Decomposition of K_n into Cycles of Length At Most Fifty

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ABSTRACT. It is shown that the necessary conditions for the existence of a k-cycle system of order n are sufficient for $k \in \{20, 24, 28, 30, 33, 35, 36, 39, 40, 42, 44, 45, 48\}$, thus settling the problem for all $k \leq 50$.

1 Introduction

The complete graph K_n is *partitionable into k-cycles* if the edges of K_n can be partitioned into sets, each of which induces a cycle of length k. The resulting partition is called a *k-cycle system.*

There are three conditions necessary for K_n to be partitionable.

i) If $n > 1$, then $n \geq k$,

ii) n is odd, and

iii) $2k | n(n-1)$.

These three necessary conditions have been shown to be sufficient when,

a) $k = p^r$ (p is prime) [6], [8],

b) $k = 2p^r$ (p is prime) [1],

c) $k \leq 31, k$ odd [4],

d) $k \leq 18, k$ even [2],

e) $n \equiv 1 \pmod{2k}$, [6], [8], and

f) $n \equiv k \pmod{2k}$, k odd [5].

If we set $K = \{20, 24, 28, 30, 33, 35, 36, 39, 40, 42, 44, 45, 48\}$ then the results above show that the necessary conditions for the existence of a k-cycle system are sufficient for all $k \leq 50$, $k \notin K$. In this paper we use the following theorem, as well as e) and f) above, to show that the necessary conditions for the existence of a k-cycle system are also sufficient for each $k \in K$. These results are presented in Lemmas 1 to 13.

Theorem 1. *Suppose there is a k-cycle system of order n for all admissible n with $k \leq n < 3k$. Then there is a k-cycle system of order n for all admissible n.*

Proof: In the case of k-even, see [7]. For k-odd, see [4]. □

When $k \in \{20, 24, 28, 30, 33, 35, 36, 39, 40, 42, 44, 45, 48\}$ the following cycle systems are given.

2 Constructions

In all of the designs, the underlying set is $\mathcal{X} = (\mathbf{Z}_\alpha \times \mathbf{Z}_\sigma) \bigcup I$, where I is either ϕ or $\{\infty\}$.

Permutations π_1, μ_s, and ρ_s are defined by

$$\begin{aligned} \pi_1(i,j) &= (i+1,j) \\ \mu_s(i,j) &= (i,sj) \\ \text{and } \rho_s(i,j) &= (i,j+s). \end{aligned}$$

The point ∞ when it exists, is fixed by these.

In all cases, ij is the ordered pair (i,j).

Lemma 1. *A 33-cycle system of order n exists if and only if $n \equiv 1, 33, 45$ or 55 (mod 66).*

Proof: It is only necessary to construct designs of order 45 and 55.

For $n = 45$, the underlying set $\mathcal{X} = \mathbf{Z}_{15} \times \mathbf{Z}_3$ is used. Applying $\langle \pi_1 \rangle$ to the representative cycles [00, 101, 10, 91, 20, 81, 30, 71, 40, 61, 50, 51, 102, 41, 112, 31, 122, 21, 132, 11, 142, 01, 02, 100, 52, 90, 62, 80, 72, 70, 82, 60, 92] and [00, 140, 10, 40, 80, 30, 120, 50, 11, 20, 141, 02, 121, 12, 90, 22, 130, 42, 32, 52, 122, 62, 112, 72, 102, 81, 01, 91, 111, 61, 51, 21, 131], we have two orbits with fifteen 33-cycles each.

For $n = 55$, the underlying set is $\mathcal{X} = (\mathbf{Z}_9 \times \mathbf{Z}_6) \bigcup \{\infty\}$. The first two orbits with nine 33-cycles each are obtained by applying $\langle \pi_1 \rangle$ to the representative cycles [05, 12, 85, 22, 14, 32, 04, 60, 84, 70, 62, 80, 52, 64, 20, 72, 35, 21, 45, 11, 44, 01, 54, 81, 03, 71, 13, 50, 23, 40, 33, 15, 43] and [05, 62, 24, 30, 42, 64, 32, 74, 00, 54, 80, 12, 60, 02, 45, 71, 65, 61, 15, 31, 44, 21, 13, 51, 23, 41, 43, 10, 53, 55, 33, 25, 83]. The remaining twenty seven 33-cycles are obtained by applying $\langle\pi_1\rangle$ to $\beta, \rho_2\beta$, and $\rho_4\beta$, where $\beta =$ [00, ∞, 01, 62, 81, 72, 63, 82, 53, 02, 25, 12, 15, 04, 35, 14, 55, 50, 45, 60, 61, 21, 51, 31, 41, 83, 44, 03, 34, 30, 40, 80, 20]. □

Lemma 2. *A 35-cycle system of order n exists if and only if $n \equiv 1, 15, 21$, or 35 (mod 70).*

Proof: It is only necessary to construct designs of order 85 and 91.

For $n = 85$, the underlying set is $\mathcal{X} = \mathbf{Z}_{17} \times \mathbf{Z}_5$. Apply $\langle\pi_1\rangle$ to the six 35-cycles [00, 161, 10, 151, 122, 141, 132, 123, 142, 113, 84, 103, 94, 12, 104, 02, 70, 162, 80, 03, 90, 163, 61, 153, 71, 164, 81, 154, 152, 140, 143, 131, 134, 160, 124], [00, 81, 160, 91, 22, 101, 12, 93, 02, 103, 34, 113, 24, 122, 14, 132, 80, 142, 70, 03, 60, 13, 131, 23, 121, 54, 111, 64, 52, 50, 43, 41, 104, 10, 114], [00, 111, 160, 121, 82, 131, 72, 13, 62, 23, 154, 33, 144, 102, 134, 112, 90, 122, 80, 63, 100, 73, 51, 83, 41, 24, 61, 34, 92, 150, 43, 101, 94, 10, 74], [00, 01, 160, 34, 150, 163, 140, 03, 130, 13, 14, 113, 42, 43, 91, 53, 81, 63, 71, 84, 61, 94, 51, 104, 152, 114, 142, 124, 132, 131, 32, 60, 12, 30, 162], [00, 21, 160, 31, 41, 61, 91, 131, 142, 121, 152, 162, 12, 42, 82, 93, 72, 103, 113, 133, 163, 33, 44, 23, 54, 64, 84, 114, 154, 120, 144, 130, 110, 150, 140], and [00, 51, 160, 61, 111, 01, 101, 11, 72, 21, 62, 02, 52, 122, 42, 83, 32, 93, 153, 103, 13, 113, 04, 123, 164, 44, 154, 64, 134, 130, 124, 140, 90, 150, 80], for a total of 102 cycles in six orbits of seventeen 35-cycles each.

For $n = 91, \mathcal{X} = \mathbf{Z}_{13} \times \mathbf{Z}_7$ is the underlying set. Apply $\langle\pi_1\rangle$ to $\beta, \mu_2\beta$, and $\mu_3\beta$, where $\beta =$ [00, 01, 120, 11, 110, 21, 62, 31, 52, 41, 42, 43, 32, 53, 22, 63, 104, 73, 94, 83, 84, 85, 74, 95, 64, 105, 16, 115, 06, 125, 126, 90, 116, 100, 106] and [00, 81, 120, 91, 110, 101, 92, 111, 82, 121, 72, 23, 62, 33, 52, 43, 34, 53, 24, 63, 14, 95, 04, 105, 124, 115, 106, 125, 96, 05, 86, 20, 66, 10, 126]. The remaining thirty-nine 35-cycles are obtained by applying $\langle\pi_1\rangle$ to the 35-cycles [00, 53, 116, 52, 105, 41, 94, 20, 83, 26, 72, 15, 61, 124, 40, 112, 50, 102, 34, 92, 14, 76, 04, 56, 101, 36, 91, 23, 81, 03, 65, 13, 85, 10, 95], [00, 51, 102, 23, 74, 125, 46, 80, 11, 71, 21, 61, 122, 32, 112, 42, 103, 33, 123, 43, 104, 14, 94, 24, 85, 15, 95, 05, 66, 106, 56, 116, 30, 120, 70] and [00, 73, 126, 52, 115, 41, 114, 20, 91, 81, 61, 31, 102, 112, 92, 62, 03, 13, 33, 63, 04, 14, 34, 64, 05, 35, 45, 25, 96, 106, 06, 116, 40, 10, 120]. □

Lemma 3. *A 39-cycle system of order n exists if and only if $n \equiv 1, 13, 27$, or 39 (mod 78).*

Proof: It is only necessary to construct designs of order 91 and 105.

For $n = 91$, the underlying set is $\mathcal{X} = \mathbf{Z}_7 \times \mathbf{Z}_{13}$. Applying $\langle\pi_1\rangle$ to $\mu_s\beta, 1 \le s \le 6$, where $\beta =$ [00, 41, 60, 51, 42, 61, 32, 03, 22, 13, 04, 23, 64, 55, 14, 65, 56, 15, 66, 47, 06, 67, 38, 57, 48, 39, 58, 29, 610, 19, 010, 611, 110, 511, 212, 411, 312, 10, 412] and [00, 11, 60, 21, 52, 31, 42, 53, 32, 63, 24, 03, 14, 25, 04, 35, 56, 45, 06, 37, 16, 27, 38, 17, 48, 09, 58, 69, 010, 59, 110, 211, 610, 111, 212, 011, 312, 20, 012], we get twelve orbits with seven 39-cycles each. The remaining three orbits can be obtained by applying $\langle \pi_1 \rangle$ to the 39-cycles [00, 06, 012, 05, 011, 04, 010, 03, 09, 02, 08, 01, 07, 10, 15, 110, 12, 17, 112, 412, 44, 49, 39, 59, 29, 21, 26, 36, 66, 46, 411, 43, 33, 53, 23, 28, 30, 20, 40], [00, 03, 06, 09, 012, 02, 05, 08, 011, 01, 04, 07, 010, 10, 14, 18, 28, 58, 38, 312, 112, 13, 17, 111, 211, 511, 311, 32, 52, 22, 12, 16, 110, 11, 15, 55, 45, 65, 69], and [00, 02, 04, 06, 08, 010, 012, 01, 03, 05, 07, 09, 011, 10, 11, 21, 51, 31, 32, 33, 34, 44, 24, 54, 55, 56, 57, 27, 47, 37, 38, 39, 310, 410, 210, 510, 511, 512, 612].

For $n = 105$, the underlying set is $\mathcal{X} = \mathbf{Z}_{35} \times \mathbf{Z}_3$. Applying $\langle \pi_1 \rangle$ to the 39-cycles [00, 341, 10, 331, 20, 321, 30, 311, 40, 301, 50, 291, 60, 281, 152, 271, 162, 261, 172, 251, 182, 241, 192, 231, 202, 221, 212, 200, 222, 190, 232, 180, 242, 170, 252, 160, 262, 150, 272], [00, 01, 340, 11, 330, 21, 320, 31, 310, 41, 300, 51, 290, 61, 182, 71, 172, 81, 162, 91, 152, 101, 142, 111, 132, 121, 122, 120, 92, 150, 62, 180, 32, 210, 02, 20, 12, 80, 302], [00, 131, 340, 141, 292, 151, 282, 150, 302, 140, 301, 121, 311, 111, 321, 101, 331, 91, 341, 81, 242, 72, 232, 82, 222, 92, 212, 102, 202, 112, 210, 40, 200, 100, 220, 80, 170, 20, 130], and [00, 211, 10, 201, 20, 191, 181, 161, 131, 91, 41, 101, 171, 251, 72, 241, 82, 231, 92, 172, 112, 42, 32, 12, 322, 02, 52, 90, 332, 120, 312, 130, 210, 260, 250, 310, 270, 340, 320], we get four orbits with thirty-five 39-cycles each. □

Lemma 4. *A 45-cycle system of order n exists if and only if* $n \equiv 1, 45, 55,$ *or* 81 *(mod 90).*

Proof: Only designs of order 55 and 81 need to be constructed.

For $n = 55$ the underlying set is $\mathcal{X} = \mathbf{Z}_{11} \times \mathbf{Z}_5$. Three orbits with eleven 45-cycles each can be obtained by applying $\langle \pi_1 \rangle$ to the 45-cycles [00, 102, 10, 92, 64, 82, 74, 61, 84, 51, 23, 41, 33, 20, 53, 30, 32, 34, 31, 43, 40, 01, 50, 101, 60, 91, 22, 81, 42, 11, 72, 03, 62, 13, 52, 83, 44, 93, 14, 73, 04, 70, 94, 100, 54], [00, 101, 10, 91, 62, 81, 72, 63, 82, 53, 24, 43, 34, 20, 54, 90, 01, 02, 03, 04, 30, 42, 80, 32, 100, 12, 74, 22, 64, 102, 14, 21, 104, 51, 94, 31, 33, 71, 13, 61, 83, 40, 23, 50, 103], and [00, 01, 74, 91, 13, 81, 92, 40, 72, 84, 52, 63, 100, 53, 64, 60, 71, 21, 31, 101, 11, 41, 62, 32, 102, 42, 22, 12, 33, 23, 03, 43, 103, 73, 94, 34, 104, 24, 44, 54, 70, 10, 50, 80, 90].

For $n = 81$, the underlying set is $\mathcal{X} = \mathbf{Z}_9 \times \mathbf{Z}_9$. Apply $\langle \pi_1 \rangle$ to $\beta, \mu_2\beta$ and $\mu_4\beta$ where $\beta =$ [00, 81, 10, 71, 20, 61, 32, 51, 42, 01, 52, 03, 22, 73, 82, 53, 24, 43, 34, 83, 44, 35, 84, 45, 74, 55, 46, 65, 36, 75, 26, 07, 16, 67, 06, 47, 28, 37, 88, 27, 68, 50, 18, 80, 38] to obtain three orbits of nine 45-cycles each. The remaining orbits are obtained by applying $\langle \pi_1 \rangle$ to [00, 14, 38, 63, 67, 72, 06, 31, 35, 40, 64, 08, 03, 17, 32, 66, 10, 56, 80, 23, 70, 33, 26, 43, 16, 11, 44, 01, 54, 57, 34, 47, 71, 37, 81, 05, 78, 85, 58, 12, 68, 02, 65, 82, 75], [00, 04, 18, 33, 67, 62, 76, 01, 35, 30, 64, 68, 73, 07, 32, 36, 40, 46, 60, 43, 50, 23, 26, 03, 16, 21, 14, 41, 24, 17, 44, 27, 31, 37, 51, 75, 08, 85, 88, 82, 78, 02, 55, 22, 65], [00, 01, 02, 03, 04, 05, 06, 07, 08, 10, 12, 14, 16, 18, 11, 13, 15, 82, 85, 72, 88, 52, 78, 25, 68, 35, 67, 34, 87, 44, 41, 54, 31, 57, 21, 37, 40, 63, 60, 73, 26, 83, 46, 20, 36], [00, 12, 80, 22, 82, 42, 74, 52, 64, 04, 54, 66, 44, 76, 16, 56, 68, 46, 78, 48, 08, 31, 18, 21, 51, 01, 13, 81, 23, 53, 03, 15, 83, 25, 55, 05, 07, 85, 17, 47, 87, 10, 77, 70, 40], and [00, 11, 80, 70, 50, 81, 02, 71, 61, 41, 72, 83, 62, 52, 32, 63, 74, 53, 43, 23, 54, 65, 44, 34, 14, 45, 56, 35, 25, 05, 36, 47, 26, 16, 86, 27, 38, 17, 07, 77, 18, 40, 48, 68, 78]. □

Lemma 5. *A 20-cycle system of order n exists if and only if $n \equiv 1$ or 25 (mod 40).*

Proof: It is only necessary to construct the design of order 25.

The underlying set is $\mathcal{X} = \mathbf{Z}_5 \times \mathbf{Z}_5$. Apply $\langle\pi_1\rangle$ to β and $\mu_2\beta$ where $\beta =$ [00, 01, 02, 03, 04, 10, 21, 40, 11, 32, 41, 22, 13, 42, 23, 14, 33, 44, 20, 34], next apply $\langle\pi_1\rangle$ to [00, 41, 02, 13, 34, 30, 22, 12, 32, 44, 04, 24, 31, 01, 11, 33, 43, 23, 20, 40] for three orbits of five 20-cycles each. □

Lemma 6. *A 24-cycle system of order n exists if and only if $n \equiv 1$ or 33 (mod 48).*

Proof: Only the design of order 33 needs to be constructed.

The underlying set is $\mathcal{X} = \mathbf{Z}_{11} \times \mathbf{Z}_3$. The permutation $\langle\pi_1\rangle$ is applied to the 24-cycles [00, 81, 100, 91, 82, 101, 72, 60, 92, 70, 11, 80, 01, 90, 41, 42, 21, 32, 71, 22, 50, 12, 10, 102] and [00, 01, 100, 61, 51, 71, 41, 81, 31, 62, 21, 72, 52, 22, 82, 12, 02, 50, 102, 60, 70, 90, 40, 80] to obtain two orbits of eleven 24-cycles each. □

Lemma 7. *A 28-cycle system of order n exists if and only if $n \equiv 1$ or 49 (mod 56).*

Proof: It is necessary to construct only the design of order 49.

The underlying set $\mathcal{X} = \mathbf{Z}_7 \times \mathbf{Z}_7$ is used. Three of the six orbits with seven 28-cycles each are found by applying $\langle\pi_1\rangle$ to $\beta, \mu_2\beta$, and $\mu_3\beta$, where $\beta =$ [00, 61, 42, 13, 04, 55, 26, 10, 41, 60, 31, 22, 51, 12, 43, 52, 33, 14, 53, 24, 65, 34, 25, 56, 05, 66, 40, 36]. The remaining three orbits are obtained by applying $\langle\pi_1\rangle$ to [00, 21, 42, 63, 14, 35, 56 , 10, 33, 30, 43, 46, 23, 36, 32, 16, 22, 25, 02, 15, 11, 65, 01, 04, 51, 64, 60, 44], [00, 11, 22, 33, 44, 55, 66, 10, 32, 30, 42, 54, 52, 04, 26, 14, 16, 31, 36, 41, 53, 51, 03, 15, 63, 65, 20, 05], and [00, 01, 11, 61, 21, 22, 32, 12, 42, 43, 13, 33, 23, 24, 34, 14, 44, 45, 55, 35, 05, 06, 16, 66, 26, 20, 60, 10]. □

Lemma 8. *A 30-cycle system of order n exists if and only if $n \equiv 1, 21, 25$, or 45 (mod 60).*

Proof: Only designs of order 45, 81, and 85 need to be constructed.

For $n = 45$, the underlying set is $\mathcal{X} = \mathbf{Z}_{15} \times \mathbf{Z}_3$. First construct three 30-cycles applying $\rho_s, 0 \leq s \leq 2$, to the cycle [00, 11, 10, 21, 20, 31, 30, 41, 40, 51, 50, 61, 60, 71, 70, 81, 80, 91, 90, 101, 100, 111, 110, 121, 120, 131, 130, 141, 140, 01]. The remaining thirty 30-cycles can be constructed by applying $\langle\pi_1\rangle$ to [00, 21, 42, 60, 11, 70, 01, 80, 141, 10, 131, 140, 101, 62, 91, 72, 81, 142, 71, 02, 61, 12, 130, 32, 110, 122, 30, 92, 50, 22] and [00, 51, 10, 41, 31, 11, 131, 91, 141, 81, 01, 52, 21, 62, 72, 92, 122, 12, 112, 22, 102, 20, 132, 30, 70, 40, 100, 50, 60, 80].

For $n = 81$, the underlying set $\mathcal{X} = \mathbf{Z}_{27} \times \mathbf{Z}_3$ was used. Apply $\langle\pi_1\rangle$ to [00, 261, 252, 240, 141, 230, 151, 220, 161, 210, 171, 200, 181, 162, 191, 152, 201, 142, 211, 132, 221, 122, 70, 172, 130, 222, 160, 242, 260, 22], [00, 161, 52, 250, 251, 240, 261, 230, 01, 220, 11, 210, 21, 102, 31, 92, 41, 82, 51, 72, 61, 62, 60, 32, 70, 22, 30, 242, 40, 192], [00, 91, 182, 40, 141, 30, 151, 11, 161, 01, 171, 261, 181, 62, 191, 52, 192, 42, 202, 32, 212, 22, 110, 12, 120, 200, 20, 140, 10, 170], and [00, 131, 260, 141, 151, 171, 201, 241, 191, 251, 181, 12, 161, 02, 22, 52, 92, 82, 132, 72, 142, 30, 152, 20, 60, 130, 150, 140, 110, 50], for four orbits of twenty-seven 30-cycles each.

For $n = 85$, the underlying set used is $\mathcal{X} = \mathbf{Z}_{17} \times \mathbf{Z}_5$. Sixty-eight 30-cycles can be constructed by applying $\langle\pi_1\rangle$ to β and $\mu_2\beta$, where $\beta =$ [00, 161, 152, 143, 134, 120, 61, 110, 71, 100, 81, 62, 91, 52, 101, 42, 153, 32, 163, 22, 03, 144, 33, 14, 63, 24, 160, 34, 10, 64] and [00, 101, 32, 133, 64, 160, 41, 150, 51, 140, 61, 152, 71, 142, 81, 132, 13, 122, 23, 112, 33, 124, 43, 114, 53, 104, 20, 24, 30, 154]. The remaining fifty-one cycles can be obtained by applying $\langle\pi_1\rangle$ to the 30-cycles [00, 01, 160, 11, 32, 21, 22, 23, 12, 33, 54, 43, 44, 100, 34, 110, 112, 90, 10 2, 104, 92, 114, 121, 124, 141, 163, 151, 153, 40, 143], [00, 32, 64, 91, 123, 160, 21, 101, 01, 61, 11, 42, 92, 152, 52, 142, 03, 83, 13, 73, 23, 54, 104, 44, 114, 24, 50, 150, 60, 120], and [00, 42, 84, 121, 163, 70, 111, 151, 141, 161, 131, 02, 12, 32, 62, 102, 143, 153, 03, 33, 73, 114, 124, 144, 04, 44, 80, 40, 10, 160]. □

Lemma 9. *A 36-cycle system of order n exists if and only if $n \equiv 1$ or 9 (mod 72).*

Proof: It is only necessary to construct the design of order 81. The underlying set is $\mathcal{X} = \mathbf{Z}_9 \times \mathbf{Z}_9$. Three orbits of nine 36-cycles each can be obtained by applying $\langle\pi_1\rangle$ to $\beta, \mu_2\beta$, and $\mu_4\beta$, where $\beta =$ [00, 81, 72, 63, 54, 45, 36, 27, 18, 80, 41, 70, 51, 32, 61, 22, 73, 12, 83, 64, 13, 74, 55, 04, 65, 46, 85, 56, 37, 76, 47, 28, 67, 38, 20, 58]. The remaining sixty-three 36-cycles can be obtained by applying $\langle\pi_1\rangle$ to the seven 36-cycles [06, 47, 88, 40, 81, 32, 73, 24, 17, 34, 07, 61, 87, 71, 64, 01, 74, 25, 08, 35, 28, 12, 38, 02, 65, 82, 75, 26, 10, 36, 00, 63, 70, 53, 16, 23], [07, 50, 02, 44, 86, 38, 71, 23, 66, 33, 06, 30, 76, 20, 63, 10, 43, 85, 48, 15, 58, 12, 78, 22, 75, 32, 65, 17, 61, 27, 51, 84, 41, 04, 57, 54], [00, 44, 88, 33, 77, 22, 66, 11, 14, 01, 24, 57, 34, 47, 41, 37, 51, 05, 08, 85, 18, 32, 38, 42, 55, 52, 75, 30, 43, 20, 23, 26, 03, 16, 10, 86], [00, 01, 12, 33, 64, 65, 76, 07, 38, 30, 42, 20, 52, 84, 82, 14, 16, 74, 86, 88, 66, 78, 11, 08, 21, 23, 81, 13, 15, 83, 25, 27, 05, 17, 10, 87], [00, 02, 14, 36, 68, 61, 73, 85, 27, 50, 54, 40, 64, 88, 74, 78, 03, 08, 13, 47, 23, 37, 42, 17, 32, 46, 22, 56, 51, 26, 31, 45, 11, 15, 20, 05], [00, 11, 80, 21, 61, 62, 41, 72, 22, 23, 12, 43, 83, 14, 03, 04, 44, 55, 34, 65, 15, 46, 25, 26, 66, 67, 56, 87, 37, 58, 47, 48, 08, 30, 28, 40], and [00, 34, 44, 24, 84, 28, 38, 18, 48, 73, 83, 63, 03, 07, 17, 87, 27, 22, 32, 12, 42, 46, 36, 56, 26, 41, 31, 51, 21, 45, 55, 25, 05, 30, 60, 80]. □

Lemma 10. *A 40-cycle system of order n exists if and only if $n \equiv 1$ or 65 (mod 80).*

Proof: Only the design of order 65 needs to be constructed.

The underlying set used is $\mathcal{X} = \mathbf{Z}_{13} \times \mathbf{Z}_5$. Apply $\langle \pi_1 \rangle$ to β and $\mu_2\beta$, where $\beta =$ [00, 121, 112, 103, 94, 80, 61, 90, 51, 100, 41, 110, 31, 32, 21, 42, 11, 52, 01, 62, 43, 72, 33, 82, 23, 92, 13, 14, 03, 24, 123, 34, 113, 44, 40, 64, 70, 104, 120, 54] to obtain twenty-six 40-cycles. The remaining twenty-six 40-cycles can be constructed by applying $\langle \pi_1 \rangle$ to the 40-cycles [00, 01, 120, 11, 110, 21, 81, 31, 121, 62, 111, 72, 101, 82, 22, 102, 12, 13, 02, 23, 122, 33, 93, 43, 83, 24, 73, 34, 63, 44, 104, 54, 94, 50, 124, 40, 04, 30, 60, 20] and [00, 02, 120, 12, 110, 22, 32, 52, 82, 24, 72, 34, 62, 44, 54, 74, 104, 101, 94, 111, 84, 121, 01, 21, 51, 33, 61, 23, 71, 13, 43, 63, 53, 80, 123, 40, 113, 20, 10, 60]. □

Lemma 11. *A 42-cycle system of order n exists if and only if $n \equiv 1, 21, 49$ or 57 (mod 84).*

Proof: It is necessary only to construct designs of order $49, 57$, and 105.

For $n = 49$, the underlying set used is $\mathcal{X} = \mathbf{Z}_7 \times \mathbf{Z}_7$. The first three orbits of seven 42-cycles each can be constructed by applying $\langle \pi_1 \rangle$ to $\beta, \mu_2\beta$, and $\mu_3\beta$, where β is [00, 61, 52, 43, 34, 25, 16, 60, 01, 50, 11, 40, 41, 42, 31, 62, 21, 02, 33, 12, 23, 22, 63, 04, 03, 54, 13, 44, 55, 14, 15, 64, 45, 46, 05, 56, 35, 66, 10, 06, 30, 36]. The remaining seven 42-cycles can be obtained by applying $\langle \pi_1 \rangle$ to the 42-cycle [00, 52, 04, 56, 01, 33, 45, 30, 13, 36, 12, 22, 42, 02, 25, 51, 64, 50, 31, 21, 61, 41, 62, 43, 23, 63, 03, 24, 34, 14, 44, 05, 65, 15, 55, 66, 46, 16, 26, 10, 20, 40].

For $n = 57$, the underlying set is $\mathcal{X} = \mathbf{Z}_{19} \times \mathbf{Z}_3$. Two orbits of nineteen 42-cycles each may be constructed applying $\langle \pi_1 \rangle$ to the 42-cycles [00, 181, 10, 171, 20, 161, 30, 151, 40, 141, 50, 131, 121, 101, 71, 152, 61, 162, 51, 172, 41, 182, 31, 02, 21, 12, 42, 22, 32, 180, 52, 170, 62, 160, 72, 150, 82, 140, 92, 130, 70, 120] and [00, 71, 10, 61, 20, 51, 82, 41, 92, 31, 102, 90, 112, 80, 132, 160, 181, 170, 171, 101, 11, 91, 151, 111, 161, 162, 141, 152, 72, 142, 42, 02, 62, 12, 30, 32, 40, 120, 150, 130, 140, 100].

For $n = 105$, the underlying set is $\mathcal{X} = \mathbf{Z}_{21} \times \mathbf{Z}_5$. Four cycles are constructed by applying $\rho_s, 0 \leq s \leq 3$, to the cycle [00, 01, 200, 201, 190, 191, 180, 181, 170, 171, 160, 161, 150, 151, 140, 141, 130, 131, 120, 121, 110, 111, 100, 101, 90, 91, 80, 81, 70, 71, 60, 61, 50, 51, 40, 41, 30, 31, 20, 21, 10, 11]. Eighty-four of the remaining 42-cycles can be constructed applying $\langle\pi_1\rangle$ to the 42-cycles [00, 121, 10, 111, 20, 101, 182, 91, 192, 81, 202, 113, 02, 103, 12, 93, 174, 83, 184, 73, 194, 100, 04, 90, 14, 80, 82, 70, 92, 114, 102, 104, 131, 94, 141, 193, 151, 183, 190, 173, 200, 163], [00, 71, 142, 03, 152, 203, 44, 193, 54, 60, 34, 50, 112, 200, 62, 160, 32, 140, 22, 144, 82, 194, 122, 14, 132, 04, 81, 204, 61, 154, 51, 164, 11, 123, 21, 113, 31, 153, 91, 163, 10, 143], [00, 61, 10, 51, 20, 41, 151, 01, 131, 201, 111, 132, 101, 142, 91, 152, 02, 72, 202, 112, 12, 53, 32, 63, 123, 193, 113, 23, 133, 174, 143, 164, 104, 34, 114, 24, 124, 180, 190, 170, 200, 160], and [00, 32, 200, 42, 52, 72, 102, 142, 192, 14, 182, 24, 34, 54, 84, 44, 94, 91, 74, 81, 71, 51, 101, 61, 31, 53, 41, 43, 203, 163, 133, 113, 103, 190, 73, 170, 134, 180, 70, 160, 80, 150]. The last two orbits of twenty-one 42-cycles each are constructed applying $\langle\pi_1\rangle$ to β and $\mu_2\beta$, where $\beta =$ [00, 201, 10, 191, 162, 181, 172, 163, 182, 153, 124, 143, 134, 120, 144, 110, 154, 100, 21, 90, 31, 80, 41, 02, 51, 202, 61, 192, 113, 152, 83, 132, 73, 14, 63, 204, 33, 164, 160, 34, 170, 64]. □

Lemma 12. *A 44-cycle system of order n exists if and only if $n \equiv 1$ or 33 (mod 88).*

Proof: It is only necessary to construct the design of order 121.

The underlying set used is $\mathcal{X} = \mathbf{Z}_{11} \times \mathbf{Z}_{11}$. Ten orbits of eleven 44-cycles each can be constructed applying $\langle\pi_1\rangle$ to $\mu_s\beta, 1 \leq s \leq 5$, where $\beta =$ [00, 101, 92, 83, 74, 65, 56, 47, 38, 29, 110, 100, 81, 10, 91, 52, 71, 42, 03, 32, 13, 104, 23, 94, 75, 04, 85, 66, 105, 76, 97, 86, 87, 48, 77, 58, 39, 78, 49, 210, 69, 310, 20, 510] and [00, 61, 12, 73, 24, 85, 36, 97, 48, 109, 510, 10, 41, 100, 31, 72, 21, 52, 83, 42, 93, 34, 03, 44, 95, 64, 105, 46, 15, 56, 107, 76, 07, 58, 27, 68, 09, 88, 19, 610, 39, 710 80, 810]. The remaining fifty-five 44-cycles can be constructed applying $\langle\pi_1\rangle$ to the 44-cycles [00, 05, 110, 34, 39, 43, 68, 22, 27, 31, 56, 100, 104, 80, 94, 108, 84, 88, 81, 78, 91, 95, 71, 85, 89, 75, 99, 02, 109, 102, 66, 92, 76, 910, 86, 810, 83, 610, 73, 97, 93, 107, 10, 77], [00, 15, 310, 34, 49, 63, 68, 32, 47, 61, 66, 100, 103, 80, 93, 06, 03, 16, 39, 26, 29, 21, 09, 11, 24, 01, 04, 17, 14, 37, 1010, 27, 010, 02, 910, 102, 05, 92, 95, 98, 85, 108, 10, 78], [00, 25, 210, 34, 59, 53, 68, 42, 67, 61, 76, 90, 92, 80, 102, 14, 02, 04, 06, 104, 16, 38, 26, 28, 410, 48, 510, 51, 310, 41, 03, 31, 13, 35, 33, 45, 57, 55, 77, 99, 87, 89, 100, 69], [00, 21, 42, 63, 84, 105, 16, 87, 108, 19, 310, 50, 51, 91, 41, 52, 92, 32, 33, 73, 23, 34, 74, 24, 25, 65, 05, 06, 76, 26, 107, 37, 97, 98, 28, 78, 89, 49, 99, 910, 410, 810, 10, 70], and [00, 11, 21, 41, 71, 72, 62, 42, 12, 23, 33, 53, 83, 84, 74, 54, 24, 35, 45, 25, 55, 66, 76, 56, 86, 67, 77, 57, 87, 98, 108, 88, 58, 59, 69, 89, 09, 110, 010, 210, 1010, 40, 50, 20]. □

Lemma 13. *A 48-cycle system of order n exists if and only if $n \equiv 1$ or 33 (mod 96).*

Proof: Only the design of order 129 needs to be constructed.

On the underlying set $\mathcal{X} = \mathbf{Z}_{43} \times \mathbf{Z}_3$ apply $\langle \pi_1 \rangle$ to the 48-cycles [00, 421, 10, 411, 20, 401, 30, 391, 40, 381, 50, 371, 60, 361, 232, 351, 242, 341, 252, 331, 262, 321, 272, 311, 282, 301, 292, 280, 302, 270, 312, 260, 322, 250, 332, 240, 342, 230, 352, 220, 61, 210, 71, 222, 81, 212, 420, 202], [00, 261, 10, 251, 20, 241, 30, 231, 40, 221, 50, 211, 60, 201, 372, 191, 382, 181, 392, 171, 402, 161, 412, 151, 422, 141, 02, 200, 12, 190, 22, 180, 32, 170, 42, 160, 52, 150, 62, 140, 271, 270, 281, 302, 291, 362, 420, 142], [00, 121, 10, 111, 20, 101, 182, 91, 192, 81, 202, 270, 02, 260, 12, 250, 271, 61, 261, 71, 321, 151, 421, 141, 281, 411, 291, 22, 242, 122, 252, 112, 262, 102, 272, 92, 282, 82, 80, 290, 90, 280, 100, 220, 390, 120, 410, 130] and [00, 71, 10, 61, 20, 51, 41, 21, 421, 381, 331, 271, 201, 281, 191, 291, 181, 182, 151, 192, 141, 202, 312, 302, 362, 282, 372, 322, 422, 402, 02, 392, 32, 80, 42, 70, 52, 60, 170, 270, 360, 280, 210, 150, 100, 90, 50, 30]. □

Theorem 2. *The necessary conditions for a k-cycle system are also sufficient for all $k \leq 50$.*

Proof: For $k \notin K$ refer to points a) to f) and Theorem 1. For $k \in K$ refer to Lemmas 1 to 13. □

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