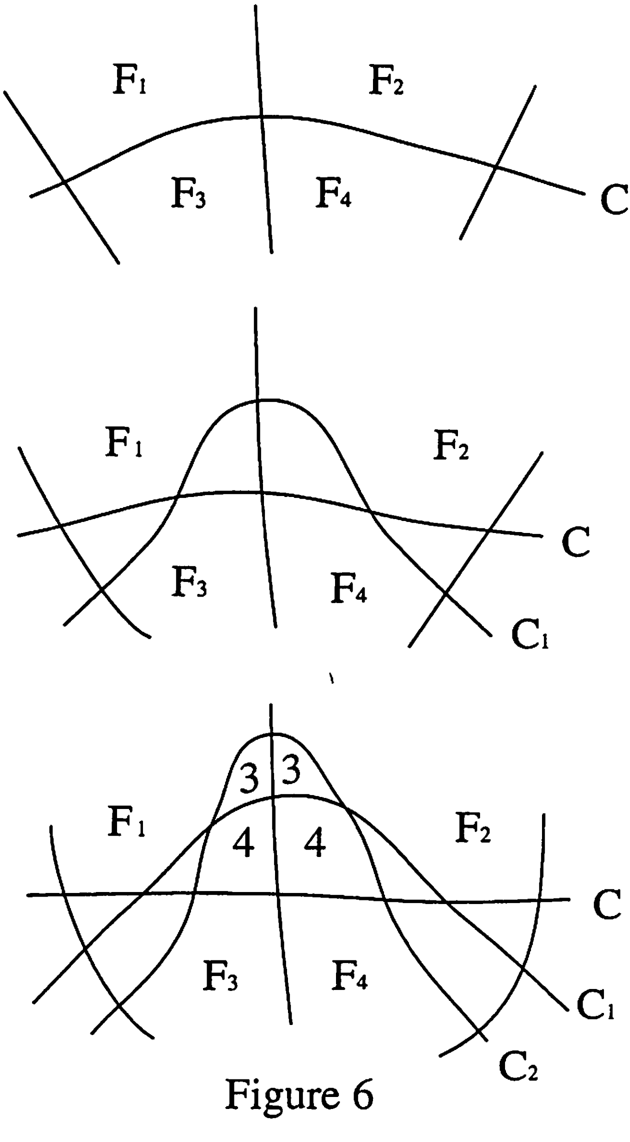


Figure 6