On Involutions Acting on Symmetric (81,16,3)-Block Designs

Ljubo Marangunić

Department of Mathematics
University of Zagreb
41000 Zagreb
Croatia

Abstract. We show that $(81,16,3)$-block designs have no involutionary automorphisms that fix just 13 points. Since the nonexistence of $(81,16,3)$-designs with involutionary automorphism fixing 17 points has already been proved, it follows that any involution that an $(81,16,3)$-design may have must fix just 9 points.

1. Introduction and preliminaries

At the beginning we recall some basic definitions and facts related with symmetric block designs (see for example [1,4]). In the following, we assume all sets under consideration to be finite.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an incidence structure with point set $\mathcal{P}$, line set $\mathcal{B}$ and incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$. For $P \in \mathcal{P}$, $x \in \mathcal{B}$ denote

$$\langle P\rangle = \{y \in \mathcal{B} \mid (P,y) \in I\}, \quad \langle x\rangle = \{Q \in \mathcal{P} \mid (Q,x) \in I\},$$
$$|P| = |\langle P\rangle|, \quad |x| = |\langle x\rangle|.$$

The number $|x|$ is called the block length.

Definition 1: A *symmetric block design* (v,k,λ), $v,k,\lambda \in N$, is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ such that:

(i) $|\mathcal{P}| = |\mathcal{B}| = v = k(k-1)/\lambda + 1$
(ii) $|x| = |P| = k$, for all $x \in \mathcal{B}$, $P \in \mathcal{P}$
(iii) $|\langle x\rangle \cap \langle y\rangle| = |\langle P\rangle \cap \langle Q\rangle| = \lambda$, for all $x,y \in \mathcal{B}$, $P,Q \in \mathcal{P}$ with $x \neq y$, $P \neq Q$.

The conditions (iii) we call the *consistence conditions*.

For two symmetric block designs $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$ an *isomorphism* from $\mathcal{D}_1$ onto $\mathcal{D}_2$ is a bijection which maps points onto points and lines onto lines and preserves incidences.

In this article we deal with the $(81,16,3)$-symmetric block design, which is one of the most interesting block designs whose existence is still in doubt. We try to make a design under the assumption that it has an involutionary automorphism ρ. Suppose ρ fixes exactly F points and, consequently, exactly F lines. Using the known result for an upper bound for F (see [4],p.33), and a lower bound for F (see [3],p.155), we have:

$$1 + \frac{k-1}{\lambda} \leq F \leq k + \sqrt{k-\lambda}.$$

For a 2-(81, 16, 3) design, since ρ can operate only with an odd number of orbits (see [3], p.98), we obtain $F \in \{9, 13, 17\}$ as the only possibilities for F. The case $F = 17$ is eliminated in [2]. In our paper we eliminate the case $F = 13$ and thus prove the following :

Theorem. *Let $\mathcal{D}$ be a symmetric block design* (81, 16, 3) *admitting an involution ρ. Then ρ fixes exactly nine points of $\mathcal{P}$ and exactly nine lines of $\mathcal{B}$.*

This result we obtain by means of combinatorial methods and with the help of an exhaustive computer search.

2. The fixed structure of $\mathcal{D}$ for ρ

For the action of ρ on $\mathcal{D}$ denote by $\mathcal{P}_\infty = \{\infty_1, \infty_2, \dots, \infty_{13}\}$ and $\mathcal{B}_\infty = \{(p_\infty)_1, (p_\infty)_2, \dots, (p_\infty)_{13}\}$ the sets of fixed points and fixed lines respectively. Among the $v - F$ non-fixed lines of $\mathcal{D}$ we distinguish two types of lines: those that contain just one fixed point, and those that contain just three fixed points. Denote by q_1 the lines of the first type, and by q_3 the lines of the second. Let M_1,M_3 be the numbers of corresponding 2-orbits $\{q_1, q_1\rho\}$, $\{q_3, q_3\rho\}$. Then:

(1) $$M_1 + M_3 = \frac{1}{2}(v - F) = 34\,.$$

We define the fixed structure $\mathcal{F}(\rho)$ in the usual way:

(2) $$\mathcal{F}(\rho) = (\mathcal{P}_\infty, \mathcal{B}_\infty, I)\,.$$

Since the number of fixed points on a line from $\mathcal{B}_\infty$ need not be constant, the structure $\mathcal{F}(\rho)$ is not necessarily uniform. We say the block of $\mathcal{F}(\rho)$ is of p_r-type if its length is r. Denote by N_r the number of p_r-blocks in $\mathcal{F}(\rho)$. In our case $k = 16$ is even, which implies the sizes r are also even. Putting $r = 2i$, with $F = 13$, we obtain $i \in A = \{0, 1, \dots, i_{\max} = 6\}$, and so:

(3) $$\sum_{i \in A} N_{2i} = F = 13\,.$$

Definition 2: An *l-distribution* is the vector $\Delta = (N_{2i}, M_j \mid i \in \{6, \dots, 0\}, j \in \{3, 1\})$ where N_{2i},M_j satisfy (1) and (3).

In our further considerations we study only those l-distributions which can be induced by a fixed structure $\mathcal{F}(\rho)$. For this purpose we impose some conditions on the components of Δ.

Since $\lambda = 3$ is odd, every two blocks of $\mathcal{F}(\rho)$ intersect in an odd number of fixed points, which means that every p_r-block must contain at least one fixed point. This fact eliminates the existence of p_0-blocks, i.e. $N_0 = 0$.

Let x and y be any two blocks of $\mathcal{F}(\rho)$ with (not necessarily) distinct lengths $|x| = r$ and $|y| = s$. Then the condition $|x \cap y| \leq \lambda$ implies $F \geq |x \cup y| = |x| + |y| - |x \cap y| \geq r + s - \lambda$, i.e.

$$(4) \qquad r + s \leq F + \lambda.$$

Similarly, if r, s, t are the lengths of any three blocks of $\mathcal{F}(\rho)$, then using $|x \cup y \cup z| = |x| + |y| + |z| - |x \cap y| - |x \cap z| - |y \cap z| + |x \cap y \cap z|$ we get

$$(4)' \qquad r + s + t \leq F + 3\lambda.$$

Since $2 \cdot M_1 + 2 \cdot 3 \cdot M_3$ is the number of occurences of fixed points in $\mathcal{B} \setminus \mathcal{B}_\infty$, we have:

$$(5) \qquad \sum_{i \in A} 2i\, N_{2i} = kF - (2M_1 + 6M_3) = (k+1)F - v - 4M_3.$$

Next, since $2 \cdot \binom{3}{2} \cdot M_3$ is the total number of pairs of fixed points in $\mathcal{B} \setminus \mathcal{B}_\infty$, we have:

$$(6) \qquad \sum_{i \in A} \binom{2i}{2} N_{2i} = \binom{F}{2}\lambda - 6M_3.$$

Lemma 1. *Let $\mathcal{F}(\rho)$ and Δ be as stated above. We have exactly 12 solutions for Δ satisfying the conditions (4),(4)′,(5) and (6), as displayed in Table I.*

Proof: From (5) and (6), by the elimination of M_3, we find that:

$$(7) \qquad N_2 = \frac{1}{4}[-\lambda F^2 + (3k + \lambda + 3)F - 3v] + \sum_{i=3}^{i_{max}} i(i-2)N_{2i}.$$

Inserting this expression into (3) we get:

$$(8) \qquad N_4 = \frac{1}{4}[\lambda F^2 - (3k + \lambda - 1)F + 3v] - \sum_{i=3}^{i_{max}} (i-1)^2 N_{2i}.$$

Therefore, from (7),(8) and (5) it follows that:

$$(9) \qquad \sum_{i=3}^{i_{max}} \binom{i-1}{2} N_{2i} = \frac{1}{8}[\lambda F^2 - (5k + \lambda - 3)F + 5v] + M_3.$$

When $v = 81, k = 16, \lambda = 3, F = 13, i_{max} = 6$, the last three equations can be rewritten as:

$$(*) \qquad \begin{aligned} &N_6 + 3N_8 + 6N_{10} + 10N_{12} = -16 + M_3, \\ &N_2 = -12 + (3N_6 + 8N_8 + 15N_{10} + 24N_{12}), \\ &N_4 = 25 - (4N_6 + 9N_8 + 16N_{10} + 25N_{12}). \end{aligned}$$

The solutions of the above equation system (*), *taking into account* (4) *and* (4)′, *give the proof of Lemma* 1.

Table I
Possible l-distributions for Constructing $\mathcal{F}(\rho)$

case	N_{12}	N_{10}	N_8	N_6	N_4	N_2	M_3	M_1
1.)	1	0	0	0	0	12	26	8
2.)	0	1	0	2	1	9	24	10
3.)	0	1	0	1	5	6	23	11
4.)	0	1	0	0	9	3	22	12
5.)	0	0	2	1	3	7	23	11
6.)	0	0	2	0	7	4	22	12
7.)	0	0	1	4	0	8	23	11
8.)	0	0	1	3	4	5	22	12
9.)	0	0	1	2	8	2	21	13
10.)	0	0	0	6	1	6	22	12
11.)	0	0	0	5	5	3	21	13
12.)	0	0	0	4	9	0	20	14

3. Constructing all possible fixed structures

For two fixed structures $\mathcal{F}_1(\rho) = (\mathcal{P}_\infty, \mathcal{B}_{\mathcal{F}_1}, I_1)$ and $\mathcal{F}_2(\rho) = (\mathcal{P}_\infty, \mathcal{B}_{\mathcal{F}_2}, I_2)$, we define an isomorphism from $\mathcal{F}_1$ onto $\mathcal{F}_2$ as a bijection which maps points onto points and lines onto lines and preserves incidences.

Next, we introduce a lexicographical order among the fixed structures. We define it in terms of the incidence matrix of $\mathcal{F}(\rho)$. Suppose that the points of $\mathcal{F}(\rho)$ are indexed by $\infty_1, \infty_2, \ldots, \infty_F$ and that the blocks are $x_1, x_2, \ldots, x_F$. Let us recall that the incidence matrix $\Gamma = [f_{ij}]$ of $\mathcal{F}(\rho)$ is an $F \times F$ matrix, where $f_{ij} = 1$ if ∞_j is incident with the block x_i, and $f_{ij} = 0$ otherwise.

Definition 3: Let x_i,x_j be two blocks of $\mathcal{F}(\rho)$, with the same length, and $\Gamma = [f_{ij}]$ be the incidence matrix of $\mathcal{F}(\rho)$. Then x_i *precedes* x_j, $x_i \prec x_j$, if there is some w, $1 \leq w \leq F$, such that $f_{it} = f_{jt}$ for $t < w$ and $f_{iw} > f_{jw}$.

Definition 4: We say $\mathcal{F}(\rho)$ is in the *canonical form* if:

(i) the sizes of its blocks are in the reverse lexicographical order with respect to the usual ordering within the natural numbers, and
(ii) within any set of blocks of the same block size, the blocks are ordered in terms of the precedence introduced in Definition 3.

For every $\mathcal{F}(\rho)$ the canonical form of $\mathcal{F}(\rho)$ is uniquely determined. The sequence of the block sizes of this canonical form we denote by $\{s_i\}$, $1 \leq i \leq F$.

Definition 5: For two fixed structures $\mathcal{F}_1$,$\mathcal{F}_2$ we say that $\mathcal{F}_1$ *precedes* $\mathcal{F}_2$, $\mathcal{F}_1 \prec \mathcal{F}_2$, if the canonical form of $\mathcal{F}_1$ precedes that of $\mathcal{F}_2$ in terms of the precedence of their blocks.

Now we sketch our algorithm for constructing all fixed structures $\mathcal{F}$ of $\mathcal{D}$ for ρ. We produce the structures in a canonical form, by building up the schemes level by level.

Thus, let Δ be the l-distribution under consideration, and let $\{s_i\}$ be the sequence of block sizes of the canonical form of $\mathcal{F}$. Denote by f_i the i-th row of Γ. The *i-th layer of $\mathcal{F}$*, denoted by $f^{(i)}$, consists of all possible rows f_i with s_i entries equal to 1, and the remaining entries being 0.

A *partial fixed structure of the i-th level* is a $(0,1)$-matrix $\Gamma(i) = [\gamma_{rs}]$ with s_r units in the r-th row, $1 \leq r \leq i$, $1 \leq s \leq 13$, and satisfying the conditions:

$$(**) \qquad \begin{aligned} &\sum_{t=1}^{13} \gamma_{rt}\gamma_{st} \in \{1,3\} && r,s \in \{1,\ldots,i\},\ r \neq s, \\ &\sum_{t=1}^{i} \gamma_{tr}\gamma_{ts} \leq 3 && r,s \in \{1,\ldots,13\},\ r \neq s. \end{aligned}$$

Denote by $\Gamma^{(i)}$ the set of all i-th level partial structures which we construct in our procedure.

We begin in the following way:

1) $\Gamma^{(1)}$ consists of the initial row of the first layer.
2) We construct $\Gamma^{(i)}$ from $\Gamma^{(i-1)}$ by adjoining, as the next row, each possible i-th layer row $f_i \in f^{(i)}$ to each $\Gamma(i-1)$, in such a way that matrix $\Gamma(i)$ so obtained is an i-th level partial structure. In this way we obtain partial structures of the i-th level.

Let $\alpha \in S(\mathcal{P}_\infty) \times S(\{x_1,\ldots,x_i\})$, $S(S)$ denoting the symmetric group on the set S. We include the matrix $\Gamma(i)$ in the set $\Gamma^{(i)}$, if it cannot be eliminated by finding some α such that $\Gamma(i)\alpha \prec \Gamma(i)$, in terms of the precedence of partial structures considered as parts of the whole fixed structures $\mathcal{F}(\rho)$.
On the 13-th level, we also have to check that $\Gamma(13)$ fulfils the column conditions:

$$(**)' \qquad \begin{aligned} &\sum_{t=1}^{13} \gamma_{tr} \in \{2,4,6,8,10,12\} && r \in \{1,\ldots,13\}, \\ &\sum_{t=1}^{13} \gamma_{tr}\gamma_{ts} \in \{1,3\} && r,s \in \{1,\ldots,13\},\ r \neq s. \end{aligned}$$

At the end of this procedure $\Gamma^{(13)}$ becomes the set of all possible incident matrices Γ of $\mathcal{F}(\rho)$, for the given l- distribution Δ. Carrying out this construction for all l-distributions Δ we get all the required fixed structures $\mathcal{F}(\rho)$.

The described procedure was carried out by computer. It turned out that only two l-distributions, namely the 1-st and the 12-th from the Table I, produced an $\mathcal{F}(\rho)$. In the first case the solution is unique, while in the second there are two nonisomorphic solutions. So we proved:

Table II
Possible Fixed Structures of $\mathcal{D}$ for ρ

A			*B*			*C*		
lev.	block length		lev.	block length		lev.	block length	
1	12	1111111111110	1	6	1111110000000	1	6	1111110000000
2	2	1000000000001	2	6	1110001110000	2	6	1110001110000
3	2	0100000000001	3	6	1001101101000	3	6	1001101101000
4	2	0010000000001	4	6	0101011011000	4	6	0000010011111
5	2	0001000000001	5	4	1110000000100	5	4	1110000000100
6	2	0000100000001	6	4	1001100000010	6	4	1001100000010
7	2	0000010000001	7	4	1000001100001	7	4	1000001100001
8	2	0000001000001	8	4	0101010000001	8	4	0100000001011
9	2	0000000100001	9	4	0100001010010	9	4	0010000001011
10	2	0000000010001	10	4	0010000001011	10	4	0001000010101
11	2	0000000001001	11	4	0001001001100	11	4	0000100010101
12	2	0000000000101	12	4	0000100010101	12	4	0000011000110
13	2	0000000000011	13	4	0000010100110	13	4	0000010100110

Lemma 2. *Let $\mathcal{D}(81, 16, 3)$ be a symmetric block design admitting an involution ρ fixing 13 points. Then there exist exactly 3 nonisomorphic fixed structures $\mathcal{F}(\rho)$ of $\mathcal{D}$ for ρ, as displayed in Table II.*

4. Proof of the theorem

Denote by $N_{\mathcal{F}}(\infty_r)$, $\infty_r \in \mathcal{P}_\infty$, the number of occurences of ∞_r on the blocks of $\mathcal{F}(\rho)$, and by $C_{\mathcal{F}}(\infty_r)$ the number of pairs $\infty_r\infty_s$, $\infty_s \in \mathcal{P}_\infty$, $s \neq r$, in the blocks of $\mathcal{F}(\rho)$. Let $m_1(\infty_r)$ be the number of $\{q_1, q_1\rho\}$-orbits and $m_3(\infty_r)$ the number of $\{q_3, q_3\rho\}$-orbits in which the point ∞_r appears. By counting the total numbers of occurences and pairs containing ∞_r in all the lines of $\mathcal{B}$, denoted by $N_{\mathcal{D}}(\infty_r)$ and $C_{\mathcal{D}}(\infty_r)$, we get:

(10) $$N_{\mathcal{D}}(\infty_r) = k = N_{\mathcal{F}}(\infty_r) + 2\, m_1(\infty_r) + 2\, m_3(\infty_r) = 16,$$

(11) $$C_{\mathcal{D}}(\infty_r) = \lambda\,(F-1) = C_{\mathcal{F}}(\infty_r) + 4\, m_3(\infty_r) = 36.$$

We prove that none of the fixed structures, constructed in the previous paragraph, can be extended to a full design. We consider each of the three solutions in turn.

Case A:
When $\infty_r \equiv \infty_{13}$, $N_{\mathcal{F}}(\infty_{13}) = 12$, $C_{\mathcal{F}}(\infty_{13}) = 12$, the equations (10) and (11) imply $m_3(\infty_{13}) = 6$, $m_1(\infty_{13}) = -4$. So A is eliminated.

Cases B and C:

Denote by:

$\mathcal{P}_1, \dots, \mathcal{P}_{13}, \mathcal{P}_{14}, \dots, \mathcal{P}_{47} \equiv \infty_1, \dots, \infty_{13}, \{1_0, 1_1\}, \dots, \{34_0, 34_1\}$,

$\mathcal{B}_1, \dots, \mathcal{B}_{13}, \mathcal{B}_{14}, \dots, \mathcal{B}_{47} \equiv (p_\infty)_1, \dots, (p_\infty)_{13}, \{x_1, x_1\rho\}, \dots, \{x_{34}, x_{34}\rho\}$,

the ρ-orbits of points and lines of $\mathcal{D}$ in the defined order. These two partitions are to be considered as the point classes and the block classes of a tactical decomposition of $\mathcal{D}$ (see e.g.[1,2]). The known parameters of our decomposition are $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_r| = \Omega_r$. Thus $\omega_r = \Omega_r = 1$ for $1 \leq r \leq 13$ and $\omega_r = \Omega_r = 2$ for $14 \leq r \leq 47$. The unknown parameters μ_{ir} (the number of points from $\mathcal{P}_r$ contained in each line of $\mathcal{B}_i$), satisfy the well known relations (see e.g. [1,2]):

$$\sum_{r=1}^{47} \mu_{ir} = k = 16 \qquad 1 \leq i \leq 47, \tag{12}$$

$$\sum_{i=1}^{47} \frac{\Omega_i}{\omega_r} \mu_{ir} = k = 16 \qquad 1 \leq r \leq 47, \tag{13}$$

$$\sum_{r=1}^{47} \frac{\Omega_j}{\omega_r} \mu_{ir}\mu_{jr} = \lambda\Omega_j + \delta_{ij}(k - \lambda) \qquad 1 \leq i,j \leq 47, \tag{14}$$

$$\sum_{i=1}^{47} \frac{\Omega_i}{\omega_r} \mu_{ir}\mu_{is} = \lambda\omega_s + \delta_{rs}(k - \lambda) \qquad 1 \leq r,s \leq 47, \tag{15}$$

δ_{ij}, δ_{rs} being the correspondent Kronecker symbols.

The 47×47 matrix $\mathcal{S} = [\mu_{ir}]$ we call the *multiplicity matrix* (or *orbital structure*) of $\mathcal{D}$ for ρ. The entries of $\mathcal{S}$ in the top left corner are already determined by the fixed structure under consideration, i.e. $\mu_{ir} = \gamma_{ir}$ for $1 \leq i, r \leq 13$.

We proceed with the construction of the orbits $\mathcal{B}_1, \dots, \mathcal{B}_{13}$ (the fixed lines of $\mathcal{D}$), by joining the nontrivial point orbits to the blocks of $\mathcal{F}(\rho)$. Obviously, since λ is odd, the number of appearences of every nontrivial point orbit inside $\mathcal{B}_\infty$ must be also odd. Since we have $N_6 = 4$, $N_4 = 9$ in the both considered cases, there exist exactly 5 values of r with $\mu_{ir} = 2$ for each i, $1 \leq i \leq 4$, and exactly 6 such values for each i, $5 \leq i \leq 13$, the remaining μ_{ir} being 0. Denote by $f_\infty(i,j) = \sum_{r=1}^{13} \mu_{ir}\mu_{jr}$ the number of points incident with both $(p_\infty)_i$ and $(p_\infty)_j$, this number being determined by the $\mathcal{F}(\rho)$ under consideration. Now we construct $\mathcal{B}_\infty$ using the following relations:

$$\begin{aligned} &\sum_{r=14}^{47} \mu_{ir}\mu_{jr} = 2\,(\,\lambda - f_\infty(i,j)\,) && i,j \in \{1, \dots, 13\},\ i \neq j, \\ &\sum_{t=1}^{13} \mu_{tr} \in \{2, 6\} && r \in \{14, \dots, 47\}. \end{aligned} \tag{16}$$

With the help of a computer, for case B we obtained 336 nonisomorphic solutions for $\mathcal{B}_\infty$. Case C produced no solutions. So we proved:

Lemma 3. *Let $\mathcal{D}$ be a* $(81, 16, 3)$*-design and* $\rho \in \mathrm{Aut}\,\mathcal{D}$ *an involution fixing* 13 *points. Then, up to isomorphism, there are* 336 *structures* $\mathcal{B}_\infty$ *for the fixed lines of* $\mathcal{D}$. *They all have the same fixed structure* $\mathcal{F}(\rho)$, *case B in Table II.*

In the Table III we display, as examples, only two of these solutions : lexicographically the first and the last one (the first 5 lines in all solutions coincide).

Table III
Two Examples of The Fixed Lines of $\mathcal{D}$

level		scheme 1.
1	1111110000000	2222200000000000000000000000000000
2	1110001110000	0000022222000000000000000000000000
3	1001101101000	0000000000222220000000000000000000
4	0101011011000	0000000000000002222200000000000000
5	1110000000100	0000000002000020000222200000000000
6	1001100000010	0000020000000002000000022220000000
7	1000001100001	2000000000000000200020002000220000
8	0101010000001	0000020000020000000002000000202200
9	0100001010010	2000000000002000000000200200002020
10	0010000001011	0200002000200000200000000020002000
11	0001001001100	0020000200000000000000200020200002
12	0000100010101	0200000200020000200002000200000000
13	0000010100110	0020000200000200020002002000000000

level		scheme 336.
6	1001100000010	0000020000000000200020002220000000
7	1000001100001	2000000000000000200020020022000000
8	0101010000001	0000002000020000000002000000202200
9	0100001010010	0200000000002000000000202000020200
10	0010000001011	0020020000200000020000000000020020
11	0001001001100	0020000200000000000000200202000002
12	0000100010101	0200000200020000200002000000000020
13	0000010100110	2000002000002002000000000200000020

In this way, the case C fails and we proceed with the construction of multiplicity matrices in the case B. With $\infty_r \in \{\infty_1, \infty_2, \infty_3\}$, $N_{\mathcal{F}}(\infty_1) = N_{\mathcal{F}}(\infty_2) = 6$, $C_{\mathcal{F}}(\infty_1) = C_{\mathcal{F}}(\infty_2) = 24$, $N_{\mathcal{F}}(\infty_3) = 4$, $C_{\mathcal{F}}(\infty_3) = 16$, equations (10) and (11) imply $m_3(\infty_1) = m_3(\infty_2) = 3$, $m_1(\infty_1) = m_1(\infty_2) = 2$, $m_3(\infty_3) = 5$, $m_1(\infty_3) = 1$.

Define a lexicographical order among the line orbits in which the orbits of levels $14-18$ contain ∞_1, the orbits of levels $19-23$ contain ∞_2 and those of levels $24-29$ contain ∞_3. Also, let the $\{q_3, q_3\rho\}$-orbits precede lexicographically the $\{q_1, q_1\rho\}$-orbits. The i-th row of S, denoted by $[\mu_{ir}]_i$, corresponding to any $\{q_3, q_3\rho\}$-orbit, has exactly three 1's for $1 \le r \le 13$ and exactly thirteen 1's for $14 \le r \le 47$, the remaining entries μ_{ir} being 0. If $[\mu_{ir}]_i$ corresponds to a $\{q_1, q_1\rho\}$-orbit then we have $\mu_{ir} = 2$ for exactly one value $r \in \{14, \dots, 47\}$, exactly one 1 for $r \in \{1, \dots, 13\}$ and exactly thirteen 1's for $r \in \{14, \dots, 47\}$. The lexicographical order among the rows of the same level is introduced in the natural way : $[\mu_{ir}]_i$ precedes $[\nu_{ir}]_i$ if there is some w, $1 \le w \le 47$, such that $\mu_{it} = \nu_{it}$ for $t < w$ and $\mu_{iw} > \nu_{iw}$.

Using an algorithm similar to the one described in paragraph 3, we start at level 14 and build multiplicity matrices, row by row. At level i, $i \ge 14$, we exhaust all the possibilities for this level by generating the corresponding rows $[\mu_{ir}]_i$ in the above defined lexicographical order. For a particular $[\mu_{ir}]_i$, after testing that the row conditions (14) are satisfied and column conditions (13), (15) are not violated, we include such a row into the i-th level scheme. The described procedure was carried out by computer and it required a big expenditure of computer time. An IBM RISC/6000 machine was used and nearly 6 000 computer-hours were spent. The biggest number, aproximately 20 000 000, of possible schemes was observed at level 16. At level 23 that number was reduced to 17 schemes, and on level 24 there was no possible continuation. So we proved :

Lemma 4. *An* (81, 16, 3) *-design cannot have an involutionary automorphism fixing* 13 *points.*

Since we proved in [2] that ρ cannot fix 17 points of $\mathcal{D}$, then by Lemma 4 we have proved the Theorem of Section 1.

References

1. V. Ćepulić, *On symmetric block designs* (45, 12, 3) *with automorphisms of order* 5, Ars Combinatoria. (to appear).
2. Lj. Marangunić, *On symmetric block designs* (81, 16, 3) *with involutory automorphisms fixing* 17 *points.* preprint, Zagreb (1992). (submitted to *Discrete Mathematics* in March 1992).
3. E. Lander, "Symmetric designs: an algebraic approach", Cambridge University Press, Cambridge, 1983.
4. J.J. Seidel, A. Blokhuis and H.A. Wilbrink, *Graphs and association schemes, algebra and geometry*, EUT–Report 83–WSK-02 (1983).