Numbers of common triples in simple balanced ternary designs

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ABSTRACT. In this note the numbers of common triples in two simple balanced ternary designs with block size 3, index 3 and $\rho_2 = 3$ are determined.

1 Preliminaries and notation

A *balanced ternary design* is a collection of multi-sets of size k, chosen from a v-set in such a way that each element occurs 0, 1 or 2 times in any one block, each pair of non-distinct elements, $\{x, x\}$, occurs in ρ_2 blocks of the design and each pair of distinct elements, $\{x, y\}$, occurs λ times throughout the design. We denote a design with these parameters by $(v; \rho_2; k, \lambda)$ BTD. A BTD on element set V is denoted by (V,B), where B is the collection of multi-subsets of V. It is easy to see that each element must occur singly in a constant number of blocks, say ρ_1 blocks, and so each element occurs altogether $r = \rho_1 + 2\rho_2$ times. Also if b is the number of blocks then

$$vr = bk \quad \text{and} \quad \lambda(v-1) = r(k-1) - 2\rho_2. \tag{1}$$

(For further information [1] should be consulted.) A BTD is called *simple* if it has no repeated blocks. Finally, let (V_1, B_1) and (V_2, B_2) be two BTDs. We say the design (V_1, B_1) contains the design (V_2, B_2) as a sub-design if $V_2 \subseteq V_1$ and $B_2 \subseteq B_1$. In this note we deal with BTDs with $k = \lambda = 3$. Thus all blocks are of the form xyz or xxy. It is straightforward to show that since $k = \lambda = 3$ such a design must satisfy the following properties:

(i) if $\rho_2 \equiv 1$ or 2 (mod 3), then $v \equiv 3 \pmod 6$;

(ii) if $\rho_2 \equiv 0 \pmod 3$, then $v \equiv 1$, 3 or 5 (mod 6). Moreover, since $k = \lambda = 3$ we have $b > \rho_2 v$. Now equalities given in (1) can be used to show $v > 2\rho_2 - 1$. But since v is odd we obtain $v \geq 2\rho_2 + 1$.

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The intersection problem for triple systems has been considered in the past. Some of the results obtained are as follows. Let $I_{\rho_2,\lambda}(v)$ denote the set of integers n such that there exist two simple $(v;\rho_2;3,\lambda)$ BTDs based on a common v-set and having n common blocks. Note that when $\rho_2 = 0$ and $\lambda = 1$, a $(v;0;3,1)$ BTD is a Steiner triple system of order v (STS(v)).

Theorem 1 ([5]). *Let $v \equiv 1$ or $3 \pmod 6$. Then for all $v \geq 13$, $I_{0,1}(v) = \{0,1,2,3,...,\alpha\} \setminus \{\alpha - j \mid j = 1,2,3,5\}$, where $\alpha = v(v-1)/6$, and $I_{0,1}(3) = \{1\}$, $I_{0,1}(7) = \{0,1,3,7\}$ and $I_{0,1}(9) = \{0,1,2,3,4,6,12\}$.*

Theorem 2 ([2]). *Let $v \equiv 0 \pmod 3$. Then $I_{1,2}(v) = \{0,1,2,3,...,v^2/3\} \setminus \{v^2/3-1, v^2/3-2\}$ with the one exception: $5 \notin I_{1,2}(6)$.*

Theorem 3 ([3]). *Let $v \equiv 0$ or $2 \pmod 3$. Then for all $v > 6$, $I_{2,2}(v) = \{0,1,2,3,...,\alpha\} \setminus \{\alpha-1,\alpha-2\}$, where $\alpha = v(v+1)/3$, and $I_{2,2}(5) = \{0,3,4,5,6,7,10\}$.*

Theorem 4 ([4]).

(i) Let $v \equiv 0$ or $1 \pmod 3$. Then for all $v \geq 9$, $I_{3,2}(v) = \{0,1,2,3,...,\alpha\} \setminus \{\alpha-1,\alpha-2\}$, where $\alpha = v(v+2)/3$, and $I_{3,2}(7) = \{0,3,4,5,...,17, 18,21\}$.

(ii) Let $v \equiv 0 \pmod 3$. Then for all $v \geq 11$, $I_{4,2}(v) = \{0,1,2,3,...,\beta\} \setminus \{\beta-1,\beta-2\}$, where $\beta = v(v+3)/3$, and $I_{4,2}(9) = \{0,3,4,5,...,32, 33,36\}$.

We use the Stern and Lenz Lemma (Theorem 5 below) and thus provide the following necessary notation for this lemma. Using the notation of difference methods, if G is a graph with V(G)= $\{0,1,2,...,g-1\}$ then the edge $\{u,v\}$ in G is defined to have difference $|u-v| = \min\{u-v, g-(u-v)\}$. Let $D \subseteq \{1,2,...,\lfloor g/2 \rfloor\}$, and let G$(D,g)$ be the graph with vertex set $\{0,1,2,...,g-1\}$ and edge set containing all edges having a difference in D; that is, the edge set of G(D,g) is $\{\{u,v\} \mid |u-v| \in \text{D}\}$.

Theorem 5 ([6]). *If D contains an element d where $g/gcd(\{d,g\})$ is even, then $G(D,g)$ has a 1-factorization.*

In this paper we prove the following result:

Theorem 6. *Let $v \equiv 1 \pmod 2$ and $v \geq 9$. Then $I_{3,3}(v) = \{0,1,2,3,...,\alpha\} \setminus \{\alpha-1,\alpha-2\}$, where $\alpha = v(v+1)/2$, and $I_{3,3}(7) = \{0,1,3,4,5,...,25,28\}$.*

For brevity, we use the notation $T_{i_1 i_2 i_3 \dots i_k}$ instead of $T_{i_1} \cup T_{i_2} \cup T_{i_3} \cup \dots \cup T_{i_k}$ and we define $J_3(v) = \{0,1,2,3,...,v(v+1)/2\} \setminus \{v(v+1)/1-1, v(v+1)/2-2\}$. Finally, the graph $G^{++\cdots+}$ is the graph G with n loops per vertex (denoted by ++...+, n times).

2 Construction of designs

In this section we use the techniques of [2], to determine $I_{3,3}(v)$ (so I assume familiarity with this paper). However since $\lambda = 3$, there are more small cases to be determined for the recursive constructions. In order to achieve this we take a $(v;3;3,3)$ BTD and use it to construct a $(2v-1;3;3,3)$ BTD and a $(2v+1;3;3,3)$ BTD. Note that for any BTD with parameters set $(v;3;3,3)$, we have $v \equiv 1$, 3 or 5 (mod 6) and so $2v-1 \equiv 1$, 5 or 3 (mod 6), and $2v+1 \equiv 3$, 1 or 5 (mod 6). So $2v-1$ and $2v+1$ are both admissible orders for our designs here.

2.1 v to $2v-1$

Let $\{F_i\}_{i=0}^{v-3}$ be a 1-factorization of the complete graph K_{v-1}. If we define $H_i = F_i \cup F_{i+1} \cup F_{i+2}$ for $0 \leq i \leq v-3$ (addition is mod $v-2$), then $\{H_i\}_{i=0}^{v-3}$ is a 3-factorization of $3K_{v-1}$ such that none of these H_i's has a repeated edge. Now we may use the set $\{F_0^+, F_1^+, F_2^+, H_1, H_2, H_3, ..., H_{v-3}\}$, to obtain v 3-factors for $3K_{v-1}^{+++}$. Suppose that (V, B) is a simple $(v;3;3,3)$ BTD on the v-set $V = \{x_1, x_2, x_3, ..., x_v\}$. We associate one of the v elements $\{x_1, x_2, x_3, ..., x_v\}$ with each of these v 3-factors. For each edge ab and each loop cc in a 3-factor, we take the block abx_i and ccx_i, if x_i is the element associated with this 3-factor. Of course we also take the blocks of the simple BTD of order v based on the set $\{x_1, x_2, x_3, ..., x_v\}$. Since no 3-factor has a repeated edge, the resulting BTD of order $2v-1$ is simple. Moreover it contains the BTD (V, B) as a sub-design.

2.2 v to $2v+1$

Let $G_i = G(\{i\}, v+1)$ for $1 \leq i \leq (v+1)/2$ and $w = (v-7)/2$. Now consider the graph $G(\{i, i+1\}, v+1)$. For $i \neq w+3$ ($i = w+3$) this is a 4-regular (3-regular) graph on $v+1$ vertices. By Theorem 5, this graph has a 1-factorization for $1 \leq i \leq w+3$, say $\{F_1^i, F_2^i, F_3^i, F_4^i\}$ when $i \neq w+3$ and $\{F_1, F_2, F_3\}$ when $i = w+3$. Let $v \equiv 1$ (mod 4) and α be the cyclic permutation $(w\ w-2\ w-4\ ...\ 5\ 3\ 1)$. Then the set

$$\{F_1^+, F_2^+, G_{w+2} \cup G_{w+4}, G_{w+2} \cup G_{w+4}, G_{w+2} \cup F_3, G_{2i-1} \cup F_1^{\alpha(2i-1)}, \\ G_{2i-1} \cup F_2^{\alpha(2i-1)}, G_{2i} \cup F_3^{\alpha(2i-1)}, G_{2i} \cup F_4^{\alpha(2i-1)} \mid 1 \leqslant i \leqslant (w+1)/2\}$$

consists of v 3-factors of $3K_{v+1}^{+++}$ such that none of these 3-factors has a repeated edge. Let $v \equiv 3$ (mod 4) and β be the cyclic permutation $(w+1\ w-1\ w-3\ ...\ 5\ 3\ 1)$. Then similarly the set

$$\{G_{w+4}^+, G_{w+4}^+, F_1 \cup F_2 \cup F_3, G_{2i-1} \cup F_1^{\beta(2i-1)}, G_{2i-1} \cup F_2^{\beta(2i-1)}, \\ G_{2i} \cup F_3^{\beta(2i-1)}, G_{2i} \cup F_4^{\beta(2i-1)} \mid 1 \leqslant i \leqslant (w+2)/2\}$$

consists of v 3-factors of $3K_{v+1}^{+++}$ such that none of these 3-factors has a repeated edge. Note that the remaining differences are $\{0, w+3, w+3\}$. Now suppose that (V, B) is a simple $(v; 3; 3, 3)$ BTD on the v-set $\{x_1, x_2, x_3, ..., x_v\}$. As before, we associate one of the v elements $\{x_1, x_2, x_3, ..., x_v\}$ with each of these v 3-factors. For each edge ab and each loop cc in a 3-factor, we take the blocks abx_i and ccx_i, if x_i is the element associated with this 3-factor. Moreover we take $v+1$ blocks from the initial block 0 0 $w+3$ or 0 0 $w+5$ cyclically (mod $v+1$). Of course we also take the blocks of the simple BTD of order v based on the set $\{x_1, x_2, x_3, ..., x_v\}$. Since no 3-factor has a repeated edge, the resulting BTD of order $2v+1$ is simple. Moreover it contains the BTD (V, B) as a sub-design.

2.3 Pairs of designs

Now we use these constructions to produce two BTDs based on the same set of elements (of size $2v-1$ or $2v+1$). Having constructed one BTD, the number of common blocks in the second design can be adjusted by

(i) changing the allocation of the v elements in the sub-design of order v to the v 3-factors;

(ii) changing the sub-design of order v;

(iii) in v to $2v+1$ cases, possibly trading the single orbit of $v+1$ triples outside the sub-design of order v.

By this method, for $v \geq 9$ and v odd, we obtain

$$3(v-1)/2.\{0, 1, 2, 3, ..., v-2, v\} + I_{3,3}(v) \subseteq I_{3,3}(2v-1) \qquad (2)$$

and for $v \geq 19$ and v odd, we obtain

$$3(v+1)/2.\{0, 1, 2, 3, ..., v-2, v\} + \{0, v+1\} + I_{3,3}(v) \subseteq I_{3,3}(2v+1) (3)$$

When $v \geqslant 11$, by the construction in Section 2.2 we can construct a simple $(2v+1; 3; 3, 3)$ BTD, having the original $(v; 3; 3, 3)$ BTD as a sub-design. But when $11 \leqslant v \leqslant 17$, this construction can not determine all possible intersection numbers. For these cases we proceed as follows. When $v = 9$, 11, 13, 15 or 17, by Tables 3.5, 3.6, 3.7, 3.8 and 3.9 respectively, we can decompose $3K_{v+1}^{+++}$ into v 3-factors and an extra $v+1$ triangles which we obtain by cycling the base blocks 013 (mod $v+1$) or 023 (mod $v+1$). Note that none of these 3-factors has repeated edges. Now by the method described above we find that Equation (3) is also valid for $v = 9$, 11, 13, 15 or 17.

Remark 7: It is quite straightforward to verify that there do not exist two $(v; \rho_2; 3, \lambda)$ BTDs based on the same v-set which have all but one block the same, or all but two blocks the same. So for $v \geq 9$, $I_{3,3}(v) = J_3(v)$ implies $I_{3,3}(2v-1) = J_3(2v-1)$ and $I_{3,3}(2v+1) = J_3(2v+1)$.

3 The small cases

In this section we show that $I_{3,3}(v) = J_3(v)$ for $v = 9$, 11, 13 and 15 and $I_{3,3}(7) = J_3(7) \setminus \{2\}$. So Theorem 6 is proved by Remark 7.

3.1 The case $v = 7$

Let D_1 be a (7;3;3,2) BTD and D_2 be a STS(7) on the same 7-set V. Since all blocks in D_1 are of the form xxy, we have $|D_1 \cap D_2|$=0. Thus $D_1 \cup D_2$ is a simple (7;3;3,3) BTD on V. Now by Theorems 1 and 4, we obtain $\{0, 3, 4, ..., 18, 21\} + \{0, 1, 3, 7\} \subseteq I_{3,3}(7)$. Moreover we assert that $2 \notin I_{3,3}(7)$. To prove this, let (X, B) be a (7;3;3,3) BTD, and define $B_1 = \{b \in B \mid b = \{xxy\}$ for some $x, y \in X\}$ and $B_2 = B \setminus B_1$. Since $\rho_2 = \lambda = k = 3$ we find $|B_1| = 21$, $|B_2| = 7$ and each distinct pair $\{x, y\}$ occurs at least once in the blocks of B_2. On the other hand B_2 contains at most 21 distinct pairs. Thus (X, B_2) is a STS(7) and (X, B_1) is a (7;3;3,2) BTD. Now let (X, B) and (X, B') be two (7;3;3,3) BTDs such that $|B \cap B'| = 2$. Suppose that $B = B_1 \cup B_2$ and $B' = B'_1 \cup B'_2$ such that (X, B_1) and (X, B'_1) are two (7;3;3,2) BTDs and (X, B_2) and (X, B'_2) are two STS(7). Since $|B \cap B'| = 2$ we obtain $|B_1 \cap B'_1| = 0$, 1 or 2. If $|B_1 \cap B'_1| = 1$ or 2 then this is a contradiction by Theorem 3, and if $|B_1 \cap B'_1| = 0$ then $|B_2 \cap B'_2| = 2$ and this is also a contradiction by Theorem 1. Therefore $2 \notin I_{3,3}(7)$. So $I_{3,3}(7) = J_3(7) \setminus 2$.

3.2 The case $v = 9$

Let D_1, D_2 and D_3 be three (9;3;3,3) BTDs which we obtain from the initial blocks 003, 005, 008, 027, 023 and 001, 002, 006, 045, 025 and 003, 007, 008, 045, 047 cyclically (mod 9), respectively. If we define $T_1 = \{001, 113, 330\}$, $T_2 = \{006, 667, 778, 880\}$, $T_3 = \{002, 223, 335, 557, 770\}$, $T_4 = \{112, 224, 445, 556, 668, 881\}$, $T_5 = \{117, 774, 441, 228, 885, 552, 334, 446, 663\}$, $S_1 = \{005, 554, 440\}$, $S_2 = \{887, 776, 662, 227, 773, 338\}$, $S_3 = \{003, 336, 660, 116, 665, 551\}$, $S_4 = \{884, 443, 332, 221, 110, 008\}$, $S_5 = \{114, 447, 771, 225, 558, 882\}$, $T'_i = \{aab|\ bba \in T_i\}$ and $S'_i = \{aab|\ bba \in S_i\}$ for $1 \leq i \leq 5$, then by Table 3.1 we verify $I_{3,3}(9) = J_3(9)$.

$\lvert D_1 \cap D_2\rvert = 0$	$\lvert[(D_2 - T_{25}) \cup T'_{25}] \cap D_3\rvert = 22$
$\lvert[(D_1 - S_1) \cup S'_1] \cap D_2\rvert = 1$	$\lvert[(D_2 - T_{35}) \cup T'_{35}] \cap D_3\rvert = 23$
$\lvert[(D_1 - S_2) \cup S'_2] \cap D_2\rvert = 2$	$\lvert[(D_2 - T_{45}) \cup T'_{45}] \cap D_3\rvert = 24$
$\lvert[(D_1 - S_{12}) \cup S'_{12}] \cap D_2\rvert = 3$	$\lvert[(D_2 - T_{125}) \cup T'_{125}] \cap D_3\rvert = 25$
$\lvert[(D_1 - S_3) \cup S'_3] \cap D_2\rvert = 4$	$\lvert[(D_2 - T_{135}) \cup T'_{135}] \cap D_3\rvert = 26$
$\lvert[(D_1 - S_4) \cup S'_4] \cap D_2\rvert = 5$	$\lvert[(D_2 - T_{145}) \cup T'_{145}] \cap D_3\rvert = 27$
$\lvert[(D_1 - S_5) \cup S'_5] \cap D_2\rvert = 6$	$\lvert[(D_2 - T_{245}) \cup T'_{245}] \cap D_3\rvert = 28$
$\lvert[(D_1 - S_{15}) \cup S'_{15}] \cap D_2\rvert = 7$	$\lvert[(D_2 - T_{345}) \cup T'_{345}] \cap D_3\rvert = 29$
$\lvert[(D_1 - S_{25}) \cup S'_{25}] \cap D_2\rvert = 8$	$\lvert[(D_2 - T_{45}) \cup T'_{45}] \cap D_2\rvert = 30$
$\lvert[(D_1 - S_{34}) \cup S'_{34}] \cap D_2\rvert = 9$	$\lvert[(D_2 - T_{35}) \cup T'_{35}] \cap D_2\rvert = 31$
$\lvert[(D_1 - S_{35}) \cup S'_{35}] \cap D_2\rvert = 10$	$\lvert[(D_2 - T_{25}) \cup T'_{25}] \cap D_2\rvert = 32$
$\lvert[(D_1 - S_{45}) \cup S'_{45}] \cap D_2\rvert = 11$	$\lvert[(D_2 - T_{15}) \cup T'_{15}] \cap D_2\rvert = 33$
$\lvert[(D_1 - S_{145}) \cup S'_{145}] \cap D_2\rvert = 12$	$\lvert[(D_2 - T_{34}) \cup T'_{34}] \cap D_2\rvert = 34$
$\lvert[(D_1 - S_{245}) \cup S'_{245}] \cap D_2\rvert = 13$	$\lvert[(D_2 - T_{24}) \cup T'_{24}] \cap D_2\rvert = 35$
$\lvert[(D_1 - S_{1245}) \cup S'_{1245}] \cap D_2\rvert = 14$	$\lvert[(D_2 - T_5) \cup T'_5] \cap D_2\rvert = 36$
$\lvert[(D_1 - S_{345}) \cup S'_{345}] \cap D_2\rvert = 15$	$\lvert[(D_2 - T_{13}) \cup T'_{13}] \cap D_2\rvert = 37$
$\lvert[(D_1 - S_{1345}) \cup S'_{1345}] \cap D_2\rvert = 16$	$\lvert[(D_2 - T_{12}) \cup T'_{12}] \cap D_2\rvert = 38$
$\lvert[(D_1 - S_{2345}) \cup S'_{2345}] \cap D_2\rvert = 17$	$\lvert[(D_2 - T_4) \cup T'_4] \cap D_2\rvert = 39$
$\lvert D_1 \cap D_3\rvert = 18$	$\lvert[(D_2 - T_3) \cup T'_3] \cap D_2\rvert = 40$
$\lvert[(D_2 - T_{24}) \cup T'_{24}] \cap D_3\rvert = 19$	$\lvert[(D_2 - T_2) \cup T'_2] \cap D_2\rvert = 41$
$\lvert[(D_2 - T_{34}) \cup T'_{34}] \cap D_3\rvert = 20$	$\lvert[(D_2 - T_1) \cup T'_1] \cap D_2\rvert = 42$
$\lvert[(D_2 - T_{15}) \cup T'_{15}] \cap D_3\rvert = 21$	$\lvert D_2 \cap D_2\rvert = 45$

Table 3.1

3.3 The case $v = 11$

Let D_1, D_2 and D_3 be three (11;3;3,3) BTDs which we obtain from the initial blocks 001, 002, 005, 047, 025, 018 and 0 0 10, 009, 006, 047, 028, 014 and 003, 004, 0 0 10, 029, 056, 028 cyclically (mod 11), respectively. As before, we define $T_1 = \{001, 116, 660\}$, $T_2 = \{002, 227, 779, 990\}$, $T_3 = \{112,223,338,8\ 8\ 10,10\ 10\ 1\}$, $T_4 = \{005,557,778,889,9\ 9\ 10,10\ 10\ 0\}$, $T_5 = \{334, 446, 668, 882, 224, 449, 993\}$, $T_6 = \{771,113,335,5\ 5\ 10,10\ 10\ 4,445,556,667\}$, $T_7 = \{334, 445, 556, 667, 779, 993\}$, $T_8 = \{778,889,9\ 9\ 10,10\ 10\ 0,002,227\}$, $T_9 = \{0\ 0\ 10,10\ 10\ 9,998,887,776,665,554,443,332,220\}$, $T_{10} = \{221, 117, 772\}$ and $T'_i = \{aab \mid bba \in T_i\}$ for $1 \leq i \leq 10$. Now by Table 3.2 we verify $I_{3,3}(11) = J_3(11)$.

$\|D_1 \cap D_3\| = 0$	$\|[(D_1 - T_{1256}) \cup T'_{1256}] \cap D_2\| = 33$
$\|[(D_1 - T_1) \cup T'_1] \cap D_3\| = 1$	$\|[(D_1 - T_{1356}) \cup T'_{1356}] \cap D_2\| = 34$
$\|[(D_1 - T_3) \cup T'_3] \cap D_3\| = 2$	$\|[(D_1 - T_{1456}) \cup T'_{1456}] \cap D_2\| = 35$
$\|[(D_1 - T_{13}) \cup T'_{13}] \cap D_3\| = 3$	$\|[(D_1 - T_{2456}) \cup T'_{2456}] \cap D_2\| = 36$
$\|[(D_1 - T_7) \cup T'_7] \cap D_3\| = 4$	$\|[(D_1 - T_{3456}) \cup T'_{3456}] \cap D_2\| = 37$
$\|[(D_1 - T_{17}) \cup T'_{17}] \cap D_3\| = 5$	$\|[(D_1 - T_{12356}) \cup T'_{12356}] \cap D_2\| = 38$
$\|[(D_1 - T_{37}) \cup T'_{37}] \cap D_3\| = 6$	$\|[(D_1 - T_{12356}) \cup T'_{12356}] \cap D_1\| = 39$
$\|[(D_1 - T_{137}) \cup T'_{137}] \cap D_3\| = 7$	$\|[(D_1 - T_{3456}) \cup T'_{3456}] \cap D_1\| = 40$
$\|[(D_1 - T_{78}) \cup T'_{78}] \cap D_3\| = 8$	$\|[(D_1 - T_{2456}) \cup T'_{2456}] \cap D_1\| = 41$
$\|[(D_1 - T_{178}) \cup T'_{178}] \cap D_3\| = 9$	$\|[(D_1 - T_{1456}) \cup T'_{1456}] \cap D_1\| = 42$
$\|[(D_1 - T_{378}) \cup T'_{378}] \cap D_3\| = 10$	$\|[(D_1 - T_{1356}) \cup T'_{1356}] \cap D_1\| = 43$
$\|D_1 \cap D_2\| = 11$	$\|[(D_1 - T_{1256}) \cup T'_{1256}] \cap D_1\| = 44$
$\|[(D_2 - T_{910}) \cup T'_{910}] \cap D_3\| = 12$	$\|[(D_1 - T_{456}) \cup T'_{456}] \cap D_1\| = 45$
$\|[(D_2 - T_9) \cup T'_9] \cap D_3\| = 13$	$\|[(D_1 - T_{356}) \cup T'_{356}] \cap D_1\| = 46$
$\|[(D_1 - T_1) \cup T'_1] \cap D_2\| = 14$	$\|[(D_1 - T_{256}) \cup T'_{256}] \cap D_1\| = 47$
$\|[(D_1 - T_2) \cup T'_2] \cap D_2\| = 15$	$\|[(D_1 - T_{246}) \cup T'_{246}] \cap D_1\| = 48$
$\|[(D_1 - T_3) \cup T'_3] \cap D_2\| = 16$	$\|[(D_1 - T_{236}) \cup T'_{236}] \cap D_1\| = 49$
$\|[(D_1 - T_4) \cup T'_4] \cap D_2\| = 17$	$\|[(D_1 - T_{136}) \cup T'_{136}] \cap D_1\| = 50$
$\|[(D_1 - T_5) \cup T'_5] \cap D_2\| = 18$	$\|[(D_1 - T_{56}) \cup T'_{56}] \cap D_1\| = 51$
$\|[(D_1 - T_6) \cup T'_6] \cap D_2\| = 19$	$\|[(D_1 - T_{46}) \cup T'_{46}] \cap D_1\| = 52$
$\|[(D_1 - T_{23}) \cup T'_{23}] \cap D_2\| = 20$	$\|[(D_1 - T_{36}) \cup T'_{36}] \cap D_1\| = 53$
$\|[(D_1 - T_{24}) \cup T'_{24}] \cap D_2\| = 21$	$\|[(D_1 - T_{26}) \cup T'_{26}] \cap D_1\| = 54$
$\|[(D_1 - T_{25}) \cup T'_{25}] \cap D_2\| = 22$	$\|[(D_1 - T_{25}) \cup T'_{25}] \cap D_1\| = 55$
$\|[(D_1 - T_{26}) \cup T'_{26}] \cap D_2\| = 23$	$\|[(D_1 - T_{24}) \cup T'_{24}] \cap D_1\| = 56$
$\|[(D_1 - T_{36}) \cup T'_{36}] \cap D_2\| = 24$	$\|[(D_1 - T_{23}) \cup T'_{23}] \cap D_1\| = 57$
$\|[(D_1 - T_{46}) \cup T'_{46}] \cap D_2\| = 25$	$\|[(D_1 - T_6) \cup T'_6] \cap D_1\| = 58$
$\|[(D_1 - T_{56}) \cup T'_{56}] \cap D_2\| = 26$	$\|[(D_1 - T_5) \cup T'_5] \cap D_1\| = 59$
$\|[(D_1 - T_{136}) \cup T'_{136}] \cap D_2\| = 27$	$\|[(D_1 - T_4) \cup T'_4] \cap D_1\| = 60$
$\|[(D_1 - T_{236}) \cup T'_{236}] \cap D_2\| = 28$	$\|[(D_1 - T_3) \cup T'_3] \cap D_1\| = 61$
$\|[(D_1 - T_{246}) \cup T'_{246}] \cap D_2\| = 29$	$\|[(D_1 - T_2) \cup T'_2] \cap D_1\| = 62$
$\|[(D_1 - T_{256}) \cup T'_{256}] \cap D_2\| = 30$	$\|[(D_1 - T_1) \cup T'_1] \cap D_1\| = 63$
$\|[(D_1 - T_{356}) \cup T'_{356}] \cap D_2\| = 31$	$\|D_1 \cap D_1\| = 66$
$\|[(D_1 - T_{456}) \cup T'_{456}] \cap D_2\| = 32$	

Table 3.2

3.4 The case $v = 13$

First, by the construction given in Section 2.1, we decompose $3K_6^{+++}$ into seven 3-factors:

H_1: 00, 11, 22, 33, 44, 55, 01, 25, 34;
H_2: 00, 11, 22, 33, 44, 55, 02, 13, 45;
H_3: 00, 11, 22, 33, 44, 55, 03, 15, 24;
H_4: (0,2,4,5,1,3), 04, 35, 12;
H_5: (0,3,5,1,2,4), 05, 14, 23;
H_6: (0,4,1,2,3,5), 01, 25, 34;
H_7: (0,5,2,3,4,1), 02, 13, 45.

Secondly, we take a copy of a (7;3;3,3) BTD on the set $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. If we let $j \in I_{3,3}(7)$, then the following permutations give possible assignments of the seven elements $\{x_i\}$ to the seven 3-factors of $3K_6^{+++}$.

							no. of common blocks
H_1	H_2	H_3	H_4	H_5	H_6	H_7	63+j
H_4	H_2	H_3	H_1	H_5	H_6	H_7	45+j
H_4	H_5	H_3	H_1	H_2	H_6	H_7	27+j
H_6	H_5	H_7	H_1	H_2	H_4	H_3	6+j

So we get $\{6, 27, 45, 63\} + I_{3,3}(7) \subseteq I_{3,3}(13)$. Thus we need to show that $\{0, 1, 2, 3, 4, 5, 8\} \subset I_{3,3}(13)$. For this, let D_1 and D_2 be two (13;3;3,3) BTDs which we obtain from initial blocks 001, 004, 008, 024, 036, 067, 035 and 002, 007, 0 0 12, 058, 049, 037, 023 cyclically (mod 13), respectively. Moreover, let $T_1 = \{001, 115, 550\}$, $T_2 = \{112, 226, 661\}$, $T_3 = \{223,334,445,559,994,4\ 4\ 12,12\ 12\ 7,772\}$, $T_4 = \{556,667,778,889,9\ 9\ 10,10\ 10\ 5\}$ and $T_i' = \{aab |\ bba \in T_i\}$. Then we find:

$\lvert D_1 \cap D_2 \rvert = 0$	$\lvert[(D_1 - T_{13}) \cup T_{13}'] \cap D_2\rvert = 4$
$\lvert[(D_1 - T_1) \cup T_1'] \cap D_2\rvert = 1$	$\lvert[(D_1 - T_4) \cup T_4'] \cap D_2\rvert = 5$
$\lvert[(D_1 - T_{12}) \cup T_{12}'] \cap D_2\rvert = 2$	$\lvert[(D_1 - T_{34}) \cup T_{34}'] \cap D_2\rvert = 8$
$\lvert[(D_1 - T_3) \cup T_3'] \cap D_2\rvert = 3$	

Table 3.3

So $I_{3,3}(13) = J_3(13)$.

3.5 The case $v = 15$

Here, we decompose $3K_8^{+++}$ into seven 3-factors $\{H_i\}_{i=1}^7$ as follows and eight blocks which we may obtain from the initial block 003 or 005 cyclically (mod 8).

H_1: 00, 11, 22, 33, 44, 55, 66, 77, 03, 16, 47, 25;
H_2: 00, 11, 22, 33, 44, 55, 66, 77, 36, 14, 27, 05;
H_3: (0,1,2,3,4,5,6,7), 02, 46, 13, 57;
H_4: (0,1,2,3,4,5,6,7), 24, 06, 35, 17;
H_5: (0,1,2,3,4,5,6,7), 04, 15, 26, 37;
H_6: (0,2,4,6),(1,3,5,7), 04, 15, 26, 37;
H_7: (0,2,4,6),(1,3,5,7), 04, 15, 26, 37.

Now we can assign seven new elements to the seven 3-factors in different ways, where in the table below, $j \in I_{3,3}(7)$:

							no. of common blocks
H_1	H_2	H_3	H_4	H_5	H_6	H_7	84+j
H_3	H_2	H_1	H_4	H_5	H_6	H_7	60+j
H_3	H_4	H_1	H_2	H_5	H_6	H_7	36+j
H_6	H_4	H_1	H_2	H_7	H_3	H_5	12+j

With the extra switch of the base blocks 003 and 005, we obtain $\{12, 36, 60, 84\} + \{0, 8\} + I_{3,3}(7) \subseteq I_{3,3}(15)$. For the remaining intersection numbers, let D_1 and D_2 be two (15;3;3,3) BTDs which we obtain from the initial blocks 002, 005, 007, 0 1 14, 0 3 12, 048, 0 6 12, 045 and 003, 004, 0 0 10, 079, 067, 068, 0 10 11, 013 cyclically (mod 15), respectively. Now if $T_1 = \{005, 557, 779, 991, 118, 880\}$, $T_2 = \{007$,7 7 12,12 12 4,449,9 9 11,11 11 13,13 13 0$\}$, $T_3 = \{116$,6 6 11,11 11 1$\}$, $T_4 = \{338$,8 8 13,13 13 3,5 5 10,10 10 0,002,227,7 7 14,14 14 4,4 4 11,11 11 3,335$\}$, $T_5 = \{12$ 12 2,224,446,668,8 8 10,10 10 12$\}$ and $T_i' = \{aab \mid bba \in T_i\}$ for $1 \le i \le 5$, then by Table 3.4 we obtain $\{0, 1, 2, 3, ..., 11, 14\} \subset I_{3,3}(15)$. So $I_{3,3}(15) = J_3(15)$.

$\lvert D_1 \cap D_2 \rvert = 0$	$\lvert [(D_1 - T_4) \cup T_4'] \cap D_2 \rvert = 7$
$\lvert [(D_1 - T_1) \cup T_1'] \cap D_2 \rvert = 1$	$\lvert [(D_1 - T_{14}) \cup T_{14}'] \cap D_2 \rvert = 8$
$\lvert [(D_1 - T_2) \cup T_2'] \cap D_2 \rvert = 2$	$\lvert [(D_1 - T_{24}) \cup T_{24}'] \cap D_2 \rvert = 9$
$\lvert [(D_1 - T_3) \cup T_3'] \cap D_2 \rvert = 3$	$\lvert [(D_1 - T_{34}) \cup T_{34}'] \cap D_2 \rvert = 10$
$\lvert [(D_1 - T_{13}) \cup T_{13}'] \cap D_2 \rvert = 4$	$\lvert [(D_1 - T_{134}) \cup T_{134}'] \cap D_2 \rvert = 11$
$\lvert [(D_1 - T_{23}) \cup T_{23}'] \cap D_2 \rvert = 5$	$\lvert [(D_1 - T_{12345}) \cup T_{12345}'] \cap D_2 \rvert = 14$
$\lvert [(D_1 - T_{123}) \cup T_{123}'] \cap D_2 \rvert = 6$	

Table 3.4

Nine 3-factors of $3K_{10}^{+++}$ on Z_{10}

H_1	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 05, 16, 27, 38, 49;
H_2	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 05, 16, 27, 38, 49;
H_3	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 05, 16, 27, 38, 49;
H_4	(0,1,2,3,4,5,6,7,8,9), 36, 29, 58, 14, 07;
H_5	(0,2,4,6,8),(1,3,5,7,9), 36, 29, 58, 14, 07;
H_6	(0,2,4,6,8),(1,3,5,7,9), 03, 69, 25, 18, 47;
H_7	(0,4,8,2,6),(1,5,9,3,7), 03, 69, 25, 18, 47;
H_8	(0,4,8,2,6),(1,5,9,3,7), 01, 23, 45, 67, 89;
H_9	(0,4,8,2,6),(1,5,9,3,7), 12, 34, 56, 78, 09.

Table 3.5

Eleven 3-factors of $3K_{12}^{+++}$ on Z_{12}

H_1	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 06, 17, 28, 39, 4 10, 5 11;
H_2	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 06, 17, 28, 39, 4 10, 5 11;
H_3	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 06, 17, 28, 39, 4 10, 5 11;
H_4	(0,1,2,3,4,5,6,7,8,9,10,11), 03, 69, 14, 7 10, 25, 8 11;
H_5	(0,2,4,6,8,10),(1,3,5,7,9,11), 01, 23, 45, 67, 89, 10 11;
H_6	(0,2,4,6,8,10),(1,3,5,7,9,11), 12, 34, 56, 78, 9 10, 0 11;
H_7	(0,3,6,9),(1,4,7,10),(2,5,8,11), 05, 3 10, 18, 6 11, 49, 27;
H_8	(0,5,10,3,8,1,6,11,4,9,2,7), 09, 36, 1 10, 47, 2 11, 58;
H_9	(0,4,8),(1,5,9),(2,6,10),(3,7,11), 5 10, 38, 16, 4 11, 29, 07;
H_{10}	(0,4,8),(1,5,9),(2,6,10),(3,7,11), 5 10, 38, 16, 4 11, 29, 07;
H_{11}	(0,4,8),(1,5,9),(2,6,10),(3,7,11), 05, 3 10, 18, 6 11, 49, 27.

Table 3.6

Thirteen 3-factors of $3K_{14}^{+++}$ on Z_{14}

H_1	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 12 12, 13 13, 07, 18, 29, 3 10, 4 11, 5 12, 6 13;
H_2	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 12 12, 13 13, 07, 18, 29, 3 10, 4 11, 5 12, 6 13;
H_3	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 12 12, 13 13, 07, 18, 29, 3 10, 4 11, 5 12, 6 13;
H_4	(0,1,2,3,4,5,6,7,8,9,10,11,12,13), 03, 69, 1 12, 47, 10 13, 25, 8 11;
H_5	(0,2,4,6,8,10,12),(1,3,5,7,9,11,13), 01, 23, 45, 67, 89, 10 11, 12 13;
H_6	(0,2,4,6,8,10,12),(1,3,5,7,9,11,13), 12, 34, 56, 78, 9 10, 11 12, 0 13;
H_7	(0,3,6,9,12,1,4,7,10,13,2,5,8,11), 05, 1 10, 6 11, 27, 3 12, 8 13, 49;
H_8	(0,4,8,12,2,6,10),(1,5,9,13,3,7,11), 36, 9 12, 14, 7 10, 2 13, 58, 0 11;
H_9	(0,4,8,12,2,6,10),(1,5,9,13,3,7,11), 05, 1 10, 6 11, 27, 3 12, 8 13, 49;
H_{10}	(0,4,8,12,2,6,10),(1,5,9,13,3,7,11), 05, 1 10, 6 11, 27, 3 12, 8 13, 49;
H_{11}	(0,6,12,4,10,2,8),(1,7,13,5,11,3,9), 5 10, 16, 2 11, 7 12, 38, 4 13, 09;
H_{12}	(0,6,12,4,10,2,8),(1,7,13,5,11,3,9), 5 10, 16, 2 11, 7 12, 38, 4 13, 09;
H_{13}	(0,6,12,4,10,2,8),(1,7,13,5,11,3,9), 5 10, 16, 2 11, 7 12, 38, 4 13, 09.

Table 3.7

Fifteen 3-factors of $3K_{16}^{+++}$ on Z_{16}

H_1	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 12 12, 13 13, 14 14, 15 15, 08, 19, 2 10, 3 11, 4 12, 5 13, 6 14, 7 15;
H_2	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 12 12, 13 13, 14 14, 15 15, 08, 19, 2 10, 3 11, 4 12, 5 13, 6 14, 7 15;
H_3	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10; 11 11, 12 12, 13 13, 14 14, 15 15, 08, 19, 2 10, 3 11, 4 12, 5 13, 6 14, 7 15;
H_4	(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15), 05, 10 15, 49, 3 14, 8 13, 27, 1 12, 6 11;
H_5	(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15), 05, 10 15, 49, 3 14, 8 13, 27, 1 12, 6 11;
H_6	(0,2,4,6,8,10,12,14),(1,3,5,7,9,11,13,15), 05, 10 15, 49, 3 14, 8 13, 27, 1 12, 6 11;
H_7	(0,2,4,6,8,10,12,14),(1,3,5,7,9,11,13,15), 5 10, 4 15, 9 14, 38, 2 13, 7 12, 16, 0 11;
H_8	(0,3,6,9,12,15,2,5,8,11,14,1,4,7,10,13), 5 10, 4 15, 9 14, 38, 2 13, 7 12, 16, 0 11;
H_9	(0,3,6,9,12,15,2,5,8,11,14,1,4,7,10,13), 5 10, 4 15, 9 14, 38, 2 13, 7 12, 16, 0 11;
H_{10}	(0,4,8,12),(1,5,9,13),(2,6,10,14),(3,7,11,15), 07, 5 14, 3 12, 1 10, 8 15, 6 13, 4 11, 29;
H_{11}	(0,4,8,12),(1,5,9,13),(2,6,10,14),(3,7,11,15), 07, 5 14, 3 12, 1 10, 8 15, 6 13, 4 11, 29;
H_{12}	(0,4,8,12),(1,5,9,13),(2,6,10,14),(3,7,11,15), 07, 5 14, 3 12, 1 10, 8 15, 6 13, 4 11, 29;
H_{13}	(0,6,12,2,8,14,4,10),(1,7,13,3,9,15,5,11), 7 14, 5 12, 3 10, 18, 6 15, 4 13, 2 11, 09;
H_{14}	(0,6,12,2,8,14,4,10),(1,7,13,3,9,15,5,11), 7 14, 5 12, 3 10, 18, 6 15, 4 13, 2 11, 09;
H_{15}	(0,6,12,2,8,14,4,10),(1,7,13,3,9,15,5,11), 7 14, 5 12, 3 10, 18, 6 15, 4 13, 2 11, 09.

Table 3.8

Seventeen 3-factors of $3K_{18}^{+++}$ on Z_{18}

H_1	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 12 12, 13 13, 14 14, 15 15, 16 16, 17 17, 09, 1 10, 2 11, 3 12, 4 13, 5 14, 6 15, 7 16, 8 17;
H_2	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 12 12, 13 13, 14 14, 15 15, 16 16, 17 17, 09, 1 10, 2 11, 3 12, 4 13, 5 14, 6 15, 7 16, 8 17;
H_3	00, 11, 22, 33, 44, 55, 66, 77, 88, 99, 10 10, 11 11, 12 12, 13 13, 14 14, 15 15, 16 16, 17 17, 09, 1 10, 2 11, 3 12, 4 13, 5 14, 6 15, 7 16, 8 17;
H_4	(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17), 05, 10 15, 27, 12 17, 49, 1 14, 6 11, 3 16, 8 13;
H_5	(0,2,4,6,8,10,12,14,16),(1,3,5,7,9,11,13,15,17), 05, 10 15, 27, 12 17, 49, 1 14, 6 11, 3 16, 8 13;
H_6	(0,2,4,6,8,10,12,14,16),(1,3,5,7,9,11,13,15,17), 05, 10 15, 27, 12 17, 49, 1 14, 6 11, 3 16, 8 13;
H_7	(0,3,6,9,12,15),(1,4,7,10,13,16),(2,5,8,11,14,17), 5 10, 2 15, 7 12, 4 17, 9 14, 16, 16 11, 38, 0 13;
H_8	(0,3,6,9,12,15),(1,4,7,10,13,16),(2,5,8,11,14,17), 5 10, 2 15, 7 12, 4 17, 9 14, 16, 16 11, 38, 0 13;
H_9	(0,4,8,12,16,2,6,10,14),(1,5,9,13,17,3,7,11,15), 5 10, 2 15, 7 12, 4 17, 9 14, 16, 16 11, 38, 0 13;
H_{10}	(0,4,8,12,16,2,6,10,14),(1,5,9,13,17,3,7,11,15), 07, 3 14, 10 17, 6 13, 29, 5 16, 1 12, 8 15, 4 11;
H_{11}	(0,4,8,12,16,2,6,10,14),(1,5,9,13,17,3,7,11,15), 07, 3 14, 10 17, 6 13, 29, 5 16, 1 12, 8 15, 4 11;
H_{12}	(0,6,12),(1,7,13),(2,8,14),(3,9,15),(4,10,16),(5,11,17), 07, 3 14, 10 17, 6 13, 29, 5 16, 1 12, 8 15, 4 11;
H_{13}	(0,6,12),(1,7,13),(2,8,14),(3,9,15),(4,10,16),(5,11,17), 7 14, 3 10, 6 17, 2 13, 9 16, 5 12, 18, 4 15, 0 11;
H_{14}	(0,6,12),(1,7,13),(2,8,14),(3,9,15),(4,10,16),(5,11,17), 7 14, 3 10, 6 17, 2 13, 9 16, 5 12, 18, 4 15, 0 11;
H_{15}	(0,8,16,6,14,4,12,2,10),(1,9,17,7,15,5,13,3,11), 7 14, 3 10, 6 17, 2 13, 9 16, 5 12, 18, 4 15, 0 11;
H_{16}	(0,8,16,6,14,4,12,2,10),(1,9,17,7,15,5,13,3,11), 01, 23, 45, 67, 89, 10 11, 12 13, 14 15, 16 17;
H_{17}	(0,8,16,6,14,4,12,2,10),(1,9,17,7,15,5,13,3,11), 12, 34, 56, 78, 9 10, 11 12, 13 14, 15 16, 0 17.

Table 3.9

4 Conclusion

We now have our required result:

Theorem 6. *There exist two simple balanced ternary designs of order $v \equiv 1 \pmod 2$, $v \geq 7$, with block size 3, index 3 and $\rho_2 = 3$, having n common blocks, for all $n \in \{0, 1, 2, \ldots, (v(v+1)/2) - 3, (v(v+1)/2)\}$, with the one exception that there do not exist two such BTDs of order 7 having 2 common blocks.*

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