A Characterization of Parity Graphs Containing no Cycle of Order Five or Less*

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ABSTRACT. The class of parity graphs, those in which the cardinality of every maximal independent subset of vertices has the same parity, contains the well covered graphs and arose in connection with the PSPACE-complete game "Generalized Kayles". In 1983 [5] we characterized parity graphs of girth 8 or more. This is extended to a characterization of the parity graphs of girth greater than 5. We deduce that these graphs can be recognized in polynomial time.

1 Introduction

Let G be a simple graph with vertex set $V(G)$. For $x \in V(G)$ and each integer k we define: $N_k(x) = \{v \in V(G) : d(x,v) = k\}$ and $N_k[x] = \{v \in V(G) : d(x,v) \le k\}$ where $d(x,v)$ is the edge distance between x and v. When $k = 1$ the subscript is usually dropped. If S is a set of vertices, we write $N(S) = \cup\{N(v) : v \in S\}$ and $N[S] = \cup\{N[v] : v \in S\}$. A set of mutually nonadjacent vertices of a graph is called *independent*. An independent set S *covers* a vertex v if $v \in N[S]$. An independent set S *isolates* a vertex v if $N[v] - N[S] = \{v\}$.

A graph is called *well covered* if every maximal independent set is of the same size. The class of well covered graphs, in which the NP-complete problem (see M.R. Garey and D.S. Johnson [10], p.194 [GT20]) of finding a maximum independent set is trivial, was first studied by M. D. Plummer [11] in 1970. Since then considerable effort has been expended towards the

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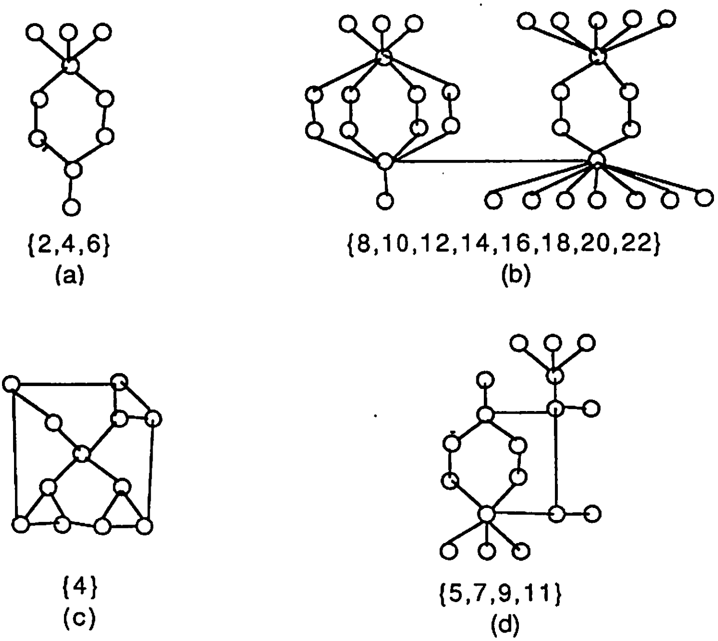
elucidation of this class (among others [1]–[4], [6]–[9], [12], [13], [16], and [17].)

A graph G is said to be a ***parity graph*** if every pair of maximal independent subsets S and T of $V(G)$ satisfy $|S| \equiv |T|$ (mod 2), where $|S|$ and $|T|$ are the cardinalities of S and T respectively. See Figure 1 for some examples. This class too is one in which an NP-complete problem, indeed a PSPACE-complete, problem is trivial.



The set accompanying each graph is the set of cardinalities of maximal independent sets. Note that graph (c) is well covered.

Figure 1

The game *"Generalized Kayles"* is played by two on an arbitrary graph G. They alternate removing a vertex and its neighbours from G, the winner being the last player with a nonempty vertex set from which to choose. The problem of whether the first player can force a win is PSPACE-complete (see T.J. Schaefer [14] and Garey and Johnson [10], p.254 [GP3]), but if G is known to be a parity graph the problem becomes easy to solve.

Recently, R.S. Sankaranarayana and L.K. Stewart [13], V. Chvatal, as well as P.J. Slater [15] have independently shown that the recognition problem for well covered graphs is co NP-complete. Their arguments apply equally to the recognition problem for parity graphs showing that it is also co NP-complete.

In this paper we characterize the parity graphs of girth greater than 5 in such a way as to show that these graphs can be recognized in polynomial time. This extends the work of A. Finbow and B. Hartnell [5] which characterized parity graphs of girth 8 or more and parallels the work of A. Finbow, B. Hartnell and R. Nowakowski [6] which characterized well covered graphs of girth 5 or more (see Theorem 2.)

2 Results

The main result is Theorem 1. For ease of exposition, the proof is broken down into a series of lemmas.

Lemma 1. *If G is a parity graph then so is $G - N[v]$ and $G - N[S]$ for vertex v and independent set S.*

Proof: If $G - N[S]$ were not a parity graph, then there would exist maximal independent subsets R and T of $V(G - N[S])$ of different parity. However, this cannot happen since then $R \cup S$ and $T \cup S$ would be maximal independent subsets of $V(G)$ of different parity. □

The following terminology will be useful in what follows. A *leaf* is a vertex of degree one while a *stem* is a vertex adjacent to a leaf. A *bush* is a subgraph induced by a stem and its leaves and is called *odd* or *even* when the number of leaves involved is respectively odd or even. Finally, a vertex which cannot be isolated in G is called *extendable (with respect to G)*. When no confusion will arise, the parenthetical modifier is dropped.

Lemma 2. *If G is a parity graph of girth ≥ 6 then $v \in V(G)$ is extendable if and only if v is a stem with an odd number of leaves attached.*

Proof: If v is a stem it cannot be isolated and thence is extendable. On the other hand if v is extendable then, since by the girth restriction $N_2(v)$ is independent, v has neighbours in $G - N[N_2(v)]$ which must be leaves. Extend $N_2(v)$ to a maximal independent set T which does not contain v. Then $G - N[T]$ consists of v and its set of leaves L. By Lemma 1 this graph is a parity graph and clearly $\{v\}$ and L are its only maximal independent sets. Hence $|L|$ is odd. □

Since any vertex of G is either extendable or not extendable and since any stem must be extendable, it follows that:

Lemma 3. *If G is a parity graph of girth ≥ 6 then each $v \in V(G)$ has either an odd number of leaves attached or has no leaves attached.*

The role of nonextendable vertices in this setting turns out to be very special.

Lemma 4. *Let v be a nonextendable vertex in a parity graph G of girth ≥ 6. Then $\deg(v) \leq 2$.*

Proof: Assume that the Lemma is false and suppose that H is a parity graph with the minimum number of vertices and with a nonextendable vertex v of degree strictly greater than two. Observe that $\deg(v) = 3$, for if S were an independent set isolating v and $w \in S \cap N_2(v)$ then $H - N[w]$ would be a smaller counter example. We can also assume that $N_3[v] \supseteq H$ for if some vertex, say w, was at a distance greater than three from v then the component of $H - N[w]$ containing v would be a smaller counter example.

Let $N(v) = \{x_1, x_2, x_3\}$ and $Y_i = N(x_i) - \{v\}$, $i = 1, 2, 3$. Since v is nonextendable, Y_i is not empty. We note that if $N_3(v)$ were empty then the girth conditions would imply that Y_i is a set of leaves for x_i and hence $|Y_i|$ would be odd by Lemma 3. But then the odd set $\{x_1, x_2, x_3\}$ as well as the even set $\{v\} \cup Y_1 \cup Y_2 \cup Y_3$ would both be maximal independent sets of H which would contradict the fact that H is a parity graph. Thus $N_3(v)$ is not empty.

Now since H is minimal, for each $u \in N_3(v)$, v is extendable in $H - N[u]$ and thus each u must be adjacent to all members of Y_j for some $j \in \{1, 2, 3\}$. It follows, by the lack of 4-cycles, that $|Y_j| = 1$. Since $N_3(v) \neq \emptyset$ at least one such j exists. Without loss of generality, $j = 1$ and we let $Y_1 = \{y_1\}$.

If every vertex in $N_3(v)$ were adjacent to y_1, then by considering $H - N[\{x_2, x_3\}]$ in which $N_3(v) \cup \{x_1\}$ would be leaves at y_1 we observe that $|N_3(v)|$ would have to be even. Now $N_3(v) \cup \{x_1, x_2, x_3\}$ and $\{y_1, v\} \cup Y_1 \cup Y_2$ are both maximal independent sets of the same parity forcing $|Y_2| + |Y_3|$ to be odd. This implies that one of Y_2 and Y_3, say Y_2, is of even parity. But then in $H - N[y_1]$ x_2 has an even number of leaves attached (namely, the set Y_2). Hence there must be some vertex, say $u_3 \in N_3(v)$, which is not adjacent to y_1.

Again, since v is extendable in $H - N[u_3]$, u_3 must be adjacent to all of Y_2 or Y_3 forcing that set, say Y_3, to be just one vertex, say y_3.

Next observe that if there is a vertex $u_2 \in N_3(v)$ adjacent to neither y_1 nor y_3 then $Y_2 = \{y_2\}$ and u_2 must be adjacent to y_2. If u_3 were adjacent to y_2 then v would have two leaves in $H - N[u_3]$ (recall that u_3 is not adjacent to y_1). Hence u_3 is adjacent to y_3 but not y_1 nor y_2. Since $G - N[\{u_2, u_3\}]$ would contain two leaves at v, u_2 must be adjacent to u_3.

Now by assumption y_1 has a neighbour u_1 in $N_3(v)$. If it were adjacent to both y_2 and y_3 a 5-cycle would result. If it were adjacent to y_1 and to exactly one of y_2 and y_3 then$H - N[u_1]$ would contain two leaves at v. Hence u_1 is adjacent to y_1 and to neither y_2 nor y_3. By the girth condition either u_1 and u_3 or u_1 and u_2 are independent, say u_1 and u_i. But then $H - N[\{u_1, u_i\}]$ contains two leaves at v.

Hence $N(y_1) \cup N(y_3) \supseteq N_3(v)$.

Observing that $H - N[\{y_1, y_3\}]$ contains the leaves $\{v\} \cup Y_2$ at x_2 we note that $|Y_2|$ is even. Hence $|Y_2| \geq 2$ (as $Y_i \neq \emptyset$).

Now $H - N[Y_2 \cup \{y_3\}]$ must contain a neighbour of y_1 in $N_3(v)$, say u_1, for otherwise there would be two leaves at x_1. Furthermore u_1 is adjacent to all u adjacent to y_3 for otherwise $H - N[\{u_1, u\}]$ contains two leaves at v. Therefore y_3 can have only one neighbour in $N_3(v)$, say u_3. Similarly $H - N[Y_2 \cup \{y_1\}]$ must contain a vertex in $N_3(v)$ which by the previous paragraph is u_3 and u_1 must be the unique neighbour of y_1 in $N_3(v)$. But then, in $H - N[\{u_1, y_3\}]$, Y_2 would be the set of leaves at x_2 which is impossible as $|Y_2|$ is even. □

Now we investigate the nonextendable vertices of degree two.

Lemma 5. *Let v be a nonextendable vertex of degree two in a parity graph G of girth at least six where G is not C_7. Then one of the neighbours of v must be extendable and the other must be nonextendable and of degree two.*

Proof: Let the neighbours of v be u and x and assume that both are extendable. Then u must have an odd number of leaves attached (by Lemma 2). Let a be a leaf attached to x and consider $G' = G - N[a]$. By the girth condition v is the only common neighbour of u and x and since $\deg(v) = 2$, G' has an even number of leaves at u which is a contradiction. Hence v has at most one extendable neighbour.

Next assume that both neighbours, say u and x, of v are nonextendable. Since v is nonextendable neither u nor x can be a leaf and hence, by Lemma 4, each has degree two. Let $N_2(v) = \{u_2, x_2\}$ where u_2 is adjacent to u and x_2 is adjacent to x. Since u and x are nonextendable neither u_2 nor x_2 can be leaves. But if u_2 has a neighbour $a \neq u$ and x_2 has a neighbour $b \neq x$, then $G - N[\{a, b\}]$ (or $G - N[a]$ if $a = b$) has two leaves at v forcing a and b to be adjacent. This must hold for all such pairs a and b which with the girth condition (no 4-cycles) implies that u_2 and x_2 must be of degree two. Furthermore if either a or b, say a, has degree greater than two, then if $a' \notin \{b, u_2\}$ is a neighbour of a, $G - N[\{a', x_2\}]$ has two leaves at u. But then, with a and b forced to be of degree two, G must be C_7 which has been excluded by hypothesis. This completes the proof. □

Lemma 6. *Suppose G is not C_7 but is a parity graph of girth at least six and that v_1 is a nonextendable vertex in G of degree two. Then there exist extendable vertices x and y in G and distinct nonextendable vertices of degree two $\{u_i, v_i\}$, for $i = 1, 2, \ldots, k$, where k is an even integer such that $N(u_i) = \{x, v_i\}$ and $N(v_i) = \{u_i, y\}$ for each $i = 1, 2, \ldots, k$.*

Proof: By Lemma 5, v_1 must have an extendable neighbour, say y, and a nonextendable neighbour, say u_1, of degree two. Similarly, u_1 must have an extendable neighbour which we can call x (x and y are distinct by the girth condition). Now consider the set S of leaves of x in G and observe that if T

is the set of leaves of x in $H = G - N[y]$, then $T = S \cup \{z : \deg(z) = 2$ and $z \in N(x) \cap N_2(y)\}$. But any $z \in T - S$ must be nonextendable in G since it can be isolated by $\{y\} \cup S$. Since $|T|$ and $|S|$ are both odd (Lemma 2), $|T - S|$ must be even. Let $T - S = \{u_1, u_2, \ldots, u_k\}$ noting that u_1, as defined, must belong to $T - S$. By Lemma 5 each u_i has a neighbour v_i which is nonextendable and of degree two. By the girth condition, $v_i \neq v_j$ if $i \neq j$. Furthermore $u_i \in N(x) \cap N_2(y)$ so that the extendable neighbour of each v_i must be y. □

In view of these results it is natural to make the following definition: vertices u and v of G are said to be connected by a 2-bridge if there are vertices x and $y \in V(G)$ with $\deg(x) = \deg(y) = 2$ and both $N(x) = \{u, y\}$ and $N(y) = \{x, v\}$. We can now summarize our results as follows:

Theorem 1. *Let G be a connected parity graph of girth ≥ 6. Then G is K_1, C_7 or G is a connected graph of girth at least six which consists of a finite union of odd bushes B_i each with stem v_i where, for each i and j, one and only one of the following hold:*

i) v_i and v_j are joined by an edge and any other path, if there is one, joining v_i and v_j must include at least one stem other than v_i and v_j.

ii) v_i and v_j are connected by $2n$ 2-bridges and any other path joining v_i and v_j must include another stem besides v_i and v_j.

iii) every path joining v_i and v_j contains at least one stem other than v_i and v_j.

Finally, it is interesting to restate the girth ≥ 6 part of the results for well covered graphs obtained in [6]:

Theorem 2. *Let G be a connected well covered graph of girth ≥ 6. Then G is K_1, C_7 or G is a connected graph of girth at least six which consists of a finite union of leaves L_i each with stem v_i where, for each i and j, one and only one of the following hold:*

i) v_i and v_j are joined by an edge and any other path, if there is one, joining v_i and v_j must include at least one stem other than v_i and v_j.

ii) every path joining v_i and v_j contains at least one stem other than v_i and v_j.

These two theorems together beg the question: "How hard is the recognition question for well covered graphs when the graphs in question are known to be parity graphs?"

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