# Some Properties of Nonnegative Integral Matrices *

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ABSTRACT. Recently, M. Lewin proved a property of the sum of squares of row sums and column sums of an $n \times n$ $(0,1)$-matrix, which has more 1's than 0's in the entries. In this article we generalize Lewin's Theorem in several aspects. Our results are: (1) for $m \times n$ matrices, where $m$ and $n$ can be different, (2) for nonnegative integral matrices as well as $(0,1)$-matrices, (3) for the sum of any positive powers of row sums and column sums, and (4) for any distributions of values in the matrix. In addition, we also characterize the boundary cases.

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## 1 Introduction

Let $A = (a_{ij})$ be an $m \times n$ matrix, and let

$$r_i = \sum_{j=1}^{n} a_{ij}, \qquad 1 \le i \le m,$$

$$s_j = \sum_{i=1}^{m} a_{ij}, \qquad 1 \le j \le n,$$

and

$$\sigma = \sum_{1 \le i \le m, 1 \le j \le n} a_{ij}.$$

Recently, M. Lewin [3] proved the following result.

**Theorem 1.** *Let $A$ be a square $(0,1)$-matrix of order $n$. If*

$$\sigma = \left\lfloor \frac{n^2}{2} \right\rfloor, \tag{1.1}$$

*then*

$$\sum_{i=1}^{n} (r_i^2 + s_i^2) \ge n\sigma, \tag{1.2}$$

*and if*

$$\sigma > \left\lfloor \frac{n^2}{2} \right\rfloor, \tag{1.3}$$

*then*

$$\sum_{i=1}^{n} (r_i^2 + s_i^2) > n\sigma, \tag{1.4}$$

In this article we generalize Lewin's Theorem in several aspects. Our results are: (1) for $m \times n$ matrices, where $m$ and $n$ can be different, (2) for nonnegative integral matrices as well as $(0,1)$-matrices, (3) for the sum of any positive powers of row sums and column sums, and (4) for any distributions of values in the matrix. In addition, we also characterize the boundary cases.

In the following $\Phi_{m\times n}(\sigma)$ denotes the set of $m \times n$ nonnegative integral matrices with $\sigma$ being the sum of its entries, and $\Psi_{m\times n}(\sigma)$ denotes the set of $m \times n$ $(0,1)$-matrices with $\sigma$ being the sum of its entries. Let $\Phi_n(\sigma) = \Phi_{n\times n}(\sigma)$ and $\Psi_n(\sigma) = \Psi_{n\times n}(\sigma)$. Then our main results can be stated as follows.

**Theorem 2.** *Let $\sigma$ be a positive integer and $p$ be a nonnegative integer. Then*

$$\min_{A\in\Phi_{m\times n}(\sigma)}\sum_{i=1}^{m} r_i^p = \left(\sigma - m\left\lfloor\frac{\sigma}{m}\right\rfloor\right)\left(\left\lfloor\frac{\sigma}{m}\right\rfloor+1\right)^p + \left(m\left(\left\lfloor\frac{\sigma}{m}\right\rfloor+1\right)-\sigma\right)\left(\left\lfloor\frac{\sigma}{m}\right\rfloor\right)^p \tag{1.5}$$

*and*

$$\min_{A\in\Phi_{m\times n}(\sigma)}\sum_{i=1}^{n} s_i^p = \left(\sigma - n\left\lfloor\frac{\sigma}{n}\right\rfloor\right)\left(\left\lfloor\frac{\sigma}{n}\right\rfloor+1\right)^p + \left(n\left(\left\lfloor\frac{\sigma}{n}\right\rfloor+1\right)-\sigma\right)\left(\left\lfloor\frac{\sigma}{n}\right\rfloor\right)^p \tag{1.6}$$

Moreover, there is a matrix in $\Phi_{m\times n}(\sigma)$ such that the minima in (1.5) and (1.6) are simultaneously realized.

**Theorem 3.** *Let $\sigma \le mn$ be a positive integer and $p$ be a nonnegative integer. Then*

$$\min_{A\in\Psi_{m\times n}(\sigma)}\sum_{i=1}^{m} r_i^p = \left(\sigma - m\left\lfloor\frac{\sigma}{m}\right\rfloor\right)\left(\left\lfloor\frac{\sigma}{m}\right\rfloor+1\right)^p + \left(m\left(\left\lfloor\frac{\sigma}{m}\right\rfloor+1\right)-\sigma\right)\left(\left\lfloor\frac{\sigma}{m}\right\rfloor\right)^p \tag{1.7}$$

*and*

$$\min_{A\in\Psi_{m\times n}(\sigma)}\sum_{i=1}^{n} s_i^p = \left(\sigma - n\left\lfloor\frac{\sigma}{n}\right\rfloor\right)\left(\left\lfloor\frac{\sigma}{n}\right\rfloor+1\right)^p + \left(n\left(\left\lfloor\frac{\sigma}{n}\right\rfloor+1\right)-\sigma\right)\left(\left\lfloor\frac{\sigma}{n}\right\rfloor\right)^p \tag{1.8}$$

Moreover, there is a matrix in $\Psi_{m\times n}(\sigma)$ such that the minima in (1.7) and (1.8) are simultaneously realized.

**Theorem 4.** *Let $A \in \Phi_{m\times n}(\sigma)$ and $p$ be a nonnegative integer. Then*

$$\sum_{i=1}^{m} r_i^p \ge \frac{\sigma^p}{m^{p-1}} \tag{1.9}$$

*and*

$$\sum_{i=1}^{n} s_i^p \ge \frac{\sigma^p}{n^{p-1}} \tag{1.10}$$

Moreover, when $p \geq 2$, the equality in (1.9) can be realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff

$$\sigma \equiv 0 \pmod{m}, \tag{1.11}$$

the equality in (1.10) can be realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff

$$\sigma \equiv 0 \pmod{n}, \tag{1.12}$$

and the equalities in (1.9) and (1.10) can be simultaneously realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff (1.11) and (1.12) hold. Furthermore, if

$$p \geq 2, \quad \sigma = \frac{mn-1}{2}, \text{ and } m \geq n \text{ are both positive odd integers,} \tag{1.1'}$$

then

$$\sum_{i=1}^{m} r_i^p \geq \frac{n\sigma^{p-1}}{2m^{p-2}}. \tag{1.9'}$$

When $p = 2$, the equality in (1.9′) can be realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff $m = n$; when $p > 2$ and $m \geq n > 1$, (1.9′) is always a strict inequality.

We note that inequalities (9) and (10) become equalities for $p = 0, 1$.

**Theorem 5.** *Let $A \in \Psi_{m\times n}(\sigma)$, $0 < \sigma \leq mn$, and $p$ be a nonnegative integer. Then* (1.9) *and* (1.10) *hold. When $p \geq 2$, the equality in* (1.9) *(resp.* (1.10)*) can be realized by a matrix in $\Psi_{m\times n}(\sigma)$ iff* (1.11) *(resp.* (1.12)*) holds, and the equalities in* (1.9) *and* (1.10) *can be simultaneously realized by a matrix in $\Psi_{m\times n}(\sigma)$ iff* (1.11) *and* (1.12) *hold. Furthermore, if* (1.1′) *holds, then* (1.9′) *holds. When $p = 2$, the equality in* (1.9′) *can be realized by a matrix in $\Psi_{m\times n}(\sigma)$ iff $m = n$; when $p > 2$ and $m \geq n > 1$,* (1.9′) *is always a strict inequality.*

We will postpone the proofs of the above theorems until Section 3. In the following we will give a number of implications of Theorems 2–5.

**Corollary 1.** *Let $A \in \Phi_n(\sigma)$ and $p$ be a nonnegative integer. Then*

$$\sum_{i=1}^{n} (r_i^p + s_i^p) \geq \frac{2}{n^{p-1}}\sigma^p. \tag{1.13}$$

Moreover, when $p \geq 2$, the equality in (1.13) can be realized by a matrix in $\Phi_n(\sigma)$ iff (1.12) holds.

**Corollary 2.** *Let $A \in \Psi_n(\sigma)$, $0 < \sigma \leq n^2$, and $p$ be a nonnegative integer. Then* (1.13) *holds. Moreover, when $p \geq 2$, the equality in* (1.13) *can be realized by a matrix in $\Psi_n(\sigma)$ iff* (1.12) *holds.*

**Corollary 3.** *Let $A \in \Phi_{m\times n}(\sigma)$ and $p$ be a nonnegative integer. If*

$$mn \geq \sigma \geq \frac{mn}{2}, \tag{1.14}$$

*then*

$$\sum_{i=1}^{m} r_i^p \geq n\frac{\sigma^{p-1}}{2m^{p-2}} \tag{1.15}$$

*and*

$$\sum_{i=1}^{n} s_i^p \geq m\frac{\sigma^{p-1}}{2n^{p-2}} \tag{1.16}$$

Moreover, when $p \geq 2$, the equality in (1.15) can be realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff

$$\sigma = \frac{mn}{2} \quad \text{and} \quad n \equiv 0 \pmod 2, \tag{1.17}$$

the equality in (1.16) can be realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff

$$\sigma = \frac{mn}{2} \quad \text{and} \quad m \equiv 0 \pmod 2, \tag{1.18}$$

and the equalities in (1.15) and (1.16) can be simultaneously realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff

$$\sigma = \frac{mn}{2} \quad \text{and} \quad m \equiv n \equiv 0 \pmod 2, \tag{1.19}$$

Furthermore, if (1.1′) holds, then (1.9′) holds. When $p = 2$, the equality in (1.9′) can be realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff $m = n$; when $p > 2$ and $m \geq n > 1$, (1.9′) is always a strict inequality.

**Corollary 4.** *Let $A \in \Psi_{m\times n}(\sigma)$, $0 < \sigma \leq mn$, and $p$ be a nonnegative integer. If* (1.14) *holds, then* (1.15) *and* (1.16) *hold. Moreover, when $p \geq 2$, the equality in* (1.15) *can be realized by a matrix in $\Psi_{m\times n}(\sigma)$ iff* (1.17) *holds, the equality in* (1.16) *can be realized by a matrix in $\Psi_{m\times n}(\sigma)$ iff* (1.18) *holds, and the equalities in* (1.15) *and* (1.16) *can be simultaneously realized by a matrix in $\Psi_{m\times n}(\sigma)$ iff* (1.19) *holds. Furthermore, if* (1.1′) *holds, then* (1.9′) *holds. When $p = 2$, the equality in* (1.9′) *can be realized by a matrix in $\Phi_{m\times n}(\sigma)$ iff $m = n$; when $p > 2$ and $m \geq n > 1$,* (1.9′) *is always a strict inequality.*

It is easy to see that Lewin's Theorem is an immediate consequence of Corollary 4.

## 2 Preliminary Results

In this section we will prove several lemmas which will be used to prove Theorems 2–5 in the next section.

**Lemma 1.** *Let $p$, $b$ and $c$ be nonnegative integers such that $b < c$. Then*

$$b^p + c^p \geq (b+1)^p + (c-1)^p. \tag{2.1}$$

*When $p \geq 2$, the equality in* (2.1) *holds iff*

$$c = b + 1. \tag{2.2}$$

**Proof:** By induction on $p$. The cases for $p = 0, 1$ and $2$ are trivial. Suppose the lemma holds for some $p < q$, where $q$ is an integer larger than 2. We will show that the lemma holds for $q$ as well. There are two cases to consider.
**Case 1.** $q = 2v$ is even.
In this case we have

$$\begin{aligned} c^q - (c-1)^q &= c^{2v} - (c-1)^{2v} \\ &= (c^v - (c-1)^v)(c^v + (c-1)^v) \\ &\geq ((b+1)^v - b^v)((b+1)^v + b^v) \\ &= (b+1)^{2v} - b^{2v} \\ &= (b+1)^q - b^q. \end{aligned} \tag{2.3}$$

The inequality in (2.3) is due to the inductive hypothesis, and it becomes an equality iff (2.2) holds.
**Case 2.** $q = 2v + 1$ is odd.
In this case we have

$$\begin{aligned} c^q - (c-1)^q &= c^{q-1} + c^{q-2}(c-1) + c^{q-3}(c-1)^2 + \ldots \\ &\quad + c^2(c-1)^{q-3} + c(c-1)^{q-2} + (c-1)^{q-1} \\ &\geq (b+1)^{q-1} + (b+1)^{q-2}b + (b+1)^{q-3}b^2 + \ldots \\ &\quad + (b+1)^2 b^{q-3} + (b+1)b^{q-2} + b^{q-1} \\ &= (b+1)^q - b^q, \end{aligned} \tag{2.4}$$

and the inequality in (2.4) becomes an equality iff (2.2) holds. This completes the proof. □

**Lemma 2.** *Let $a$ and $k$ be positive integers such that*

$$a = kq + t, \qquad 0 \leq t < k. \tag{2.5}$$

*Then*

$$\min\left\{\sum_{i=1}^{k} a_i^p \mid \sum_{i=1}^{k} a_i = a, a_i \geq 0\right\} = t(q+1)^p + (k-t)q^p, \tag{2.6}$$

*and the minimum value is reached by and only by the sequence whose first* $t$ *components are* $q+1$ *and the remaining* $k-t$ *components are* $q$,

$$(a_1, a_2, \ldots, a_k) = (q+1, \ldots, q+1, q, \ldots, q),$$

*or its permutations.*

**Proof:** It is an immediate consequence of Lemma 1. □

**Lemma 3.** *Let* $m, r, p, s$ *be integers such that*

$$s \geq r \geq 0, \quad m > r, \quad p \geq 0. \tag{2.7}$$

*Then*

$$r(s+(m-r))^p + (m-r)(s-r)^p - ms^p \geq 0, \tag{2.8}$$

*and when* $p \geq 2$, *the equality in* (2.8) *holds iff* $r = 0$. *Furthermore, if*

$$s = \frac{mn-1}{2}, r = \frac{m-1}{2}, \ m \geq n \text{ are both positive odd integers, and } p > 1 \text{ is an integer,} \tag{2.9}$$

*then*

$$r(s+(m-r))^p + (m-r)(s-r)^p \geq \frac{m^2 n}{2} s^{p-1}. \tag{2.10}$$

When $p = 2$, the equality in (2.10) holds iff $m = n$; when $p > 2$ and $m \geq n > 1$, (2.10) is always a strict inequality.

**Proof:** We will prove the first part of the lemma by induction on $s$. Let $p$, $m$ and $r$ be arbitrary integers satisfying (2.7). As the induction base we will prove that when $s = r$, (2.8) holds, and that when $p \geq 2$, the equality in (2.8) holds iff $r = 0$.

When $p = 0$ or 1, (2.8) is trivially true. When $s = r$ and $p \geq 2$, we have

$$\begin{aligned} & r(r+(m-r))^p + (m-r)(r-r)^p - mr^p \\ & = rm(m^{p-1} - r^{p-1}) \\ \geq & \ 0, \end{aligned}$$

and when $p \geq 2$, the above inequality becomes an equality iff $r = 0$.

Now suppose $s > r$. Then

$$
\begin{aligned}
& r(s+(m-r))^p + (m-r)(s-r)^p - ms^p \\
&= r(((s-1)+(m-r))+1)^p + (m-r)(((s-1)-r)+1)^p \\
&\qquad\qquad - m((s-1)+1)^p \\
&= r\sum_{i=0}^{p} c(p,i)((s-1)+(m-r))^i + (m-r) \\
&\qquad\qquad \sum_{i=0}^{p} c(p,i)((s-1)-r)^i - m\sum_{i=0}^{p} c(p,i)(s-1)^i \\
&= \sum_{i=0}^{p} c(p,i)(r((s-1)+(m-r))^i + (m-r)((s-1)-r)^i \\
&\qquad\qquad - m(s-1)^i),
\end{aligned}
$$

where $c(p,i)$ is the combination number. By the inductive hypothesis, we have

$$r((s-1)+(m-r))^i + (m-r)((s-1)-r)^i - m(s-1)^i \geq 0, \text{ for all } i \geq 0.$$

Therefore, (2.8) holds. Notice that the equality in (2.8) holds iff $r = 0$.

Under condition (2.9), we have

$$
\begin{aligned}
& r(s+(m-r))^p + (m-r)(s-r)^p \\
&= \frac{m^p}{2^{p+1}}((m-1)(n+1)^p + (m+1)(n-1)^p).
\end{aligned}
$$

Hence, (2.10) holds iff

$$m^{p-2}((m-1)(n+1)^p + (m+1)(n-1)^p) \geq 2n(mn-1)^{p-1}. \quad (2.11)$$

We will prove (2.11) by induction on $p$. When $p = 2$, (2.11) becomes

$$(m-1)(n+1)^2 + (m+1)(n-1)^2 \geq 2n(mn-1),$$

or equivalently, $m \geq n$. Since we assume that $m \geq n$, (2.11) holds for $p = 2$, and the equality in (2.11) holds iff $m = n$. Now suppose $p > 2$ and (2.11) holds for $p-1$, we will show that it holds for $p$ as well. We have

$$
\begin{aligned}
& m^{p-1}((m-1)(n+1)^{p+1} + (m+1)(n-1)^{p+1}) \\
&= mm^{p-2}((m-1)(n+1)(n+1)^p + (m+1)(n-1)(n-1)^p) \\
&= mm^{p-2}(n(m-1)(n+1)^p + (m-1)(n+1)^p \\
&\qquad\qquad + n(m+1)(n-1)^p - (m+1)(n-1)^p) \\
&= mn[m^{p-2}((m-1)(n+1)^p + (m+1)(n-1)^p)] \\
&\quad + m^{p-1}[(m-1)(n+1)^p - (m+1)(n-1)^p]. \qquad (2.12)
\end{aligned}
$$

By the inductive hypothesis, we have

$$m^{p-2}((m-1)(n+1)^p + (m+1)(n-1)^p) \geq 2n(mn-1)^{p-1}. \quad (2.13)$$

On the other hand, since $m \geq n$, we have

$$\frac{m-1}{m+1} \geq \frac{n-1}{n+1},$$

and hence

$$\frac{m-1}{m+1} \geq \left(\frac{n-1}{n+1}\right)^p, \quad (2.14)$$

which becomes an equality iff $m = n = 1$. Combining (2.12)–(2.14), we have

$$\begin{aligned}
&m^{p-1}((m-1)(n+1)^{p+1} + (m+1)(n-1)^{p+1}) \\
&\geq 2n(mn(mn-1)^{p-1}) \\
&\geq 2n(mn-1)^p,
\end{aligned}$$

which becomes an equality iff $m = n = 1$. This completes the proof. □

We also need some other well-known results. Let $R$ and $S$ be any two nonnegative integral vectors,

$$R = (r_1, r_2, \ldots, r_m) \quad \text{and} \quad S = (s_1, s_2, \ldots, s_n), \quad (2.15)$$

with dimensions $m$ and $n$, respectively, such that

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j. \quad (2.16)$$

It is well known [2, 4] that a necessary and sufficient condition for the existence of an $m \times n$ $(0,1)$-matrix with row sum vector $R$ and column sum vector $S$ is

$$\widehat{S} \succ S, \quad r_i \leq n \ (1 \leq i \leq m), \quad s_j \leq m \ (1 \leq j \leq n), \quad (2.17)$$

where $\widehat{S}$ is the column sum vector of the *maximal* matrix with row sum vector $R$. (Note that the *maximal* matrix with row sum vector $R$ is the $m \times n$ matrix in which the $i^{\text{th}}$ row has 1's in the first $r_i$ entries and 0's in the remaining entries. Also, for any two $n$-dimensional vectors $U$ and $V$, $U \succ V$, (read as $U$ majorizes $V$), means that after rearranging their components into nonincreasing order, say $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ respectively, we have $\sum_{j=1}^{i} u_j \geq \sum_{j=1}^{i} v_j$ for each $1 \leq i \leq n$.) This result is called Gale-Ryser Theorem in the literature. A counterpart of Gale-Ryser Theorem for nonnegative integral matrices can be found in Brualdi [1] and is stated as follows for easy reference.

**Lemma 4.** *For any given m-dimensional vector R and n-dimensional vector S satisfying condition (2.16), there is an $m \times n$ nonnegative integral matrix with row sum vector R and column sum vector S.*

## 3 Proofs of Theorems and Corollaries

We are now in a position to prove our Theorems and Corollaries.

**Proof of Theorem 2:** Write

$$\sigma = mq + r, \qquad 0 \leq r \leq m. \tag{3.1}$$

Then

$$q = \left\lfloor \frac{\sigma}{m} \right\rfloor, \quad \text{and} \quad r = \sigma - m \left\lfloor \frac{\sigma}{m} \right\rfloor, \tag{3.2}$$

Since

$$\sum_{i=1}^{m} r_i = \sigma, \tag{3.3}$$

and there is a matrix in $\Phi_{m\times n}(\sigma)$ with the row sum vector whose first $r$ components are $q+1$ and the remaining components are $q$:

$$(q+1, q+1, \ldots, q+1, q, q, \ldots, q), \tag{3.4}$$

(1.5) follows from Lemma 2.

By the symmetry of rows and columns of a matrix, (1.6) holds. When we write $\sigma$ as

$$\sigma = nv + w, \qquad 0 \leq w < n, \tag{3.5}$$

the minimum value is reached by a matrix with the column sum vector whose first $w$ components are $v+1$ and the remaining components are $v$:

$$(v+1, v+1, \ldots, v+1, v, v, \ldots, v). \tag{3.6}$$

Since both the sum of the components of vector (3.4) and the sum of the components of vector (3.6) are $\sigma$, by Lemma 4, there is a matrix in $\Psi_{m\times n}(\sigma)$ with (3.4) being its row sum vector and (3.6) being its column sum vector. The minimum values in (1.5) and (1.6) are simultaneously reached by such a matrix. This completes the proof of Theorem 2. □

**Proof of Theorem 3:** By Theorem 2, it suffices to prove that there is a matrix in $\Psi_{m\times n}(\sigma)$ such that the minimum values in (1.7) and (1.8) are simultaneously reached. From the proof of Theorem 2, it suffices to show that there is a matrix in $\Psi_{m\times n}(\sigma)$ with (3.4) being its row sum vector and (3.6) being its column sum vector.

Since $\sigma \leq mn$, we have $q = n$ and $r = 0$, or $q < n$. Hence each component of (3.4) cannot exceed $n$, the number of columns of the matrix in $\Phi_{m\times n}(\sigma)$. Similarly, we have $v = m$ and $w = 0$, or $v < m$. Hence each component of (3.6) cannot exceed $m$, the number of rows of the matrix in $\Phi_{m\times n}(\sigma)$. Moreover, the column sum vector $\widehat{S}$ of the *maximal* matrix with row sum vector (3.4) is the $n$-dimensional vector with $m$ being its first $q$ components, $r$ its $(q+1)^{\text{th}}$ component and 0 its remaining components:

$$(m, m, \ldots, m, r, 0, \ldots, 0), \tag{3.7}$$

which majorizes vector (3.6). Then Gale-Ryser Theorem guarantees the existence of the desired matrices. □

**Proof of Theorem 4:** Although inequalities (1.9) and (1.10) are consequences of a well-known inequality (see, for example, "Inequalities" by G. H. Hardy, J. E. Littlewood and G. Polya), we give an alternate proof here since it is an important part of the proof of the whole theorem. Suppose we have (3.1). Then, by Theorem 2,

$$\sum_{i=1}^{m} r_i^p \geq r(q+1)^p + (m-r)q^p \tag{3.8}$$

$$= r\left(\frac{\sigma - r}{m} + 1\right)^p + (m-r)\left(\frac{\sigma - r}{m}\right)^p$$

$$= m^{-p}(r(\sigma + (m-r))^p + (m-r)(\sigma - r)^p)$$

$$\geq \frac{\sigma^p}{m^{p-1}}, \quad \text{by Lemma 3} \tag{3.9}$$

which is (1.9).

By Lemma 3, when $p \geq 2$, the equality in (3.9) holds iff $r = 0$, which is the same as (1.11). Theorem 2 guarantees the existence of a matrix in $\Phi_{m\times n}(\sigma)$ such that the equality in (3.8) holds. This proves the assertions about (1.9) and (1.11). The assertions about (1.10) and (1.12) follow from the symmetry of the rows and columns of a matrix. From Lemma 4, the assertion about the simultaneous realization of the equalities in (1.9) and (1.10) is obvious.

Under condition (1.1′), we have $r = \frac{m-1}{2}$ since

$$\sigma = \frac{mn-1}{2} = mq + \frac{m-1}{2}, \quad \text{where } q = \frac{n-1}{2}.$$

We have

$$\sum_{i=1}^{m} r_i^p \geq r(q+1)^p + (m-r)q^p \tag{3.8'}$$

$$= m^{-p}(r(\sigma + (m-r))^p + (m-r)(\sigma - r)^p)$$

$$\geq m^{-p}\frac{m^2 n}{2}\sigma^{p-1} \quad \text{by Lemma 3}$$

$$= \frac{n\sigma^{p-1}}{2m^{p-2}},$$

which is (1.9').

By Lemma 3, when $p = 2$, the inequality becomes an equality iff $m = n$. Theorem 2 guarantees the existence of a matrix in $\Phi_n\left(\frac{mn-1}{2}\right)$ such that the equality in (3.8′) holds. This completes the proof of the theorem. □

**Proof of Theorem 5:** By Theorem 4, it suffices to show that if $\sigma \leq mn$, then the following four assertions hold:

Assertion 1. If (1.11) holds, then there is a matrix in $\Psi_{m\times n}(\sigma)$ such that the equality in (1.9) holds.

Assertion 2. If (1.12) holds, then there is a matrix in $\Psi_{m\times n}(\sigma)$ such that the equality in (1.10) holds.

Assertion 3. If (1.11) and (1.12) hold, then there is a matrix in $\Psi_{m\times n}(\sigma)$ such that the equalities in (1.9) and (1.10) hold.

Assertion 4. If (1.1′) holds, then there is a matrix in $\Psi_n\left(\frac{mn-1}{2}\right)$ such that the equality in (1.9′) holds.

By Theorem 3, when $\sigma \leq mn$, there is a matrix in $\Phi_{m\times n}(\sigma)$ such that the equality in (3.8) holds. By Lemma 3, when (1.11) holds, the equality in (3.9) holds. Hence we have Assertion 1. Assertion 4 can be proved in a similar fashion.

By the symmetry of rows and columns of a matrix, Assertion 2 follows from Assertion 1. Assertion 3 follows from Gale-Ryser Theorem and the facts stated and proved in the second part of Theorem 3. □

Corollaries 1 and 2 are immediate consequences of Theorems 4 and 5, respectively.

**Proof of Corollary 3:** By (1.9) and (1.14), we have

$$\sum_{i=1}^{m} r_i^p \geq \sigma\left(\frac{\sigma}{m}\right)^{p-1} \tag{3.10}$$

$$\geq \frac{mn}{2}\left(\frac{\sigma}{m}\right)^{p-1} \tag{3.11}$$

$$= \frac{n\sigma^{p-1}}{2m^{p-2}},$$

which is (1.15). By Theorem 4, when $p \geq 2$, the equality in (3.10) can be realized by a matrix in $\Phi_{m \times n}(\sigma)$ iff (1.11) holds. On the other hand, the equality in (3.11) holds iff

$$\sigma = \frac{mn}{2}. \tag{3.12}$$

Clearly, (1.11) and (3.12) together are equivalent to (1.17).

The assertion about (1.1′) and (1.9′) can be obtained similarly. By the symmetry of rows and columns of a matrix, we have (1.16) and (1.18). The assertion about the simultaneous realization of the equalities in (1.15) and (1.16) can be obtained by using Theorem 5 and the reasoning about (3.12). This completes the proof. □

The proof of Corollary 4 is similar to that of Corollary 3, and hence will be omitted.

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