Disconnection numbers, pizza cuts, and cycle rank

A.J.W. Hilton and J.K. Dugdale
Department of Mathematics
University of Reading
Whiteknights
Reading RG6 2AX U.K.
and
Department of Mathematics
West Virginia University
Morgantown, W.V. 26506 U.S.A.

ABSTRACT. In this paper we bring out more strongly the connection between the disconnection number of a graph and its cycle rank. We also show how to associate with a pizza sliced right across in a certain way with $n-2$ cuts a graph with n vertices, and show that if the pizza is cut thereby into r pieces, then any set of $r-1$ of these pieces corresponds to a basis for the cycle space of the associated graph. Finally we use this to explain why for $n \geq 3$ the greatest number of regions that can be formed by slicing a pizza in the certain way with $n-2$ cuts, namely $\frac{1}{2}(n^2-3n+4)$, equals the disconnection number of K_n.

Introduction

If S is a compact connected metric space and there is a cardinal number $p \leq \aleph_0$ such that whenever $A \subset S$, $|A| = p$ then $S - A$ is not connected, then the smallest such p is the *disconnection number* $D(S)$ of the space S. This was introduced recently by S.B. Nadler, Jr. ([2,Chap. 9], [3]), where he also showed that if S has a disconnection number and if $D(S) \geq 2$ then S is a finite connected graph. He also showed that if G is a finite connected graph then $D(G) = 2 + |E(G)| - |V(G)| + |P(G)|$, where $E(G)$, $V(G)$ and $P(G)$ are the sets of edges, vertices and end vertices of G, respectively. For a finite graph G without isolated vertices, the number $|E(G)| - |V(G)| + |C(G)|$, where $C(G)$ is the set of components of G, is the *cycle rank* $c(G)$ of G, that is, the dimension of the cycle space of G (see [1]). Thus for a finite connected graph, $D(G) = 1 + c(G) + |P(G)|$.

In Section 1 we extend the definition of the disconnection number of finite graphs to include finite graphs which are not connected, and provide a proof of the corresponding extension of Nadler's result; this proof brings out the connection between the disconnection number and the cycle rank more vividly.

From the formula above, it follows that the disconnection number of K_n is $\frac{1}{2}(n^2-3n+4)$. For $n = 3, 4, 5, \ldots$ this yields the sequence $2, 4, 7, 11, 16, 22, \ldots$ which is known in some circles as the ***pizza slice sequence.*** It is the greatest number of pieces you can slice a pizza into with $(n-2)$ straight cuts. For a proof of this, see [4], pages 261 and 286.

The question naturally arises as to whether there is any kind of combinatorial connection between the disconnection numbers of graphs and the number of parts a pizza is sliced into. In Section 3 we explain what the connection is; we show that it lies in the fact that, for a wide class of pizza cuts, the number of parts a pizza is cut up into equals one plus the cycle rank of an associated graph.

1 The disconnection number and the cycle rank of a graph

We first remark that the graphs we consider consist of a finite number of vertices, and a finite number of arcs, or edges. Each arc is homeomorphic to the closed unit interval, and joins two distinct vertices. Thus loops are not permitted. In this paper there is no loss of generality in assuming that any two vertices are joined by at most one edge (since we can introduce extra vertices). Two distinct arcs do not meet except possibly at one of their end vertices. If a point p of G is a vertex, then the *order* of p is the number of edges it is incident with (i.e. the degree), and if p is an interior point of an edge, then its order is two. If F is a subgraph of a graph G, the notation $G\backslash E(F)$ means the subgraph of G with the same vertex set as G, but with the interior of each edge of F removed. An edge is *pendant* if it has a vertex of degree one. As remarked earlier, the notation $P(G)$ denotes the set of vertices of degree one (end vertices); we let $E(P(G))$ denote the set of interior points of edges incident with vertices of $P(G)$.

Let G be a graph without isolated vertices. The ***disconnection number*** $D(G)$ of G is the least number q such that the removal of any q points from G disconnects at least one of the components of G. Here we use the word 'disconnects' in the topological sense. Clearly $D(G)$ is one greater than the greatest number of points which can be removed from G without disconnecting any of the components. If G is a graph and S is a set of points of G, let $C(G\backslash S)$ be the set of (topological) components of $G\backslash S$. The topological foundation for the maneuvers below may be found in Nadler's paper [3]. Essentially the proofs of Lemma 1 and Theorem 1 below provide an alternative to his proof of his Theorem 5.1, and in this alternative proof the connection with the cycle space is brought out more.

Lemma 1. *Let G be a graph without isolated vertices. Let S be a finite set of points of G of order at least two, and let $|S|$ be maximal with the property that $|C(G\backslash S)| = |C(G)|$. Then $|S| = c(G)$.*

Proof: Let S be as large as possible such that $|C(G\backslash S)| = |C(G)|$. By hypothesis, the points in S have order at least two. If S contains a point of order greater than two, such a point is a vertex v of G, and it can be replaced by an interior point of an edge incident with v. Thus we may assume that each vertex of S has order exactly two. Clearly no edge of G contains two or more points of S, for if it did then $|C(G\backslash S)| > |C(G)|$. Thus each point of S is an interior point of an edge, and it is the only interior point of that edge in S. Let $(G\backslash S)^*$ denote the graph obtained from $G\backslash S$ by removing the interior points of all edges which contain a point of S. Then $|C((G\backslash S)^*)| = |C(G\backslash S)|$. Moreover $(G\backslash S)^*$ is a spanning forest of G and it contains $|C(G\backslash S^*)| = |C(G)|$ trees. Therefore

$$\begin{aligned} |S| &= |E(G)| - |E((G\backslash S)^*)| \\ &= |E(G)| - (|V(G\backslash S)^*)| - |C(G\backslash S)^*)|) \\ &= |E(G)| - (|V(G) - |C(G)|) \\ &= c(G). \end{aligned}$$

Lemma 1 now follows. □

Theorem 1. *Let G be a graph without isolated vertices. Then $D(G) = 1 + c(G) + |P(G)|$.*

Proof: We may suppose that each component of G has at least three edges (by introducing further vertices if necessary). Let S' be a set of points of G with $|C(G\backslash S')| = |C(G)|$, and let $|S'|$ be as large as possible. It is clear that S' includes all points of order 1, that is, all end vertices, and includes no interior points of any pendant edge. The set $S'\backslash P(G)$ satisfies Lemma 1 with respect to the graph $G\backslash(P(G) \cup E(P(G)))$, so

$$|S'| - |P(G)| = |S'\backslash P(G)| = c(G\backslash(P(G) \cup E(P(G)))) = c(G).$$

Since $D(G) = 1 + |S'|$, the theorem follows. □

We remark that the disconnection number of K_n can be deduced immediately:

$$\begin{aligned} D(K_n) &= 1 + c(K_n) \\ &= 2 + |E(Kn)| - |V(K_n)| \\ &= 2 + \binom{n}{2} - n \\ &= \frac{1}{2}(n^2 - 3n + 4). \end{aligned}$$

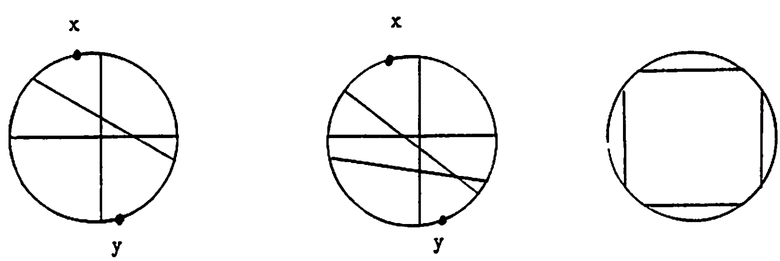
2 Pizza cuts and the cycle rank of the associated graph

We have seen that the disconnection number $D(K_n)$ of the complete graph K_n equals the maximum number of parts a pizza can be cut up into with $n-2$ cuts. Is there any combinatorial connection between these two facts, or is it a mere coincidence?

We shall show that to each cut up pizza in a fairly wide class of cut up pizzas we can associate a graph $G(P)$ whose cycle rank $c(G(P))$ and disconnection number $D(G(P))$ are closely related to the number of parts. In the case of the maximum number of parts we will see that $G(P) = K_n$, and thus the disconnection number of K_n and the number of parts a pizza can be cut up into will both be shown to be closely related in a combinatorial way to the cycle rank $c(K_n)$ of K_n. This provides the combinatorial connection we sought.

We shall consider pizzas with $n-2$ straight line cuts, each going right across starting on the boundary and finishing on the boundary. We shall suppose that no three cuts meet at the same point. Let the separated parts we obtain be referred to briefly as *parts*.

Given two points x and y on the boundary of the pizza, let the two arcs of the boundary with end points x and y be denoted by $A_1(x,y)$ and $A_2(x,y)$. For $n \geq 3$, let $P(n-2)$ be the set of all pizzas cut with $n-2$ cuts with the property that there are two points, x and y, on the boundary of the pizza, such that each of the straight line cuts meets each of $A_1(x,y)$ and $A_2(x,y)$ exactly once (and so does not go through x or y). This is illustrated in Figure 1.



In P(3) In P(4) Not in P(4).

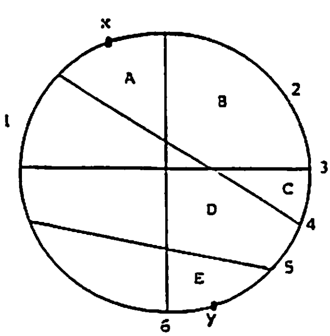
Figure 1

With each cut up pizza in $P(n-2)$, $n \geq 3$, we associate a graph with n vertices in the following way. The two circular arcs with endpoints x and y forming the boundary of the pizza are labeled 1 and 2, and the remaining $n-2$ cuts are labeled $3, \ldots, n$. The associated graph has vertex set $\{v_1, \ldots, v_n\}$. The vertices v_1 and v_2 are each joined by an edge to all other vertices, and for $3 \leq i < j < n$, the vertices v_i and v_j are joined by an edge if and only if the cut labeled i intersects the cut labeled j in the interior of the pizza. For each cut up pizza P in $\cup_{n=3}^{\infty} P(n-2)$, let $G(P)$ denote the associated graph.

Lemma 2. *For $n \geq 3$, each partition into two non-empty subsets of the set of parts of a cut up pizza $P \in P(n-2)$ corresponds to exactly one non-zero cycle vector in the cycle space of $G(P)$.*

Proof: (i) Note that the condition that each cut has x on one side and y on the other implies that there are no parts or unions of parts bounded by one cut and one of the boundary arcs. Moreover, for $n \geq 4$ each part is bounded by a distinct set of lines (or arcs). Of course, if Q is a set of parts, then Q is bounded by the same set of lines (or arcs) as is $\overline{Q}$, the complementary set of parts, except possibly for the inclusion or exclusion of a complete arc. [If $n = 3$ there are two parts, one the complement of the other, and these are both bounded by the line and the two boundary arcs.]

The labels on the lines (and arcs) bounding each part of P, taken in order going round the boundary, correspond to a cycle in $G(P)$. Thus each union of a set of parts of P corresponds to a unique cycle vector in the cycle space of $G(P)$. The union of the complementary set of parts corresponds to the same cycle vector [We assume here that neither the set of parts, nor the complementary set of parts is empty]. Thus a partition of the set of parts into two non-empty sets corresponds to a unique nonzero cycle vector in the cycle space of $G(P)$. Note that mod 2 summation (i.e. cancellation) takes place in forming the cycle vector. If a complete arc is contained in the bounding set of lines (or arcs) of the set Q of parts, it is not included in the cycle vector corresponding to Q.



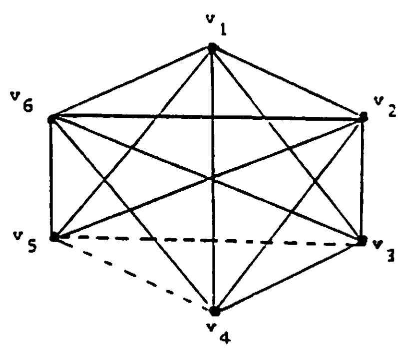


Figure 2

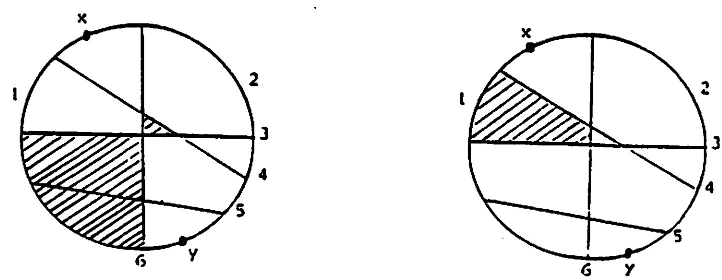
[*Aside.* We illustrate the process of forming a cycle vector from the union of cycles corresponding to the union of several parts in the following example. The example illustrates the point about the case when a complete arc is contained in the boundary of the union. In this case we actually obtain a cycle in $G(P)$, rather than the edge-disjoint union of cycles. This is to be expected since the union of A,B,C,D and E forms a connected region. Note that this cycle does not include v_2, and that the whole arc 2 is in the boundary of the union of A,B,C,D and E].

A corresponds to the cycle with edges $v_6v_2, v_2v_1, v_1v_4, v_4v_6$.
B corresponds to the cycle with edges $v_3v_2, v_2v_6, v_6v_4, v_4v_3$.
C corresponds to the cycle with edges v_2v_3, v_3v_4, v_4v_2.
D corresponds to the cycle with edges $v_2v_4, v_4v_3, v_3v_6, v_6v_5, v_5v_2$.
E corresponds to the cycle with edges $v_6v_1, v_1v_2, v_2v_5, v_5v_6$.

The modulo 2 sum of all these cycles is

$$v_1v_4, v_4v_3, v_3v_6, v_6v_1.$$

(ii) If a cycle in $G(P)$ is traversed, the corresponding lines (or arcs) bound a part, or union of parts, of the pizza. [*Aside.* Note here that the order of the vertices is important, as different orders correspond to different sets of edges in $G(P)$. This is illustrated in Figure 3.]



3 parts correspond to the cycle (v_1, v_3, v_4, v_6)

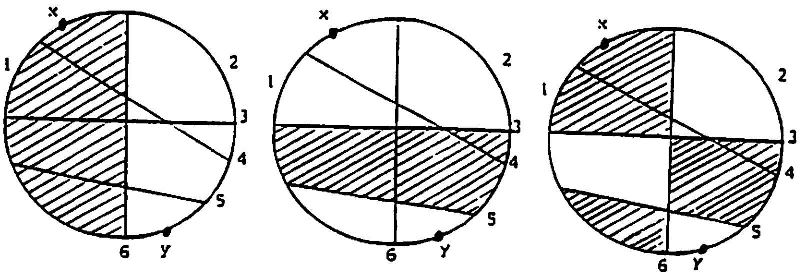
1 part corresponds to the cycle (v_1, v_4, v_6, v_3)

Figure 3

Thus a cycle corresponds to a partition of the set of parts of the pizza into two. If T is the union of parts corresponding to some cycle, let $\overline{T}$ denote the complementary set of parts; then $\{T, \overline{T}\}$ represents the partition of the set of parts corresponding to the cycle. If U is a set of parts corresponding to some other cycle, let $T\Delta U$ be the symmetric difference of T and U. Then

$$\{T\Delta U, \overline{T\Delta U}\} = \{\overline{T}\Delta U, \overline{\overline{T}\Delta U}\} = \{T\Delta \overline{U}, \overline{T\Delta \overline{U}}\} = \{\overline{T}\Delta \overline{U}, \overline{\overline{T}\Delta \overline{U}}\},$$

and this is the partition corresponding to the union of the two cycles; clearly it is unique. Of course the symmetric difference operator Δ corresponds to mod 2 summation; this is illustrated in Figure 4. This idea extends to any non-zero cycle vector, and shows that to each non-zero cycle vector there corresponds a unique partition into two non-empty parts of the set of parts of the pizza. This proves Lemma 2. □



The partition corresponding to the cycle (v_1, v_6, v_2)

The partition corresponding to the cycle (v_1, v_5, v_3)

The partition corresponding to the two cycles taken together.

Figure 4

Theorem 3. *For $P \in P(n-2)$, the dimension of the cycle space of $G(P)$ is one less than the number of parts in the cut up pizza P.*

Proof: If $p_1, \ldots, p_r$ are the set of parts of P, then any set R of $r-1$ of them are independent, since no one is the union or the complement of the union, of the remaining parts of R. However any one of $p_1, \ldots, p_r$ is the complement of the unit of the remainder, and any set Q of parts is the union, or the complement of the union, of parts of R. Thus R corresponds to a basis of the cycle space of $G(P)$. □

We can now use this connection between the dimension of the cycle space and the number of parts to provide a new proof of the fact that the greatest number of parts a pizza $P \in P(n-2)$ can have is $\frac{1}{2}(n^2 - 3n + 4)$.

Theorem 4. *Any cut up pizza $P \in P(n-2)$ with the maximum number of pieces has $P(G) = K_n$, and the number of parts is $\frac{1}{2}(n^2 - 3n + 4)$.*

It is easy to construct a cut up pizza P in $P(n-2)$ such that $G(P) = K_n$; it is only necessary to ensure that each slice cuts each other slice at a distinct point inside the pizza, and also cuts both xy-arcs.

Proof: Now let P be any cut up pizza in $P(n-2)$. Then the number of parts in P is $1 + c(G(P))$. But

$$\begin{aligned} 1 + c(G(P)) &\le 1 + c(K_n) \\ &= 1 + (|E(K_n)| - |V(K_n)| + 1) \\ &= 2 + \binom{n}{2} - n \\ &= \frac{1}{2}(n^2 - 3n + 4). \end{aligned}$$

□

To sum up, we have shown that

$$\begin{aligned}
&D(K_n)\\
&= 1 + c(K_n)\\
&= \text{The greatest number of parts a pizza } P \in P(n-2) \text{ can have}\\
&= \frac{1}{2}(n^2 - 3n + 4).
\end{aligned}$$

Acknowledgement

I would like to thank Henry Gould who pointed out to Sam Nadler that $\frac{1}{2}(n^2 - 3n + 4)$, the disconnection number of K_n, was also the pizza cut number, and Sam Nadler, who raised the question of finding a combinatorial link between these facts. I would also like to thank the referee for a number of useful comments about the first version of this note.

References

[1] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusetts, 1969.

[2] S.B. Nader, Jr., *Continuum Theory, an Introduction*, Vol. 158, Monographs and Textbooks in Mathematics, Marcel Dekker, 1992.

[3] S.B. Nader, Jr., *Continuum Theory and Graph Theory: Disconnection Numbers*, J. London Math. Soc., to appear.

[4] A.W. Tucker, Applied Combinatorics.