On Generalizing a Theorem of Jung

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ABSTRACT. For a graph G, let $\sigma_k = \min\ \{\sum_{i=1}^{k} d(v_i) \mid \{v_1, \ldots, v_k\}$ is an independent set of vertices in $G\}$. Jung proved that every 1-tough graph G with $|V(G)| = n \geq 11$ and $\sigma_2 \geq n - 4$ is hamiltonian. This result is generalized as follows: if G is a 1-tough graph with $|V(G)| = n \geq 3$ such that $\sigma_3 \geq n$ and for all $x, y \in V(G)$, $d(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{1}{2}(n - 4)$, then G is hamiltonian. It is also shown that the condition $\sigma_3 \geq n$, in the latter result, can be dropped if G is required to be 3-connected and to have at least 35 vertices.

1 Results.

We consider only finite undirected graphs without loops or multiple edges. Our terminology is standard except as indicated. A good reference for any undefined terms is [4]. By $\omega(G)$ we denote the number of components of a graph G, and by $\kappa(G)$ its connectivity. Chvátal [6] defined G to be **1-tough** if $\omega(G - S) \leq |S|$ for every subset S of $V(G)$ with $\omega(G - S) > 1$. By $\sigma_k(G)$, or just σ_k, we denote $\min\{\sum_{i=1}^{k} d(v_i) \mid \{v_1, \ldots, v_k\}$ is an independent set of vertices in $G\}$ $(k \geq 2)$.

A well-known result in hamiltonian graph theory is the following theorem of Jung.

Theorem 1 ([8]). *Let G be a 1-tough graph on $n \geq 11$ vertices such that $\sigma_2 \geq n - 4$. Then G is hamiltonian.*

The proof of Theorem 1 given in [8] is lengthy and complicated. A much simpler proof appears in [2]. Using arguments of the latter proof, Skupień recently obtained the following improvement of Theorem 1.

Theorem 2 ([9]). *Let G be a 1-tough graph on $n \geq 11$ vertices such that $\sigma_2 \geq n-4-\varepsilon(n)$, where $\varepsilon(n) = 1$ if n is even and $n \neq 12$, and $\varepsilon(n) = 0$, otherwise. Then G is hamiltonian.*

To show that Theorem 2 is best possible, we present, for each $n \geq 11$, a nonhamiltonian 1-tough graph G_n on n vertices with $\sigma_2(G_n) = n-5-\varepsilon(n)$. If n is odd, then G_n is obtained from $\overline{K}_{\frac{1}{2}(n-1)} \cup K_{\frac{1}{2}(n-5)} \cup K_3$ by joining every vertex in $K_{\frac{1}{2}(n-5)}$ to all other vertices and adding a matching between the vertices of K_3 and three vertices of $\overline{K}_{\frac{1}{2}(n-1)}$. If n is even, then G_n is obtained from G_{n-1} by adding a new vertex and joining it to a vertex of degree $\frac{1}{2}(n-6)$ in G_{n-1} and the $\frac{1}{2}(n-6)$ vertices of degree $n-2$ in G_{n-1}.

A variation of the graphs G_n for odd n (with $K_{\frac{1}{2}(n-5)}$ replaced by $\overline{K}_{\frac{1}{2}(n-5)}$ already appeared in [8], while the graphs G_n with n even appear in [9].

In fact, Skupień [9] proved more than Theorem 2. He additionally showed that for odd $n \geq 15$ every nonhamiltonian 1-tough graph on n vertices with $\sigma_2 = n-5$ is a spanning subgraph of G_n.

We will obtain a generalization of Theorem 2 by imposing two degree conditions, each of which is weaker than the degree condition of Theorem 2. One of them is of a type introduced by Fan, who established the following result.

Theorem 3 ([7]). *Let G be a 2-connected graph on n vertices and c an integer with $3 \leq c \leq n$. If, for all vertices x, y, $d(x,y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{1}{2}c$, then G has a cycle of length at least c.*

Generalizations of Theorem 3 in the case where $c = n$ were obtained in [5].

Suppose G satisfies the conditions of Theorem 2. If $d(x,y) = 2$, then obviously $\max\{d(x), d(y)\} \geq \left\{\frac{1}{2}(n-4-\varepsilon(n))\right\} = \left\{\frac{1}{2}(n-4)\right\}$, where $\{r\}$ denotes the smallest integer greater than or equal to r. Furthermore, $\sigma_3 \geq \left\{\frac{3}{2}(n-4-\varepsilon(n))\right\} \geq n$, since $n \geq 11$. Thus the following result, the proof of which is given in Section 3, generalizes Theorem 2.

Theorem 4. *Let G be a 1-tough graph on $n \geq 3$ vertices such that $\sigma_3 \geq n$ and, for all vertices x, y, $d(x,y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{1}{2}(n-4)$. Then G is hamiltonian.*

Theorem 4 is best possible in the sense that neither of the two degree conditions can be relaxed. For $n \geq 17$, the nonhamiltonian 1-tough graph G_n has $\sigma_3 \geq n$ and satisfies the second degree condition of Theorem 4 with $\frac{1}{2}(n-4)$ replaced by $\left\{\frac{1}{2}(n-4)\right\} -1$. For $t = 2, 3$, $n \geq t+5$ and

$n-t-1 \equiv 0 \pmod 2$, obtain the graph $H_{n,t}$ from $K_1+(K_t \cup 2K_{\frac{1}{2}(n-t-1)})$ by choosing one vertex from K_t and one vertex from each copy of $K_{\frac{1}{2}(n-t-1)}$ and adding the edges of a triangle between them. The graph $H_{n,t}$ is 1-tough and nonhamiltonian, while $H_{n,t}$ satisfies the second degree condition of Theorem 4 and has $\sigma_3 = n-1$.

We note that $\kappa(H_{n,t}) = 2$ for all n and t. Our next result, to be proved in Section 3, shows that the condition on σ_3 in Theorem 4 can be dropped completely if G is required to be 3-connected and large enough.

Theorem 5. *Let G be a 3-connected 1-tough graph on $n \geq 35$ vertices such that, for all vertices x, y, $d(x,y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{1}{2}(n-4)$. Then G is hamiltonian.*

The 3-connected graphs G_n show that Theorem 5 is, in a sense, best possible.

Theorem 5 generalizes Theorem 2 within the class of 3-connected graphs that are large enough. We note that within the class of graphs on at least 17 vertices with $\kappa = 2$, Theorem 2 is generalized by the following recent result.

Theorem 6 ([1]). *If G is a 2-connected graph on n vertices with $\sigma_3 \geq n+\kappa$, then G is hamiltonian.*

We do not believe that the requirement $n \geq 35$ in Theorem 5 is tight. However, examples like the Petersen graph and the graph G_{12} show that some lower bound on n has to be imposed. It would be interesting to know the value of the smallest integer n_0 by which 35 could be replaced in Theorem 5. The graph G_{12} shows that $n_0 \geq 13$.

2 Preliminaries.

For the proofs of our results we need some definitions and convenient notation. Let C be a cycle of a graph G and let $u, v \in V(C)$. C is a **dominating cycle** if $V(G) - V(C)$ is an independent set of vertices. $\vec{C}$ denotes the cycle C with a given orientation. By $u\vec{C}v$ we denote the consecutive vertices on C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We write $uv \in P_C(G)$ if u and v are connected by a path of length at least 2 with all internal vertices in $V(G)-V(C)$. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. If $S \subseteq V(C)$, then $S^+ = \{x^+ \mid x \in S\}$ and $S^- = \{x^- \mid x \in S\}$. We write u^{++}, u^{--}, S^{++}, and S^{--}, for $(u^+)^+$, $(u^-)^-$, $(S^+)^+$, and $(S^-)^-$, respectively.

Our proof of Theorem 4 heavily relies on the following two lemmas, which were established in [3]. The first one is a combination of [3, Theorem 5] and

[3, Lemma 8], while the second is implicit in the proof of [3, Theorem 9].

Lemma 7 ([3]). *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$ and let C be a longest cycle in G. Then C is a dominating cycle. Moreover, if $v \in V(G) - V(C)$ and $A = N(v)$, then $(V(G) - V(C)) \cup A^+$ is independent.*

Lemma 8 ([3]). *Let G be a nonhamiltonian 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Then G contains a longest cycle C such that* $\max\{d(x) \mid x \in V(G) - V(C)\} \geq \frac{1}{3}n$.

3 Proofs

Throughout this section we assume that G is a nonhamiltonian graph and C a longest cycle in G, with a fixed orientation, such that $\mu(C) = \max\{d(v) \mid v \in V(G) - V(C)\}$ is maximum among all longest cycles of G. Let v_0 be a vertex in $V(G) - V(C)$ of degree $\mu(C)$ and H the component of $G - V(C)$ containing v_0. Furthermore, $A = \cup_{v \in V(H)} N(v) - V(H)$ and $v_1, \ldots, v_k$ are the vertices in A, occurring on $\vec{C}$ in consecutive order. Since C is a longest cycle, we clearly have $v_i^+ \neq v_{i+1}$ ($i = 1, \ldots, k$, indices modulo k). For $i = 1, \ldots, k$, we set $u_i = v_i^+$ and $w_i = v_{i+1}^-$. The parts of C of the form $u_i\vec{C}w_i$ will be called **segments**; $u_i\vec{C}w_i$ is a t-**segment** if $|u_i\vec{C}w_i| = t$. For distinct vertices x and y of G, $x - y$ will denote a path from x to y which has all internal vertices in H and has length at least 2 unless $x, y \in V(H)$.

We state and prove four observations that will be useful in the proofs of both Theorem 4 and Theorem 5. As for observations (1), (2), and (3), we only prove the assertions obtained from them if $E(G) \cup P_C(G)$ is replaced by $E(G)$; easy variations of the arguments give the rest.

(1) If $i \neq j$, then $u_iu_j \notin E(G) \cup P_C(G)$.

Assuming the contrary to (1), the cycle $u_i\vec{C}v_j - v_i \overleftarrow{C} u_ju_i$ is longer than C, a contradiction.

(2) If $i \neq j$, then there is no vertex $z \in u_i^+\vec{C}w_{j-1}$ such that u_jz, $u_iz^+ \in E(G) \cup P_C(G)$.

Assuming the contrary to (2), the cycle $u_jz \overleftarrow{C} u_iz^+ \vec{C}v_j - v_i \overleftarrow{C} u_j$ is longer than C, a contradiction.

(3) If $i \neq j$, then there is no vertex $z \in v_j\vec{C}w_{i-1}$ such that

(a) $w_{j-1}z$, $u_iz^+ \in E(G) \cup P_C(G)$, or

(b) u_iz, $w_{j-1}z^+ \in E(G) \cup P_C(G)$.

Assuming the contrary to (a) ((b)), the cycle $u_i z^+ \vec{C} v_i - v_j \vec{C} z w_{j-1} \overleftarrow{C} u_i$ ($u_i z \overleftarrow{C} v_j - v_i \overleftarrow{C} z^+ w_{j-1} \overleftarrow{C} u_i$) is longer than C, a contradiction.

Note that observations (1) and (2), stating properties of vertices in A^+, have analogous counterparts for vertices in A^-. These counterparts will also be referred to as (1) and (2).

(4) If G is 1-tough, then $|V(C)| \geq 2|A| + 2$ and equality holds only if C contains two 2-segments.

Suppose $|V(C)| \leq 2|A| + 1$. Then all segments of C are 1-segments, except for at most one which is a 2-segment. But then by (1), $\omega(G - A) > |A|$, contradicting the assumption that G is 1-tough. Hence, $|V(C)| \geq 2|A| + 2$. Suppose $|V(C)| = 2|A| + 2$ and C does not contain two 2-segments. Then C contains a 3-segment, say with vertices u_i, u_i^+, w_i, while all other segments of C are 1-segments. If $u_i^+ u_j \in E(G) \cup P_C(G)$ for some $j \neq i$, then by (2) $u_i w_i \notin E(G) \cup P_C(G)$ and, hence, by (1), $\omega\left(G - (A \cup \{u_i^+\})\right) > |A \cup \{u_i^+\}|$, a contradiction. If $u_i^+ u_j \notin E(G) \cup P_C(G)$ for all $j \neq i$, then $\omega(G - A) > |A|$, again a contradiction.

Proof of Theorem 4: Let G satisfy the hypothesis of Theorem 4. By Lemma 7, C is a dominating cycle, so that $V(H) = \{v_0\}$ and $A = N(v_0)$. By Lemma 8, $d(v_0) \geq \frac{1}{3}n$. Using (4) we deduce that $n - 1 \geq |V(C)| \geq 2|A| + 2 \geq 2\left\{\frac{1}{3}n\right\} + 2$, which implies $n = 9$ or $n \geq 11$. We leave it to the reader to derive a contradiction if $n = 9$ and, henceforth, assume $n \geq 11$.

Suppose $d(v_0) < \frac{1}{2}(n - 4)$. Then $d(x) \geq \frac{1}{2}(n - 4)$ for all $x \in A^+ \cup A^-$, since $d(v_0, x) = 2$. If C would contain a 1-segment, say with u_i as its unique vertex, then the cycle $C' = v_i v_0 v_{i+1} \vec{C} v_i$ would satisfy $|V(C')| = |V(C)|$ and $\mu(C') \geq d(u_i) \geq \frac{1}{2}(n - 4) > d(v_0) = \mu(C)$, a contradiction with the choice of C. Hence, C contains no 1-segments. But then $d(v_0) = |A| \leq \frac{1}{3}|V(C)| < \frac{1}{3}n$. This contradiction shows that $d(v_0) \geq \frac{1}{2}(n - 4)$.

We also observe the following.

(5) At most one vertex of A^+ has degree smaller than $\frac{1}{2}(n - 4)$.

Assuming the contrary to (5), let u_i and u_j be distinct vertices in A^+ with $d(u_i)$, $d(u_j) < \frac{1}{2}(n - 4)$. By the second degree condition of the theorem we have $d(u_i, u_j) > 2$ and, hence, $N(u_i) \cap N(u_j) = \emptyset$. By Lemma 7, $N(u_i) \cup N(u_j) \subseteq V(C)$ and by (1), $(N(u_i) \cup N(u_j)) \cap A^+ = \emptyset$. It follows that $d(u_i) + d(u_j) \leq |V(C)| - |A^+| = |V(C)| - d(v_0)$ and, hence, $d(u_i) + d(u_j) + d(v_0) < n$. Since $\{u_i, u_j, v_0\}$ is an independent set, this contradicts the first degree condition of the theorem and proves (5).

Using observations (1) through (5) we now derive contradictions in all possible cases. By (4) and the fact that $|A| \geq \frac{1}{2}(n - 4)$ we have $2|A| + 2 \leq |V(C)| \leq n - 1 \leq 2|A| + 3$.

Case 1. $|V(C)| = 2|A| + 2$.
By (4), C contains two 2-segments. All other segments are 1-segments. Without loss of generality we assume that $\{u_1, w_1\}$ and $\{u_i, w_i\}$ are the vertex sets of the two 2-segments. If u_1w_i, $u_iw_1 \notin E(G) \cup P_C(G)$, then by (1) $\omega(G - A) > |A|$, contradicting the fact that G is 1-tough. Hence, assume, without loss of generality that $u_1w_i \in E(G) \cup P_C(G)$. By Lemma 7, $(V(G) - V(C)) \cup A^+$ is independent, so in fact $u_1w_i \in E(G)$.

We first show that $i = 2$. Suppose $i \geq 3$. By (1) and Lemma 7, $N(w_{i-1}) \subseteq A$ and by (2), $w_{i-1}v_1 \notin E(G)$. Hence, $d(w_{i-1}) \leq |A| - 1$ $= \frac{1}{2}(|V(C)| - 2) - 1 < \frac{1}{2}(n - 4)$. Since $d(u_i, w_{i-1}) = 2$, it follows that $d(u_i) \geq \frac{1}{2}(n-4)$. On the other hand, by (2), (3), and Lemma 7, $N(u_i) \subseteq (A \cup \{w_i\}) - \{v_{i+1}, v_1\}$, implying that $v_{i+1} = v_1$. Since $d(w_{i-1}) < \frac{1}{2}(n-4)$, we also have $d(u_2) < \frac{1}{2}(n - 4)$ by an argument of symmetry. However, $w_{i-1} = u_{i-1}$ and, since $n \geq 11$, $w_{i-1} \neq u_2$. This contradiction with (5) shows that, indeed, $i = 2$.

Now since $d(u_2, w_1) = 2$, we may assume without loss of generality that $d(u_2) \geq \frac{1}{2}(n - 4)$. By (2), (3), and Lemma 7, however, $N(u_2) \subseteq (A \cup \{w_2\}) - \{v_3, v_1\}$, a contradiction.

Case 2. $|V(C)| = 2|A| + 3$.
Then $|A| = \frac{1}{2}(n - 4)$ and $|V(C)| = n - 1$, so that v_0 is the only vertex in $V(G) - V(C)$. There are three possibilities for the segments of C that are not 1-segments.

Case 2.1. C contains one 4-segment.
Assume, without loss of generality, that u_1, u_1^+, w_1^-, w_1, are the vertices of the 4-segment. If neither u_1^+ nor w_1^- is adjacent to a vertex of a 1-segment, then $\omega(G - A) > |A|$ by (1), a contradiction. Hence, assume, without loss of generality, that $u_1^+u_i \in E(G)$, where $u_i \in A^+ \cap A^-$. By (2) and (3), w_1^- is not adjacent to any vertex in A^+, while the same is true for w_1 by (1) and (2). We conclude that $\omega\left(G - (A \cup \{u_1^+\})\right) > |A \cup \{u_1^+\}|$, a contradiction.

Case 2.2. C contains one 3-segment and one 2-segment.
Assume, without loss of generality, that u_1, u_1^+, w_1 are the vertices of the 3-segment and let u_i, w_i be the vertices of the 2-segment.

Suppose u_1w_i, $u_iw_1 \notin E(G)$. If $u_1w_1 \notin E(G)$, then $\omega\left(G - (A \cup \{u_1^+\})\right)$ $> |A \cup \{u_1^+\}|$ by (1). If $u_1w_1 \in E(G)$, then by (2) u_1^+ is not adjacent to any vertex in $(A^+ \cup A^-) - \{u_1, w_1\}$, which in combination with (1) implies $\omega(G - A) > |A|$. Thus we may assume, without loss of generality, that $u_1w_i \in E(G)$.

We now show that $i = 2$. Suppose $i \geq 3$. By (1) and (3), $N(w_{i-1}) \subseteq A$ and by (2), $w_{i-1}v_1 \notin E(G)$. Hence $d(w_{i-1}) \leq |A| - 1 < \frac{1}{2}(n - 4)$. Since $d(u_i, w_{i-1}) = 2$, it follows that $d(u_i) \geq \frac{1}{2}(n-4)$. On the other hand, by (1), (2), and (3), $N(u_i) \subseteq (A \cup \{w_i\}) - \{v_{i+1}, v_1\}$, implying that $v_{i+1} = v_1$. The argument used to prove that $d(w_{i-1}) < \frac{1}{2}(n-4)$ also yields $d(u_2) < \frac{1}{2}(n-4)$.

However, $w_{i-1} = u_{i-1}$ and, since $n \geq 11$, $w_{i-1} \neq u_2$. This contradiction with (5) shows that $i = 2$.

By (1), (2), and (3), $N(u_2) \subseteq (A \cup \{w_2\}) - \{v_3, v_1\}$, so that $d(u_2) < \frac{1}{2}(n-4)$. Hence, since $d(u_2, w_1) = 2$, $d(w_1) \geq \frac{1}{2}(n-4)$. Using (1), (2), and (3) we conclude that $N(w_1) = (A \cup \{u_1, u_1^+\}) - \{v_3, v_1\}$. In particular $w_1 v_4 \in E(G)$. Since $d(u_2) < \frac{1}{2}(n-4)$, (5) implies $d(u_3) \geq \frac{1}{2}(n-4)$. Using (1) and (2) we conclude that $N(u_3) = A$, so that, in particular, $u_3 v_2 \in E(G)$. Now the cycle $w_1 v_4 \vec{C} v_1\ v_0 v_3 u_3 v_2 u_2 w_2\ u_1 u_1^+ w_1$ is longer than C, a contradiction.

Case 2.3. C contains three 2-segments.

Without loss of generality, we assume that $\{u_1, w_1\}$, $\{u_i, w_i\}$, and $\{u_j, w_j\}$ are the vertex sets of the three 2-segments. If no vertex in any 2-segment is adjacent to a vertex in a different 2-segment, then $\omega(G - A) > |A|$. Hence we may assume, without loss of generality, that $u_1 w_j \in E(G)$. By (3), $u_j w_1 \notin E(G)$. We distinguish two subcases.

Case 2.3.1. $i < j$.

Suppose $d(w_1), d(u_j) < \frac{1}{2}(n-4)$. Then $d(w_1, u_j) > 2$ and hence $N(w_1) \cap N(u_j) = \emptyset$. By (1), (2), and (3), $(N(w_1) \cup N(u_j)) \cap ((A^+ \cup \{v_1\}) - \{u_1\}) = \emptyset$. It follows that $d(w_1) + d(u_j) \leq |V(C)| - |A^+| = |V(C)| - d(v_0)$ and hence $d(w_1) + d(u_j) + d(v_0) < n$, a contradiction since $\{w_1, u_j, v_0\}$ is independent. Thus we may assume, without loss of generality, that $d(u_j) \geq \frac{1}{2}(n-4)$. By (1), (2), and (3), $N(u_j) \subseteq (A \cup \{w_j\}) - \{v_1, v_{j+1}\}$. It follows that $v_{j+1} = v_1$ and $d(u_j) = \frac{1}{2}(n-4)$

Since $n \geq 11$, C contains at least one 1-segment. Let $\{u_p\}$ be a 1-segment. Then, by (1) and (2), $N(u_p) \subseteq A - \{v_1\}$ and hence $d(u_p) < \frac{1}{2}(n-4)$. From (5) we now deduce that C contains no other 1-segments. Thus $n = 12$, $d(v_0) = d(u_j) = 4$ and $d(u_p) \leq 3$. But then $d(v_0) + d(u_j) + d(u_p) < n$, a contradiction.

Case 2.3.2. $i > j$.

Suppose there is a 1-segment in $v_2 \vec{C} v_j$ and let $\{u_p\}$ be such a 1-segment. Then, by (1) and (2), $N(u_p) \subseteq A - \{v_1\}$ and hence $d(u_p) < \frac{1}{2}(n-4)$. Now by (5), $p = 2$ and $j = 3$. Since $d(w_1, u_2) = d(u_3, u_2) = 2$, we have $d(w_1), d(u_3) \geq \frac{1}{2}(n-4)$. By (1), (2), and (3), $N(w_1) \subseteq (A \cup \{u_1, u_i\}) - \{v_4, v_1\}$. It follows that $d(w_1) = \frac{1}{2}(n-4)$ and $w_1 u_i \in E(G)$. By the same token, $d(u_3) = \frac{1}{2}(n-4)$ and $u_3 w_i \in E(G)$. If $v_4 \neq v_i$, then, by (1) and (2), $N(u_4) \subseteq A - \{v_3\}$ and hence $d(u_4) < \frac{1}{2}(n-4)$, contradicting (5). Thus $v_4 = v_i$ and, similarly, $v_5 = v_1$. But then $n = 12$, $d(v_0) = d(w_1) = 4$ and $d(u_2) \leq 3$, implying that $d(v_0) + d(w_1) + d(u_2) < n$. From this contradiction we conclude that $j = 2$.

Since $d(w_1, u_2) = 2$, we may assume, without loss of generality, that $d(u_2) \geq \frac{1}{2}(n-4)$. Using (1), (2) and (3), we conclude that $N(u_2) = (A \cup \{w_2, w_i\}) - \{v_3, v_1\}$. In particular, $u_2 w_i \in E(G)$. The argument used to show that $j = 2$ now also applies to show that $i = 3$. Since $n \geq 11$, $v_4 \neq v_1$ and hence $u_2v_4 \in E(G)$. But then the cycle $u_2v_4\vec{C}v_1\ v_0v_2w_1u_1w_2\ v_3u_3w_3u_2$ is longer than C, our final contradiction. □

Proof of Theorem 5: Let G satisfy the hypothesis of Theorem 5. Using observations (1) through (4) we derive contradictions in all possible cases.

Case 1. $d(v_0) < \frac{1}{2}(n-4)$.
Since G is 3-connected, we have $k \geq 3$. For each $i \in \{1, \ldots, k\}$ there is a vertex $v \in V(H)$ with $d(v, u_i) = 2$, implying that $d(u_i) \geq \frac{1}{2}(n-4)$. We also observe the following.

(6) If $i \neq j$, then $u_iu_j^+ \notin E(G)$.

Assuming the contrary to (6), the cycle $C' = u_i\vec{C}v_j - v_i\overleftarrow{C}u_j^+u_i$ contradicts the choice of C, since $|V(C')| \geq |V(C)|$ and $\mu(C') \geq d(u_j) \geq \frac{1}{2}(n-4) > d(v_0) = \mu(C)$.

Note that by a similar argument C contains no 1-segments. Define

$$\begin{aligned}
S_1 &= \{x \in u_1\vec{C}v_2 \mid u_1x^{++} \in E(G)\},\\
S_2 &= \{x \in u_1\vec{C}v_2 \mid u_2x \in E(G)\},\\
S_3 &= \{x \in u_1\vec{C}v_2 \mid u_3x^{+} \in E(G)\},\\
T_1 &= \{x \in u_2\vec{C}v_3 \mid u_1x^{+} \in E(G)\},\\
T_2 &= \{x \in u_2\vec{C}v_3 \mid u_2x^{++} \in E(G)\},\\
T_3 &= \{x \in u_2\vec{C}v_3 \mid u_3x \in E(G)\},\\
U_1 &= \{x \in u_3\vec{C}v_1 \mid u_1x \in E(G)\},\\
U_2 &= \{x \in u_3\vec{C}v_1 \mid u_2x^{+} \in E(G)\},\\
U_3 &= \{x \in u_3\vec{C}v_1 \mid u_3x^{++} \in E(G)\},\\
W_i &= \{x \in V(G) - V(C) \mid u_ix \in E(G)\} \quad (i = 1, 2, 3).
\end{aligned}$$

Using (1) and (6) and taking into account that $u_iu_i^+ \in E(G)$, we have $d(u_i) = |S_i| + |T_i| + |U_i| + |W_i| + 1$ $(i = 1, 2, 3)$. By (1) and (2),

$$S_1 \cap S_3 = S_2 \cap S_3 = T_1 \cap T_2 = T_1 \cap T_3 = U_1 \cap U_2 = U_2 \cap U_3 = W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = \emptyset.$$

Noting that $v_0 \notin \cup_{i=1}^3 W_i$, we obtain

$$(7) \quad \sum_{i=1}^3 d(u_i)$$
$$= \sum_{i=1}^3 (|S_i| + |T_i| + |U_i| + |W_i| + 1)$$
$$= \left|\bigcup_{i=1}^3 S_i\right| + \left|\bigcup_{i=1}^3 T_i\right| + \left|\bigcup_{i=1}^3 U_i\right| + \left|\bigcup_{i=1}^3 W_i\right|$$
$$+ |S_1 \cap S_2| + |T_2 \cap T_3| + |U_1 \cap U_3| + 3$$
$$\leq |V(C)| + (|V(G) - v(C)| - 1)$$
$$+ |S_1 \cap S_2| + |T_2 \cap T_3| + |U_1 \cap U_3| + 3$$
$$= n + 2 + |S_1 \cap S_2| + |T_2 \cap T_3| + |U_1 \cap U_3|.$$

We now establish the following.

(8) If $v \in S_1 \cap S_2$, then v^+, $v^{++} \notin S_1 \cap S_2$.

Assume $v \in S_1 \cap S_2$. By (2), $v^+ \notin S_1 \cap S_2$. Now suppose $v^{++} \in S_1 \cap S_2$. If $v^+v^{+++} \in E(G)$, then the cycle $v_1 - v_2 \overleftarrow{C} v^{++++} u_1\vec{C}v^+ v^{+++}v^{++} u_2\vec{C}v_1$ is longer than C, a contradiction. If $v^+v^{+++} \notin E(G)$, then one of v^+ and v^{+++}, for example v^+, has degree at least $\frac{1}{2}(n-4)$, since $d(v^+, v^{+++}) = 2$. But then the cycle $C' = v_1 - v_2 \overleftarrow{C} v^{++} u_1\vec{C}vu_2\vec{C}v_1$ satisfies $|V(C')| \geq |V(C)|$ and $\mu(C') \geq d(v^+) \geq \frac{1}{2}(n-4) > \mu(C)$. This contradiction proves (8).

By (1) and (6), $u_1, u_1^+, w_1, v_2 \notin S_1 \cap S_2$. If $S_1 \cap S_2 \neq \emptyset$, then together with (8) we obtain $|u_1\vec{C}v_2| \geq 3|S_1 \cap S_2| + 2$. Clearly, this inequality is also valid if $S_1 \cap S_2 = \emptyset$. Similarly, we have $|u_2\vec{C}v_3| \geq 3|T_2 \cap T_3| + 2$ and $|u_3\vec{C}v_1| \geq 3|U_1 \cap U_3| + 2$. It follows that

$$|S_1 \cap S_2| + |T_2 \cap T_3| + |U_1 \cap U_3| \leq \frac{1}{3}(|V(C)| - 6) \leq \frac{1}{3}(n-7).$$

In combination with (7) we obtain

$$\sum_{i=1}^3 d(u_i) \leq n + 2 + \frac{1}{3}(n-7).$$

On the other hand,

$$\sum_{i=1}^3 d(u_i) \geq \frac{3}{2}(n-4).$$

It follows that $n \leq 34$, a contradiction.

Case 2. $d(v_0) \geq \frac{1}{2}(n-4)$.

Case 2.1. $V(H) - \{v_0\} \neq \emptyset$.

Case 2.1.1. There is a vertex $u_0 \in V(H) - \{v_0\}$ with $d(u_0) \geq \frac{1}{2}(n-4)$. Set $B = N(v_0) \cap N(u_0) \cap V(C)$. By Theorem 3, $|V(H)| \leq 4$, implying that

$$\begin{aligned}|A| &\geq |(N(v_0) \cup N(u_0)) \cap V(C)| \\ &= |N(v_0) \cap V(C)| + |N(u_0) \cap V(C)| - |B| \\ &\geq 2\left(\frac{1}{2}(n-4) - |V(H)| + 1\right) - |B| \geq n - 10 - |B|.\end{aligned}$$

On the other hand, by (4), $|A| \leq \frac{1}{2}(|V(C)|-2) \leq \frac{1}{2}(n-4)$. It follows that

(9) $|B| \geq \frac{1}{2}(n-16)$.

We now establish the following.

(10) If $v_i \in B$, then $u_i \vec{C} w_i$ is not a 1-segment.

Assuming the contrary to (10), let v be a vertex of H adjacent to v_{i+1}. Then $v \neq v_0$ or $v \neq u_0$, say that $v \neq v_0$. Now the cycle $v_i v_0 - v v_{i+1} \vec{C} v_i$ is longer than C. This contradiction proves (10).

Using (9) and (10) we obtain

$$n-2 \geq |V(C)| \geq 3|B| + 2(|A|-|B|) = 2|A| + |B| \geq 2\left(\frac{1}{2}(n-4) - 3\right) + \frac{1}{2}(n-16),$$

implying that $n \leq 32$, a contradiction.

Case 2.1.2. $d(x) < \frac{1}{2}(n-4)$ for all $x \in V(H) - \{v_0\}$.

Since G is 1-tough, v_0 is not a cut vertex of G, implying the existence of $v_i \in A$ and $v \in V(H)$ with $vv_i \in E(G)$ and $d(v) < \frac{1}{2}(n-4)$. Since $d(u_i, v) = 2$, we have $d(u_i) \geq \frac{1}{2}(n-4)$. Define

$$\begin{aligned}S_0 &= \{x \in V(C) \mid v_0 x \in E(G)\}, \\ S_1 &= \{x \in V(C) \mid u_i x^+ \in E(G)\}, \\ T_0 &= \{x \in V(G) - V(C) \mid v_0 x \in E(G)\}, \\ T_1 &= \{x \in V(G) - V(C) \mid u_i x \in E(G)\}.\end{aligned}$$

Using (1) we have $S_0 \cap S_1 = T_0 \cap T_1 = \emptyset$. Furthermore, if $v_j \in S_0$ and $v_j \neq v_i$, then $u_j \notin S_0$ and also $u_j \notin S_1$, otherwise the cycle $v_i v - v_0 v_j \overleftarrow{C} u_i u_j^+ \vec{C} v_i$ would be longer than C. Observing that $v_0 \notin T_0 \cup T_1$, we conclude

$$\begin{aligned}n-4 \leq d(v_0) + d(u_i) &= |S_0| + |T_0| + |S_1| + |T_1| = |S_0 \cup S_1| + |T_0 \cup T_1| \\ &\leq (|V(C)| - |S_0| + 1) + (|V(G)| - |V(C)| - 1) = n - |S_0|,\end{aligned}$$

so that $|S_0| \le 4$. On the other hand, by Theorem 3, $|S_0| \ge \frac{1}{2}(n-4) - |V(H)| + 1 \ge \frac{1}{2}(n-4) - 3$. It follows that $n \le 18$, a contradiction.

Case 2.2. $V(H) = \{v_0\}$.

Then $|A| = |N(v_0)| \ge \frac{1}{2}(n-4)$. Set $R = (V(G) - V(C)) - \{v_0\}$. By (4), $|V(C)| \ge 2|A| + 2 \ge n-2$, implying that $|R| \le 1$. The case that C contains a 4-segment is settled in exactly the same way as Case 2.1 of the proof of Theorem 4. Reasoning as in the proof of Theorem 4, we may hence assume that $u_1\vec{C}w_1$ is a 2-segment or a 3-segment, $u_i\vec{C}w_i$ is a 2-segment $(i \ne 1)$ and $u_1w_i \in E(G) \cup P_C(G)$. If $u_1w_i \in P_C(G)$, then $|R| = 1$, say that $R = \{v\}$, and u_1v, $vw_i \in E(G)$, implying that the cycle $v_1v_0\ v_i\ \overleftarrow{C}\ u_1\ vw_i\vec{C}v_1$ is longer than C. Hence, in fact, $u_1w_i \in E(G)$.

Let us call a segment of C **special** if either it is a t-segment with $t \ge 2$ or it is a 1-segment whose unique vertex is adjacent to a vertex in $V(G) - V(C)$ (which is possible only if $|R| = 1$). By (1), at most one 1-segment of C is special, implying that C contains at most four special segments.

In the remainder of the proof we will frequently use observations (1), (2), and (3) without explicitly referring to them.

Suppose $1 < r < i$ and both $\{u_r\}$ and $\{u_{r+1}\}$ are nonspecial 1-segments. Then $d(u_r, u_{r+1}) = 2$ and hence either $d(u_r) \ge \frac{1}{2}(n-4)$ or $d(u_{r+1}) \ge \frac{1}{2}(n-4)$. However, $N(u_r) \subseteq A - \{v_1\}$, so $d(u_r) \le |A| - 1 \le \frac{1}{2}(|V(C)| - 2) - 1 < \frac{1}{2}(n-4)$. Similarly, $d(u_{r+1}) < \frac{1}{2}(n-4)$. This contradiction shows that every nonspecial segment in $v_2\vec{C}v_i$ is followed by a special segment. Since with the exception of at most one 2-segment all segments in $v_{i+1}\vec{C}v_1$ are 1-segments and since n is large enough, we conclude that there exists an integer s with $i+1 < s < k$ such that both $\{u_s\}$ and $\{u_{s+1}\}$ are nonspecial 1-segments. Since $d(u_s, u_{s+1}) = 2$, either $d(u_s) \ge \frac{1}{2}(n-4)$ or $d(u_{s+1}) \ge \frac{1}{2}(n-4)$. Let u_t be a vertex in $\{u_s, u_{s+1}\}$ with $d(u_t) \ge \frac{1}{2}(n-4)$. It follows that $N(u_t) = A$, whence, in particular, $u_tv_i \in E(G)$.

If $u_iv_t \in E(G)$, then the cycle $v_1v_0v_{i+1}\vec{C}v_tu_iw_i\ u_1\vec{C}v_iu_t\vec{C}v_1$ is longer than C. Hence $u_iv_t \notin E(G)$. Similarly, $w_{i-1}v_t \notin E(G)$.

Set $B = \{v_{i+1}, v_t, v_1\}$. Then $N(u_i) \cap B = \emptyset$ and, since $i+1 < t \le k$, $|B| = 3$. Hence $|N(u_i) \cap A| \le |A| - 3$. Furthermore, $N(u_i) \cap (A^+ \cup \{v_0, w_1\}) = \emptyset$. Since $|V(G) - (A \cup A^+ \cup \{v_0, w_1\})| \le 2$, it follows that $d(u_i) \le |A| - 1 < \frac{1}{2}(n-4)$. Similarly, the observations that $|N(w_{i-1}) \cap A| \le |A| - 3$ and $N(w_{i-1}) \cap A^- \cup \{v_0, u_i\}) = \emptyset$ imply $d(w_{i-1}) < \frac{1}{2}(n-4)$. However, $d(u_i, w_{i-1}) = 2$ and hence $d(u_i) \ge \frac{1}{2}(n-4)$ or $d(w_{i-1}) \ge \frac{1}{2}(n-4)$. This contradiction completes the proof. □

Remark

One of the referees pointed out that by suitably modifying the proof of Theorem 5 the following extension of Theorem 5 can be obtained.

Theorem 5a. *Let $p \geq 4$ be an integer and G a 3-connected 1-tough graph on $n \geq 12p - 13$ vertices such that, for all vertices x, y, $d(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{1}{2}(n - p)$. Then there exists a longest cycle C of G such that C is a dominating cycle and $|V(C)| \geq n - p + 4$.*

Similarly, a simple modification of the proof of Theorem 4 yields the following extension of Theorem 4.

Theorem 4a. *Let $p \geq 4$ be an integer and G a 1-tough graph on $n \geq 3$ vertices such that $\sigma_3 \geq n$ and, for all vertices x, y, $d(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq \frac{1}{2}(n - p)$. Then every longest cycle C of G is a dominating cycle and $|V(C)| \geq n - p + 4$.*

We do not believe, however, that Theorem 4a and Theorem 5a are best possible for $p > 5$ (cf. [3]).

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