Relatively Narrow Latin Parallelepipeds That Cannot Be Extended to a Latin Cube

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Abstract. In this paper we construct a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended to a latin cube of order n for any pair of integers d, n where $d \geq 3$ and $n \geq 2d+1$. For $d = 2$, it is similar to the construction already known.

1. Introduction

One of the best known property of latin squares is that any latin $(n \times k)$-rectangle can be extended to a latin square of order n. This was proved by M. Hall [5]. Since then have arised questions whether this theorem can be extended to "more dimensional" cases. To be more precisse we introduce some notations.

Let $A^{(1)} = [a_{i,j}^{(1)}]$, $A^{(2)} = [a_{i,j}^{(2)}], \ldots, A^{(k)} = [a_{i,j}^{(k)}]$ be latin squares of elements $1, 2, \ldots, n$. The ordered k-tuple $A = (A^{(1)}, A^{(2)}, \ldots, A^{(k)})$ is called a *latin* $(n \times n \times k)$*-parallelepiped* if the elements $a_{i,j}^{(1)}, \ldots, a_{i,j}^{(k)}$ are distinct for every $1 \leq i, j \leq n$. In the case $k = n$, A is called a *latin cube* of order n.

With respect to the theorem of Hall [5] it is natural to ask the following question: Given a latin $(n \times n \times k)$-parallelepiped, do there exist $n-k$ latin squares which may be added to the given parallelepiped to form a latin cube? This problem was posed in the Sixth Hungarian Colloqium on Combinatorics, 1981. In contrast with the classical case there are known constructions of latin $(n \times n \times (n-d))$-parallelepipeds that cannot be extended to a latin cube of order n. These constructions have been presented in [7] (for $d = 2$ and $n = 2^k$, $k \geq 3$), in [4] (for $d = 2$ and $n = 6$ or $n \geq 12$), in [8] (for $d = 2$ and $n \geq 5$), in [9] (for $d \geq 3$ and $n = kd$, $k \geq 3$ or $n \geq 6d$), and in [3] (for $d = 3$ and $n = 5$). In this paper we construct a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended to a latin cube of order n for any $d \geq 3$ and $n \geq 2d+1$.

It is an easy task to prove that any latin $(n \times n \times 1)$- and $(n \times n \times (n-1))$-parallelepipeds can be extended to latin cubes of order n. We have conjectured in [9] that, if $d \geq 2$ and $n \leq 2d$, then every latin $(n \times n \times (n-d))$-parallelepiped can be extended to a latin cube of order n and, if $d \geq 2$ and $n \geq 2d+1$, then there exists a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended

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to a latin cube of order n. This conjecture was verified for $d = 2$ in [8], but disproved for $d = 3$ and $n = 5$ in [3]. In this paper we prove the second part of the conjecture. Unfortunately there are no nontrivial results regarding the first part of the conjecture, unless the contraexample presented in [3].

As a survay of properties of latin squares together with some applications we refer to the book edited and partially written by Dénes and Keedwell [2], especially the chapters written by Heinrich [6] and Lindner [10].

2. Basic notations and definitions

An *incomplete latin square* of elements $c_1, \dots, c_n$ (or, simply, of order n) is an $n\times n$ array such that the entries are the elements $c_1, \dots, c_n$, no elements of $c_1 \dots, c_n$ occurs in any row or column more than once, and some cells may be empty. Unless otherwise specified $c_i = i$ for any $i = 1, \dots, n$. If every cell is nonempty we get a *latin square* of order n. If A is an (incomplete) latin square of order n, then we write $A = [a_{i,j}]$ where $a_{i,j}$ denotes the entry in the i-th row and the j-th column.

We say that an incomplete latin square of order n can be extended to a latin square of order n if the empty cells can be filled in such that the result is a latin square of order n.

We shall use $S(c_1, \dots, c_n)$ to denote the latin square $[a_{i,j}]$ of elements $c_1, \dots, c_n$ such that $a_{i,j} = c_{i+j-1}$, where the indices are taken in $\{1, \dots, n\}$ mod n. For instance

$$S(1,2,3) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Let A be an (incomplete) latin square of order n and $1 \leq r_1 \leq r_2 \leq n$, $1 \leq s_1 \leq s_2 \leq n$. We call an $(r_1, r_2) \times (s_1, s_2)$*-subrectangle* of A the $(r_2 - r_1 + 1) \times (s_2 - s_1 + 1)$ array which arises as the intersection of the r_1th, (r_1+1)st, $\dots, r_2$th rows and the s_1th, (s_1+1)st, $\dots, s_2$th columns of A.

Let A be an (incomplete) latin square of order n, $1 \leq r_1 < r_2 \leq n$, $1 \leq s_1 < s_2 \leq n$ and B be the 2×2 array which arises as the intersection of the r_1th, r_2th rows and the s_1th, s_2th columns of A. Then B is called the (r_1, r_2) # (s_1, s_2)*-subsquare* of A.

The latin $(n \times n \times k)$-parallelepipeds have been introduced in the first section. Furthermore, if $A = (A^{(1)}, A^{(2)}, \dots, A^{(k)})$ is a latin $(n \times n \times k)$-parallelepiped then we denote by $M_{i,j}(A)$ the subset of the numbers $1, 2, \dots, n$ which do not occur in the intersection of the ith row and the jth column in any of the latin squares $A^{(1)}, A^{(2)}, \dots, A^{(k)}$.

3. The main idea of the construction

Our construction will be based on the following lemma.

Lemma 1. *Let $d \geq 2$, $n > d$ and $\mathcal{A} = (A^{(1)}, A^{(2)}, \ldots, A^{(n-d)})$ be a latin $(n \times n \times (n-d))$-parallelepiped such that*

$$M_{1,1}(\mathcal{A}) = \{2,3,\ldots,d+1\},$$
$$M_{i,1}(\mathcal{A}) = \{1,3,4,\ldots,d+1\}, \quad (2 \leq i \leq d),$$
$$M_{i,j}(\mathcal{A}) = \{1,2,\ldots,d\}, \quad (1 \leq i \leq d,\ 2 \leq j \leq d).$$

Then $\mathcal{A}$ cannot be extended to a latin cube of order n.

Proof: Suppose, to the contrary, that there exist latin squares $B^{(1)} = [b_{i,j}^{(1)}]$, $B^{(2)} = [b_{i,j}^{(2)}], \ldots, B^{(d)} = [b_{i,j}^{(d)}]$ of order n such that $\mathcal{B} = (A^{(1)}, \ldots A^{(n-d)},, B^{(1)}, \ldots, B^{(d)})$ is a latin cube (in other words, $\{b_{i,j}^{(1)}, b_{i,j}^{(2)}, \ldots, b_{i,j}^{(d)}\} = M_{i,j}(\mathcal{A})$ for any $1 \leq i,j \leq n$). We may assume without loss of generality that $b_{1,1}^{(1)} = d+1$. Since $\{2,3,\ldots,d\} \subseteq M_{1,1}(\mathcal{A}), M_{1,2}(\mathcal{A}), \ldots, M_{1,d}(\mathcal{A})$, then, for any $k \in \{2,3,\ldots,d\}$, there exists $i_k \in \{2,3,\ldots,d\}$ such that $b_{1,i_k}^{(1)} = k$ (otherwise k could not be element of the sets $M_{1,1}(\mathcal{A}), M_{1,2}(\mathcal{A}), \ldots, M_{1,d}(\mathcal{A})$). Then $\{b_{1,2}^{(1)}, b_{1,3}^{(1)}, \ldots, b_{1,d}^{(1)}\} = \{2,3,\ldots,d\}$ and $b_{1,j}^{(1)} \neq 1$ for any $j = 1, \ldots, d$.

Denote $I = \{(i,j);\ 1 \leq i,j \leq d\}$.

We have proved that $b_{1,j}^{(1)} \neq 1$ for any $j = 1, \ldots, d$. Then there exist at most $d-1$ elements $(i,j) \in I$ such that $b_{i,j}^{(1)} = 1$.

Since $b_{1,1}^{(1)} = d+1$ and $d+1 \notin M_{i,j}(\mathcal{A})$ for any $(i,j) \in I$ and $j \geq 2$, then there exists just one $(i,j) \in I$ such that $b_{i,j}^{(1)} = d+1$.

Since $2 \notin M_{i,1}(\mathcal{A})$, $2 \leq i \leq d$, and $2 \neq b_{1,1}^{(1)}$, then there exist at most $d-1$ elements $(i,j) \in I$ such that $b_{i,j}^{(1)} = 2$.

Clearly, there exist at most $d(d-2)$ elements $(i,j) \in I$ such that $b_{i,j}^{(1)} \in \{3,4,\ldots,d\}$.

Concluding, there exist at most $d^2 - 1$ elements $(i,j) \in I$ such that $b_{i,j}^{(1)} \in \{1,2,\ldots,d+1\}$. But $|I| = d^2$ and $M_{i,j}(\mathcal{A}) \subseteq \{1,2,\ldots,d+1\}$ for any $(i,j) \in I$, what is a contradiction. ∎

In the sequel we shall construct latin $(n \times n \times (n-d))$-parallelepiped $\mathcal{C}$ such that $M_{i,j}(\mathcal{C}) = \{1,\ldots,d\}$ for any $1 \leq i,j \leq d$. Then we shall do small changes in $\mathcal{C}$ and obtain a new latin $(n \times n \times (n-d))$-parallelepiped $\mathcal{E}$ having the properties of Lemma 1. In order to construct such $\mathcal{C}$ and $\mathcal{E}$ we need two incomplete latin squares introduced in the next chapter.

4. Definitions of two basic incomplete latin squares

Definition 1: Let $d \geq 3$, $n \geq 2d+1$. Denote by $K(d,n)$ the incomplete latin square of order n such that (see $K(4,9)$ and $K(3,7)$ in Table 1, where the empy cells are depicted as dots):

(a) The $(1,d) \times (1,d)$-subrectangle of $K(d,n)$ is $S(d+2,\ldots,2d+1)$.

(b) The $(1, d+1) \# (1, d+1)$-subsquare of $K(d, n)$ is equal to $S(d+2, 2)$, the $(2, d+2) \# (1, d+2)$-subsquare of $K(d, n)$ is equal to $S(d+3, 3)$, the $(d, 2d) \# (1, d+3)$-subsquare of $K(d, n)$ is equal to $S(2d+1, 1)$.

(c) If $d \geq 4$, then, for any $i = 4, \ldots, d$, the $(i-1, d+i-1) \# (1, d+3)$-subsquare of $K(d, n)$ is equal to $S(d+i, i)$.

(d) All other cells are empty.

Since any of the subsquares of $K(d, n)$ from items (b) and (c) is in fact a latin square of order 2, we can interchange the positions of the entries in any of these subsquares and $K(d, n)$ (or its extention) remains latin. This will be used in the following section.

$$K(4,9) = \begin{array}{ccccccccc} 6 & 7 & 8 & 9 & 2 & . & . & . & . \\ 7 & 8 & 9 & 6 & . & 3 & . & . & . \\ 8 & 9 & 6 & 7 & . & . & 4 & . & . \\ 9 & 6 & 7 & 8 & . & . & 1 & . & . \\ 2 & . & . & . & 6 & . & . & . & . \\ 3 & . & . & . & . & 7 & . & . & . \\ 4 & . & . & . & . & . & 8 & . & . \\ 1 & . & . & . & . & . & 9 & . & . \\ . & . & . & . & . & . & . & . & . \end{array} \qquad K(3,7) = \begin{array}{ccccccc} 5 & 6 & 7 & 2 & . & . & . \\ 6 & 7 & 5 & . & 3 & . & . \\ 7 & 5 & 6 & . & . & 1 & . \\ 2 & . & . & 5 & . & . & . \\ 3 & . & . & . & 6 & . & . \\ 1 & . & . & . & . & 7 & . \\ . & . & . & . & . & . & . \end{array}$$

Table 1

Definition 2: Let $d \geq 3, n \geq 2d+1$. Denote by $L(d, n) = [l_{i,j}]$ the incomplete latin square of order n such that (see Table 2):

(a) The $(n-d+1, n) \times (1, d)$-subrectangle of $L(d, n)$ is $S(1, \ldots, d)$.

(b) The $(1, 2) \# (d, 2d)$-subsquare of $L(d, n)$ is

$$\begin{bmatrix} d+2 & 2 \\ d+1 & 1 \end{bmatrix}.$$

(c) The $(1, d+1) \# (1, d+1)$-subsquare of $L(d, n)$ is

$$\begin{bmatrix} 2d+1 & 1 \\ d+1 & 2 \end{bmatrix}.$$

(d) For any $i = 3, \ldots, d$, the $(1, i) \# (d-i+2, 2d-i+2)$-subsquare of $L(d, n)$ is

$$\begin{bmatrix} d+i & i \\ d+1 & 2 \end{bmatrix}.$$

(e) $l_{n,d+1} = l_{n-1,d+2} = l_{n-2,d+3} = \cdots = l_{n-d+1,2d} = d+1$.

(f) All other cells are empty.

$$L(4,9) = \begin{array}{ccccccccc} 9 & 8 & 7 & 6 & 1 & 4 & 3 & 2 & . \\ . & . & . & 5 & . & . & . & 1 & . \\ . & . & 5 & . & . & . & 2 & . & . \\ . & 5 & . & . & . & 2 & . & . & . \\ 5 & . & . & . & 2 & . & . & . & . \\ 1 & 2 & 3 & 4 & . & . & . & 5 & . \\ 2 & 3 & 4 & 1 & . & . & 5 & . & . \\ 3 & 4 & 1 & 2 & . & 5 & . & . & . \\ 4 & 1 & 2 & 3 & 5 & . & . & . & . \end{array} \qquad L(3,7) = \begin{array}{ccccccc} 7 & 6 & 5 & 1 & 3 & 2 & . \\ . & . & 4 & . & . & 1 & . \\ . & 4 & . & . & 2 & . & . \\ 4 & . & . & 2 & . & . & . \\ 1 & 2 & 3 & . & . & 4 & . \\ 2 & 3 & 1 & . & 4 & . & . \\ 3 & 1 & 2 & 4 & . & . & . \end{array}$$

Table 2

In the sequel we use the following fact. Suppose $L(d,n)$ be extended to a latin square of order n and remove the last d rows from it. We get a latin rectangle. This rectangle remains latin if we interchange the positions of the entries $d+1$ and 1 in the 2nd row and the entries $d+1$ and 2 in the 3th, ..., $(d+1)$st rows.

5. The construction

The last section will be devoted to the proof of the following lemma.

Lemma 2. *Let $d \geq 3, n \geq 2d+1$. Then $K(d,n)$ and $L(d,n)$ can be extended to latin squares of order n.*

If φ is a permutation of the set $\{1,\ldots,n\}$ and $A = [a_{i,j}]$ is a latin square of order n, then by $\varphi(A) = [b_{i,j}]$ we denote the latin square of order n such that $b_{i,j} = \varphi(a_{i,j})$.

Now we can prove the main theorem.

Theorem 1. *For any $d \geq 2$ and $n \geq 2d+1$ there exists a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended to a latin cube of order n.*

Proof: This theorem was proved for $d = 2$ and $n \geq 5$ in [8].

Choose fixed integers d, n where $d \geq 3$ and $n \geq 2d+1$. By Lemma 2, there exist latin squares $R(d,n)$ and $S(d,n)$ of order n that are extentions of $K(d,n)$ and $L(d,n)$ respectively.

Take $S(d,n) = [s_{i,j}]$. This latin square defines n permutations φ_i $(1 \leq i \leq n)$ of the set $\{1,\ldots,n\}$ such that φ_i maps the 1st row of $S(d,n)$ on the ith row of $S(d,n)$, more precissely, $\varphi_i(s_{1,j}) = s_{i,j}$ for any $1 \leq j \leq n$. Note that φ_1 is the identical mapping.

Take $R(d,n)$ and let us construct a latin $(n \times n \times (n-d))$-parallelepiped $C = (C^{(1)},\ldots,C^{(n-d)})$ such that $C^{(i)} = \varphi_i(R(d,n))$ for any $1 \leq i \leq n-d$. From the fact that $R(d,n)$ and $S(d,n)$ are extentions of $K(d,n)$ and $L(d,n)$ respectively, we can easily check (1)-(3):

(1) $M_{i,j}(C) = \{1\ldots,d\}$ for any $1 \leq i,j \leq d$.

(2) $M_{i,j}(C) = \{1 \dots, d\}$ for any pair (i, j) from the set

$$\{(d+1, d+1), (d+2, d+2), (d+3, d+3), (d+4, d+3), \dots, (2d, d+3)\}.$$

(3) $d+1 \in M_{i,j}(C)$ for any pair (i, j) from the sets

$$\{(1, d+1), (2, d+2), (3, d+3), (4, d+3), \dots, (d, d+3)\},$$
$$\{(d+1, 1), (d+2, 1), (d+3, 1), \dots, (2d, 1)\}.$$

Furthermore, from Definitions 1 and 2 we can easily check: (See Table 3, where are depicted segments from $C^{(1)}, \dots, C^{(5)}$ if $d = 4$, $n = 9$ and $S(4, 9)$ is depicted in Table 4. The less important entries of $C^{(i)}$ are depicted as dots.)

(4) The $(1, d+1)$ # $(1, d+1)$-subsquare of $C^{(2)}$ is equal to $S(d+1, 1)$. (For instance this follows from the facts that the $(1, d+1)$ # $(1, d+1)$-subsquare of $K(d, n)$ is $S(d+2, 2)$ and the $(1, 2)$ # $(d, 2d)$-subsquare of $L(d, n)$ is $\begin{bmatrix} d+2 & 2 \\ d+1 & 1 \end{bmatrix}$.)

(5) The $(2, d+2)$ # $(1, d+2)$-subsquare of $C^{(3)}$ is equal to $S(d+1, 2)$.

(6) The $(i-1, d+i-1)$ # $(1, d+3)$-subsquare of $C^{(i)}$ is equal to $S(d+1, 2)$ for any $i = 4, \dots, d+1$.

As pointed out after Definition 1, we can interchange the positions of the entries of the subsquares from items (4)-(6). I.e., the subsquares equal to $S(d+1, 1)$ or $S(d+1, 2)$ can be replaced by the subsquares equal to $S(1, d+1)$ or $S(2, d+2)$, respectively. Performing all these changes we get new latin squares $E^{(2)}, \dots, E^{(d+1)}$ from $C^{(2)}, \dots, C^{(d+1)}$, respectively. Otherwise let $E^{(i)} = C^{(i)}$ (see Table 3).

From (1)-(3) it follows that if we use the notation $E^{(r)} = [e_{i,j}^{(r)}]$, $r = 1, \dots, n-d$, then the entries $e_{i,j}^{(1)}, \dots, e_{i,j}^{(n-d)}$ are distinct for every $1 \leq i, j \leq n$. (In fact we have used that $S(d, n)$ is an extention of $L(d, n)$ and the arguments of the remark after Definition 2.) Thus $\mathcal{E} = (E^{(1)}, \dots, E^{(n-d)})$ is latin $(n \times n \times (n-d))$-parallelepiped.

Finally, from (4)-(6) and (1) it follows that

$$M_{1,1}(\mathcal{E}) = \{2, 3, \dots, d+1\},$$
$$M_{i,1}(\mathcal{E}) = \{1, 3, 4, \dots, d+1\}, \quad (2 \leq i \leq d),$$
$$M_{i,j}(\mathcal{E}) = \{1, 2, \dots, d\}, \quad (1 \leq i \leq d,\ 2 \leq j \leq d).$$

Thus, by Lemma 1, $\mathcal{E}$ cannot be extended to a latin cube of order n. ∎

Unfortunatelly, the construction of Theorem 1 cannot be applied for $n \leq 2d$. We can check that if $n \leq 2d$, then there exists no latin $(n \times n \times (n-d))$-parallelepiped satisfying the conditions of Lemma 1.

$$C^{(1)} = \begin{array}{ccccccccc} 6 & 7 & 8 & 9 & 2 & . & . & . & . \\ 7 & 8 & 9 & 6 & . & 3 & . & . & . \\ 8 & 9 & 6 & 7 & . & . & 4 & . & . \\ 9 & 6 & 7 & 8 & . & . & 1 & . & . \\ 2 & . & . & . & 6 & . & . & . & . \\ 6 & . & . & . & . & 7 & . & . & . \\ 4 & . & . & . & . & . & 8 & . & . \\ 1 & . & . & . & . & . & 9 & . & . \\ . & . & . & . & . & . & . & . & . \end{array} \qquad E^{(1)} = \begin{array}{ccccccccc} 6 & 7 & 8 & 9 & 2 & . & . & . & . \\ 7 & 8 & 9 & 6 & . & 3 & . & . & . \\ 8 & 9 & 6 & 7 & . & . & 4 & . & . \\ 9 & 6 & 7 & 8 & . & . & 1 & . & . \\ 2 & . & . & . & 6 & . & . & . & . \\ 6 & . & . & . & . & 7 & . & . & . \\ 4 & . & . & . & . & . & 8 & . & . \\ 1 & . & . & . & . & . & 9 & . & . \\ . & . & . & . & . & . & . & . & . \end{array}$$

$$C^{(2)} = \begin{array}{ccccccccc} 5 & 6 & 7 & 8 & 1 & . & . & . & . \\ 6 & 7 & 8 & 5 & . & 9 & . & . & . \\ 7 & 8 & 5 & 6 & . & . & 3 & . & . \\ 8 & 5 & 6 & 7 & . & . & 4 & . & . \\ 1 & . & . & . & 5 & . & . & . & . \\ 9 & . & . & . & . & 6 & . & . & . \\ 3 & . & . & . & . & . & 7 & . & . \\ 4 & . & . & . & . & . & 8 & . & . \\ . & . & . & . & . & . & . & . & . \end{array} \qquad E^{(2)} = \begin{array}{ccccccccc} 1 & 6 & 7 & 8 & 5 & . & . & . & . \\ 6 & 7 & 8 & 5 & . & 9 & . & . & . \\ 7 & 8 & 5 & 6 & . & . & 3 & . & . \\ 8 & 5 & 6 & 7 & . & . & 4 & . & . \\ 5 & . & . & . & 1 & . & . & . & . \\ 9 & . & . & . & . & 6 & . & . & . \\ 3 & . & . & . & . & . & 7 & . & . \\ 4 & . & . & . & . & . & 8 & . & . \\ . & . & . & . & . & . & . & . & . \end{array}$$

$$C^{(3)} = \begin{array}{ccccccccc} 9 & 5 & 6 & 7 & 4 & . & . & . & . \\ 5 & 6 & 7 & 9 & . & 2 & . & . & . \\ 6 & 7 & 9 & 5 & . & . & 8 & . & . \\ 7 & 9 & 5 & 6 & . & . & 3 & . & . \\ 4 & . & . & . & 9 & . & . & . & . \\ 2 & . & . & . & . & 5 & . & . & . \\ 8 & . & . & . & . & . & 6 & . & . \\ 3 & . & . & . & . & . & 7 & . & . \\ . & . & . & . & . & . & . & . & . \end{array} \qquad E^{(3)} = \begin{array}{ccccccccc} 9 & 5 & 6 & 7 & 4 & . & . & . & . \\ 2 & 6 & 7 & 9 & . & 5 & . & . & . \\ 6 & 7 & 9 & 5 & . & . & 8 & . & . \\ 7 & 9 & 5 & 6 & . & . & 3 & . & . \\ 4 & . & . & . & 9 & . & . & . & . \\ 5 & . & . & . & . & 2 & . & . & . \\ 8 & . & . & . & . & . & 6 & . & . \\ 3 & . & . & . & . & . & 7 & . & . \\ . & . & . & . & . & . & . & . & . \end{array}$$

$$C^{(4)} = \begin{array}{ccccccccc} 8 & 9 & 5 & 6 & 3 & . & . & . & . \\ 9 & 5 & 6 & 8 & . & 1 & . & . & . \\ 5 & 6 & 8 & 9 & . & . & 2 & . & . \\ 6 & 8 & 9 & 5 & . & . & 7 & . & . \\ 3 & . & . & . & 8 & . & . & . & . \\ 1 & . & . & . & . & 9 & . & . & . \\ 2 & . & . & . & . & . & 5 & . & . \\ 7 & . & . & . & . & . & 6 & . & . \\ . & . & . & . & . & . & . & . & . \end{array} \qquad E^{(4)} = \begin{array}{ccccccccc} 8 & 9 & 5 & 6 & 3 & . & . & . & . \\ 9 & 5 & 6 & 8 & . & 1 & . & . & . \\ 2 & 6 & 8 & 9 & . & . & 5 & . & . \\ 6 & 8 & 9 & 5 & . & . & 7 & . & . \\ 3 & . & . & . & 8 & . & . & . & . \\ 1 & . & . & . & . & 9 & . & . & . \\ 5 & . & . & . & . & . & 2 & . & . \\ 7 & . & . & . & . & . & 6 & . & . \\ . & . & . & . & . & . & . & . & . \end{array}$$

Table 3

$$C^{(5)} = \begin{array}{ccccccccc} 7 & 8 & 9 & 5 & 6 & . & . & . & . \\ 8 & 9 & 5 & 7 & . & 4 & . & . & . \\ 9 & 5 & 7 & 8 & . & . & 1 & . & . \\ 5 & 7 & 8 & 9 & . & . & 2 & . & . \\ 6 & . & . & . & 7 & . & . & . & . \\ 4 & . & . & . & . & 8 & . & . & . \\ 1 & . & . & . & . & . & 9 & . & . \\ 2 & . & . & . & . & . & 5 & . & . \\ . & . & . & . & . & . & . & . & . \end{array} \qquad C^{(5)} = \begin{array}{ccccccccc} 7 & 8 & 9 & 5 & 6 & . & . & . & . \\ 8 & 9 & 5 & 7 & . & 4 & . & . & . \\ 9 & 5 & 7 & 8 & . & . & 1 & . & . \\ 2 & 7 & 8 & 9 & . & . & 5 & . & . \\ 6 & . & . & . & 7 & . & . & . & . \\ 4 & . & . & . & . & 8 & . & . & . \\ 1 & . & . & . & . & . & 9 & . & . \\ 5 & . & . & . & . & . & 2 & . & . \\ . & . & . & . & . & . & . & . & . \end{array}$$

Table 3 (continued)

$$S(4,9) = \begin{array}{ccccccccc} 9 & 8 & 7 & 6 & 1 & 4 & 3 & 2 & 5 \\ 8 & 7 & 6 & 5 & 4 & 3 & 9 & 1 & 2 \\ 7 & 6 & 5 & 9 & 3 & 8 & 2 & 4 & 1 \\ 6 & 5 & 9 & 8 & 7 & 2 & 1 & 3 & 4 \\ 5 & 9 & 8 & 7 & 2 & 1 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 & 8 & 9 & 6 & 5 & 7 \\ 2 & 3 & 4 & 1 & 9 & 6 & 5 & 7 & 8 \\ 3 & 4 & 1 & 2 & 6 & 5 & 7 & 8 & 9 \\ 4 & 1 & 2 & 3 & 5 & 7 & 8 & 9 & 6 \end{array}$$

Table 4

6. Proof of Lemma 2

As pointed out before, this part has only auxiliary character and its aim is to prove Lemma 2. Primarily we need several easy lemmas:

Lemma 3. *Let A be an incomplete latin square of order n such that*

(a) *The cells in the first d ($\leq n$) columns are occupied.*
(b) *There are occupied some (but not necessary all) cells from the first row.*
(c) *All other cells are empty.*

Then A can be extended to a latin square of order n.

Proof: Take as A' the incomplete latin square that is the subsquare of A and has only the cells of the first d rows occupied. Then, by [5], A' can be extended to a latin square A'' of order n. Using the appropriate permutation of columns of A'' we get the required extention of A. ∎

Note that Lemma 3 is in fact a special case of a more general result from [1].

Let ξ be a mapping that maps an incomplete latin square $A = [a_{i,j}]$ of order n to an incomplete latin square $\xi(A) = [b_{i,j}]$ of order n such that (see e.g. $L(4,10)$ and $\xi(L(4,10))$ in Table 10):

(a) If $a_{i,j} = k$, then $b_{i,k} = j$ for any $1 \leq i, j, k \leq n$.

(b) If $a_{i,j}$ is an empty cell, then $\xi(A)$ has no cell in the ith row with the entry equal to j.

We can check that $\xi(\xi(A)) = A$.

Lemma 4. *Let A be an incomplete latin square of order n and suppose B is an incomplete latin square of order n obtained from A by using any combination of the following processes:*

(a) *permutation of rows,*
(b) *permutation of columns,*
(c) *using the mapping ξ.*

Then A can be extended to a latin square of order n if and only if B can be.

Proof: The property of being a latin square is unchanged by any of the processes and furthermore each can be reversed. ∎

$K'(4,9) =$

6	7	8	9	2	4	3	.	.
7	8	9	6	4	3	5	.	.
8	9	6	7	3	1	4	.	.
9	6	7	8	5	2	1	.	.
2	3	4	1	6	5	7	.	.
3	4	1	5	9	7	2	.	.
4	1	5	2	7	9	8	.	.
1	5	2	3	8	6	9	.	.
5	2	3	4	1	8	6	.	.

$K'(3,7) =$

5	6	7	2	1	3	.
6	7	5	1	3	4	.
7	5	6	4	2	1	.
2	3	1	5	4	6	.
3	1	4	7	6	2	.
1	4	2	3	5	7	.
4	2	3	6	7	5	.

Table 5

$K'(4,10) =$

6	7	8	9	2	4	3	.	.	.
7	8	9	6	4	3	10	.	.	.
8	9	6	7	3	10	4	.	.	.
9	6	7	8	5	2	1	.	.	.
2	3	4	1	6	5	7	.	.	.
3	4	1	5	10	7	2	.	.	.
4	1	5	10	7	9	8	.	.	.
1	5	10	2	8	6	9	.	.	.
5	10	2	3	1	8	6	.	.	.
10	2	3	4	9	1	5	.	.	.

$K'(3,8) =$

5	6	7	2	8	3	.	.
6	7	5	1	3	4	.	.
7	5	6	4	2	1	.	.
2	3	1	5	4	8	.	.
3	1	4	8	6	2	.	.
1	4	8	3	5	7	.	.
4	8	2	7	1	6	.	.
8	2	3	6	7	5	.	.

Table 6

Lemma 5. *For any $d \geq 3$, $K(d,2d+1)$ and $K(d,2d+2)$ can be extended to a latin square of order $2d+1$ and $2d+2$ respectively.*

Proof: Let $K'(3,7)$ be the incomplete latin square depicted on Table 5. If $d \geq 4$, then define an incomplete latin square $K'(d,2d+1)$ of order $2d+1$ such that:

(a) The $(1,d) \times (1,d)$-subrectangle of $K'(d,2d+1)$ is equal to $S(d+2, \ldots, 2d+1)$.
(b) The $(d+1,2d+1) \times (1,d)$-subrectangle of $K'(d,2d+1)$ is formed from $S(2,3,\ldots,d,1,d+1)$ by deleting the last column.
(c) The $(1,2d+1) \times (d+1,d+3)$-subrectangle of $K'(d,2d+1)$ is depicted on Table 7(a). (This table can be used for any $d \geq 4$, see cf. $K'(4,9)$ on Table 5.)
(d) All other cells are empty.

It is easy to check that $K'(d,2d+1)$ is an extention of $K(d,2d+1)$ for any $d \geq 3$. By Hall [5], $K'(d,2d+1)$, $d \geq 3$, can be extended to a latin square of order $2d+1$.

Similarly, let $K'(3,8)$ be depicted on Table 6. If $d \geq 4$, then define an incomplete latin square $K'(d,2d+2)$ of order $2d+2$ such that (see cf. $K'(4,10)$ on Table 6):

(a) The $(1,d) \times (1,d)$-subrectangle of $K'(d,2d+2)$ is equal to $S(d+2, \ldots, 2d+1)$.
(b) The $(d+1,2d+2) \times (1,d)$-subrectangle of $K'(d,2d+2)$ is formed from $S(2,3,\ldots,d,1,d+1,2d+2)$ by deleting the last two columns.
(c) The $(1,2d+2) \times (d+1,d+3)$-subrectangle of $K'(d,2d+2)$ is depicted on Table 7(b).
(d) All other cells are empty.

It is easy to check that $K'(d,2d+2)$ is an extention of $K(d,2d+2)$ for any $d \geq 3$. By Hall [5], $K'(d,d+2)$, $d \geq 3$, can be extended to a latin square of order $2d+2$, completing the proof. ∎

Lemma 6. *For any $d \geq 3$, $L(d,2d+1)$ and $L(d,2d+2)$ can be extended to a latin square of order $2d+1$ and $2d+2$ respectively.*

Proof: Let $L'(d,2d+1) = [l'_{i,j}]$ $(d \geq 3)$ be an incomplete latin square of order $2d+1$ such that (see Table 8):

(a) The $(1,d+1) \times (d+1,2d+1)$-subrectangle of $L'(d,n)$ is formed by the following process: Take $S(1,d,d-1,\ldots,3,d+1,2)$ and delete all entries equal to $d+1$ from it with the exception $l'_{1,2d} = d+1$.
(b) The $(1,2d+1) \times (1,d)$-subrectangle of $L'(d,2d+1)$ is the same as in the $L(d,2d+1)$.
(c) $l'_{2d+1,d+1} = l'_{2d,d+2} = \cdots = l'_{d+3,2d-1} = l'_{d+2,2d+1} = d+1$
(d) All other cells are empty.

row:						
1	2	4	3	2	4	3
2	4	3	$d+1$	4	3	$2d+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$i+3$	$i+2$	$i+1$	$i+3$	$i+2$	$i+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d-1$	3	1	d	3	$2d+2$	d
d	5	2	1	5	2	1
$d+1$	$d+2$	$d+1$	$d+3$	$d+2$	$d+1$	$d+3$
$d+2$	$d+5$	$d+3$	2	$2d+2$	$d+3$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d+i$	$d+i+3$	$d+i+2$	$d+i+1$	$d+i+3$	$d+i+2$	$d+i+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2d-1$	$d+3$	$2d+1$	$2d$	$d+3$	$2d+1$	$2d$
$2d$	$d+4$	$d+2$	$2d+1$	$d+4$	$d+2$	$2d+1$
$2d+1$	1	$d+4$	$d+2$	1	$d+4$	$d+2$
$2d+2$				$d+5$	1	$d+1$

(a) (b)

$i \in \{3, \ldots, d-2\}$

Table 7

Let $\overline{L}(d, 2d+1)$ $(d \geq 3)$ be the incomplete latin square of order $2d+1$ that arise from $L(d, 2d+1)$ by interchanging the last two columns. Then $L'(d, 2d+1)$ is an extention of $\overline{L}(d, 2d+1)$. (See $\overline{L}(4,9)$ and $L'(4,9)$ on Table 8). We can check that $\xi(L'(d, 2d+1))$ satisfies the conditions of Lemma 3. Thus, by Lemmas 3 and 4, $L(d, 2d+1)$ can be extended to a latin square of order $2d+1$.

$$\overline{L}(4,9)=\begin{array}{ccccccccc}
9&8&7&6&1&4&3&.&2\\
.&.&.&5&.&.&.&.&1\\
.&.&5&.&.&.&2&.&.\\
.&5&.&.&.&2&.&.&.\\
5&.&.&.&2&.&.&.&.\\
1&2&3&4&.&.&.&.&5\\
2&3&4&1&.&.&5&.&.\\
3&4&1&2&.&5&.&.&.\\
4&1&2&3&5&.&.&.&.
\end{array}
\qquad
L'(4,9)=\begin{array}{ccccccccc}
9&8&7&6&1&4&3&5&2\\
.&.&.&5&4&3&.&2&1\\
.&.&5&.&3&.&2&1&4\\
.&5&.&.&.&2&1&4&3\\
5&.&.&.&2&1&4&3&.\\
1&2&3&4&.&.&.&.&5\\
2&3&4&1&.&.&5&.&.\\
3&4&1&2&.&5&.&.&.\\
4&1&2&3&5&.&.&.&.
\end{array}$$

Table 8

Let $L'(d,2d+2) = [l'_{i,j}]$ $(d \geq 3)$ be an incomplete latin square of order $2d+2$ such that (see Table 9):

(a) The $(1,d+2)\times(d+1,2d+2)$-subrectangle of $L'(d,n)$ is formed by the following process: Take $S(d,1,d-1,\dots,3,x,d+1,2)$ and delete all entries equal to x and $d+1$ from it with just two exceptions $l'_{1,2d+1} = l'_{2,2d} = d+1$.

(b) The $(d+3,2d+2)\times(1,d)$-subrectangle of $L'(d,2d+2)$ is equal to $S(1,\dots,d)$.

(c) The $(1,1)\times(1,d)$-subrectangle of $L'(d,2d+2)$ is equal to

$$[2d+1,2d,\dots,d+2].$$

(d) $l'_{d+1,1} = l'_{d+2,2} = l'_{d,3} = l'_{d-1,4} = \dots = l'_{3,d} = l'_{2d+1,d+1} = l'_{2d+2,d+2} = l'_{2d,d+3} = \dots = l'_{d+4,2d-1} = l'_{d+3,2d+2} = d+1$.

(e) All other cells are empty.

$$\overline{L}(4,10)=\begin{array}{cccccccccc}
9&8&7&6&4&1&3&.&.&2\\
.&.&.&.&.&.&.&.&.&.\\
.&.&.&5&.&.&.&.&.&1\\
.&.&5&.&.&.&2&.&.&.\\
5&.&.&.&.&2&.&.&.&.\\
.&5&.&.&2&.&.&.&.&.\\
1&2&3&4&.&.&.&.&.&5\\
2&3&4&1&.&.&5&.&.&.\\
3&4&1&2&5&.&.&.&.&.\\
4&1&2&3&.&5&.&.&.&.
\end{array}
\qquad
L'(4,10)=\begin{array}{cccccccccc}
9&8&7&6&4&1&3&.&5&2\\
.&.&.&.&1&3&.&5&2&4\\
.&.&.&5&3&.&.&2&4&1\\
.&.&5&.&.&.&2&4&1&3\\
5&.&.&.&.&2&4&1&3&.\\
.&5&.&.&2&4&1&3&.&.\\
1&2&3&4&.&.&.&.&.&5\\
2&3&4&1&.&.&5&.&.&.\\
3&4&1&2&5&.&.&.&.&.\\
4&1&2&3&.&5&.&.&.&.
\end{array}$$

Table 9

We can check that $L'(d,2d+2)$ is an extention of an incomplete latin square which we get from $L(d,2d+2)$ by permutations of rows and columns. (See

$\overline{L}(4,10)$, a permutation of $L(4,10)$, and $L'(4,10)$, which is an extention of $\overline{L}(4,10)$ in Table 9.) Clearly $\xi(L'(d,2d+2))$ satisfies the conditions of Lemma 3. Thus, by Lemmas 3 and 4, $L(d,2d+2)$ can be extended to a latin square of order $2d+2$. ∎

Let A be an incomplete latin square of order n. Denote by $A^{(k)}$ the incomplete latin square of order $n+k$ which arises from A by adding k new rows and k new columns of empty cells to A. If $A^{(k)}$ can be extended to a latin square of order $n+k$ then we say that A can be extended to a latin square of order $n+k$. In [11] (see also [10]) is proved:

Lemma 7. *Let A be an incomplete latin square of order n such that the cells in the intersections of the ith row and the jth column of A are empty for any $i,j \in \{1,\ldots,n\}$ and $i+j \geq n+2$. Suppose A can be extended to a latin square of order n. Then A can be extended to a latin square of order $n+k$ for any $k \geq 1$.*

Now we are able to prove Lemma 2.

$$L(4,10) = \begin{array}{cccccccccc} 9&8&7&6&1&4&3&2&.&. \\ .&.&.&5&.&.&.&1&.&. \\ .&.&5&.&.&.&2&.&.&. \\ .&5&.&.&.&2&.&.&.&. \\ 5&.&.&.&2&.&.&.&.&. \\ .&.&.&.&.&.&.&.&.&. \\ 1&2&3&4&.&.&.&5&.&. \\ 2&3&4&1&.&.&5&.&.&. \\ 3&4&1&2&.&5&.&.&.&. \\ 4&1&2&3&5&.&.&.&.&. \end{array} \qquad \xi(L(4,10)) = \begin{array}{cccccccccc} 5&8&7&6&.&4&3&2&1&. \\ 8&.&.&.&4&.&.&.&.&. \\ .&7&.&.&3&.&.&.&.&. \\ .&6&.&.&2&.&.&.&.&. \\ .&5&.&.&1&.&.&.&.&. \\ .&.&.&.&.&.&.&.&.&. \\ 1&2&3&4&8&.&.&.&.&. \\ 4&1&2&3&7&.&.&.&.&. \\ 3&4&1&2&6&.&.&.&.&. \\ 2&3&4&1&5&.&.&.&.&. \end{array}$$

Table 10

Proof of Lemma 2: We have verified Lemma 2 if $d \geq 3$ and $n = 2d+1, 2d+2$.

It is easily seen that $K(d,n)^{(k)}$ $(L(d,n)^{(k)})$ is equal to $K(d,n+k)$ $(L(d,n+k)$, respectively) as to permutations of rows and columns. Thus it remains to show that $K(d,2d+2)$ $(L(d,2d+2))$ can be extended to a latin square of order n for any $n > 2d+2$.

Take $K(d,2d+2)$ $(d \geq 3)$ and interchange the 2nd and the $(d+3)$th columns and the 3th and the $(d+2)$nd columns of $K(d,2d+2)$. We get an incomplete latin square which satisfies the conditions of Lemma 7. Thus, by Lemmas 4, 5 and 7, $K(d,2d+2)$ can be extended to a latin square of order n for any $n > 2d+2$.

Take $\xi(L(d,2d+2))$ for any $d \geq 3$ (see $\xi(L(4,10))$ in Table 10). We can check that there exists a permutation of rows and columns of $\xi(L(d,2d+2))$ such that the new incomplete latin square satisfies the conditions of Lemma 7. Then from Lemmas 4, 6 and 7 folows that $L(d,2d+2)$ can be extended to a latin square of order n for any $n > 2d+2$, completing the proof of Lemma 2. ∎

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