NEW SUFFICIENT CONDITIONS FOR EQUALITY OF MINIMUM DEGREE AND EDGE-CONNECTIVITY

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ABSTRACT. New sufficient conditions for equality of edge-connectivity and minimum degree of graphs are presented, including those of Chartrand, Lesniak, Plesník, Plesník and Znám, and Volkmann.

1. TERMINOLOGY AND INTRODUCTION

We consider finite, undirected, and simple graphs G of order $n = n(G)$ with the vertex set $V = V(G)$ and the edge set $E = E(G)$. For $X \subseteq V(G)$ let $G[X]$ be the subgraph induced by X. $N(x) = N(x, G)$ denotes the set of vertices adjacent to the vertex x and $N(X) = N(X, G) = \bigcup_{x \in X} N(x)$ for a subset X of $V(G)$. The degree of the vertex x and the minimum degree of G are $d(x) = d(x, G) = |N(x)|$ and $\delta = \delta(G) = \min\{d(x) \mid x \in V(G)\}$, respectively. The distance between two subsets X and Y of $V(G)$ is denoted by $d(X, Y)$ and $\mathrm{dm}(G)$ means the diameter of G. If (X, Y) is the set of edges with one end in X and the other in Y and $\bar{X} = V(G) - X$ for $X, Y \subseteq V(G)$, then the edge-connectivity of G is given by

$$\lambda = \lambda(G) = \min\{|(X, \bar{X})| \;\Big|\; X \subseteq V(G), X \neq \emptyset, V(G)\}.$$

The inequality $\lambda(G) \leq \delta(G)$ is immediate. Equality holds, if

(1) $n \leq 2\delta + 1$: Chartrand [2], 1966.
(2) $d(x) + d(y) \geq n - 1$ for all nonadjacent vertices: Lesniak [6], 1974.
(3) $\mathrm{dm}(G) \leq 2$: Plesník [7], 1975.
(4) G is connected, and there are no four vertices u_1, v_1, u_2, v_2 with

$$d(u_1, u_2),\ d(u_1, v_2),\ d(v_1, u_2),\ d(v_1, v_2) \geq 3:$$

Plesník and Znám [8], 1989.
(5) G is bipartite with $n \leq 4\delta - 1$: Volkmann [10], 1988.
(6) G is bipartite with $\mathrm{dm}(G) \leq 3$: Plesník and Znám [8], 1989.
(7) G is p-partite with $n \leq 2\lfloor \frac{p\delta}{p-1} \rfloor - 1$: Volkmann [11], 1989.

Condition (4) includes these of (3), (2), and (1), and (5) is a special case of (7), while the conditions (5) and (6) are independent of (4). For further such and similar results see the survey of Plesník and Znám [8]. Proofs of (4) and (7) can also be found in [12].

In this note we shall give new sufficient conditions for equality of minimum degree and edge-connectivity, in particular we shall generalize (4), (6), and (7).

2. Results

A pair of sets $X, Y \subseteq V(G)$ with $d(X,Y) = k$ ($k \in \mathbb{N}$) is called *k-distance maximal*, if there exist no sets $X_1 \supseteq X$ and $Y_1 \supseteq Y$ with $X_1 \neq X$ or $Y_1 \neq Y$ such that $d(X_1, Y_1) = k$.

Theorem 1. *Let G be a connected graph. If for all 3-distance maximal pairs of sets $X, Y \subseteq V(G)$ the condition $\delta(G[X \cup Y]) = 0$ is fulfilled, then $\lambda = \delta$.*

Proof. Suppose on the contrary that $\lambda < \delta$. Then there exists a set of edges E' with $|E'| = \lambda$, such that $G - E'$ consists of two components G_1 and G_2. If $S = V(G_1)$ and $T = V(G_2)$, then let $A \subseteq S$ and $B \subseteq T$ be the set of vertices incident with any edge of E'. Furthermore, we define $A_0 = S - A$ and $B_0 = T - B$.
In view of our assumption, we see $|A|, |B| \leq \lambda < \delta$. Now we shall investigate two cases.

Case 1: $A_0, B_0 \neq \emptyset$. Clearly, the distance between A_0 and B_0 is at least 3. Choose a 3-distance maximal pair (X, Y) with $A_0 \subseteq X$ and $B_0 \subseteq Y$. According to our assumption, one of the subgraphs $G[X]$ or $G[Y]$ contains an isolated vertex u. Without loss of generality, u is contained in X. If $u \in A_0$, then we obtain the following contradiction.

$$\delta \leq |N(u,G)| \leq |A| \leq \lambda < \delta$$

If $u \in A$, then the definition of A and the fact that u has neighbours only in $A \cup B$ yields

$$\begin{aligned}
\delta &\leq |N(u,G) \cap B| + |N(u,G) \cap A| \\
&\leq |N(u,G) \cap B| + \sum_{x \in N(u,G) \cap A} |N(x,G) \cap B| \\
&\leq \sum_{x \in A} |N(x,G) \cap B| \\
&= \lambda,
\end{aligned}$$

but this contradicts our assumption $\lambda < \delta$.
Assume now that $u \in B$. Again, $N(u,G) \subseteq A \cup B$ holds. Similar to the above case $u \in A$, a contradiction is derived.

Case 2: If we have without loss of generality $|A_0| = 0$, then we obtain the same contradiction for an arbitrary vertex $a \in A$ instead of u. □

Corollary 1. [8] *If in a connected graph G there exist no four vertices u_1, v_1, u_2, v_2 with*

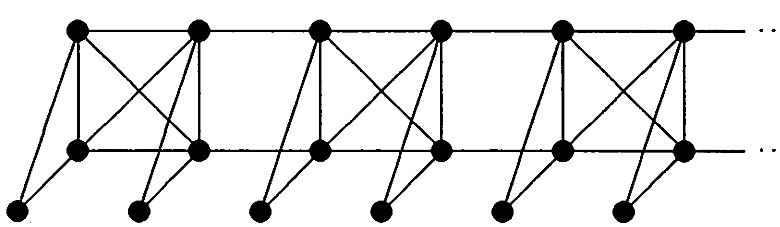
$$d(u_1,u_2),\ d(u_1,v_2),\ d(v_1,u_2),\ d(v_1,v_2) \geq 3,$$

then $\lambda = \delta$.

Proof. If $X, Y \subseteq V(G)$ is a pair of 3-distance maximal sets, then the hypothesis yields $\min\{|X|,|Y|\} \leq 1$, and the desired result is immediate by Theorem 1. □

Remark 1. If $\lambda < \delta$ and if E' is an edge set of cardinality λ, such that $G - E'$ consists of two components, then the proof of Theorem 1 shows that $|A_0|, |B_0| \geq 2$. This observation that also implies Corollary 1, can be found in a paper of Goldsmith [3].

Remark 2. Obviously, conditions (1) – (4) work only for graphs with diameter at most 4. Moreover, it is easy to prove that no graph whose degree sequence satisfies one of the conditions given by Bollobás [1], Goldsmith and Entringer [4], or Goldsmith and White [5] has diameter more than 5. The following figure shows a graph with arbitrary large diameter for which Theorem 1 guarantees equality of minimum degree and edge-connectivity.



Corresponding examples exist for every $\delta = \lambda \geq 3$.

Now we give similar results for bipartite graphs. The second one uses a weaker degree condition, but needs a diameter restriction of the graph G. The following convention will be used. If $G = (V, E)$ is a bipartite graph with bipartition $V = V' \cup V''$ and M is an arbitrary subset of V, then we define $M' = M \cap V'$ and $M'' = M \cap V''$.

In a bipartite graph G, a pair of sets $X, Y \subseteq V(G)$ with $d(X', Y') = d(X'', Y'') = k$ ($k \in \mathbb{N}$) is called *(k,k)-distance maximal*, if there exist no sets $X_1 \supseteq X$ and $Y_1 \supseteq Y$ with $X_1 \neq X$ or $Y_1 \neq Y$ such that $d(X_1', Y_1') = d(X_1'', Y_1'') = k$.

Theorem 2. *Let G be a connected bipartite graph. If for all $(4,4)$-distance maximal pairs of sets $X, Y \subseteq V(G)$ the condition $\delta(G[X \cup Y]) = 0$ is fulfilled, then $\lambda = \delta$.*

Proof. Suppose on the contrary that $\lambda < \delta$. We shall use the same notations as in the proof of Theorem 1. First we show $A_0', A_0'', B_0', B_0'' \neq \emptyset$.

Suppose that A_0' is empty. If A_0'' contains a vertex v, then we observe $\delta \leq |N(v,G)| \leq |A'| \leq \lambda$, but this is impossible. Therefore, A_0'' is also empty and we assume without loss of generality that there is a vertex $v \in A'$. Then we have

$$\begin{aligned} \delta &\leq |N(v) \cap A''| + |N(v) \cap B''| \\ &\leq |A''| + |B''| \\ &\leq \lambda, \end{aligned}$$

contradicting $\lambda < \delta$. Thus we have shown $A_0' \neq \emptyset$. Similarly, A_0'', B_0', B_0'' are nonempty.
Clearly, $d(A_0', B_0'), d(A_0'', B_0'') \geq 4$ holds. Choose a $(4,4)$-distance maximal pair (X, Y) with $A_0 \subseteq X$ and $B_0 \subseteq Y$. According to our hypothesis, one of the subgraphs $G[X]$ or $G[Y]$ contains an isolated vertex u. Without loss of generality, $u \in X'$. By our assumption, u has neither neighbours in A_0'' nor in B_0''. So we obtain the contradiction

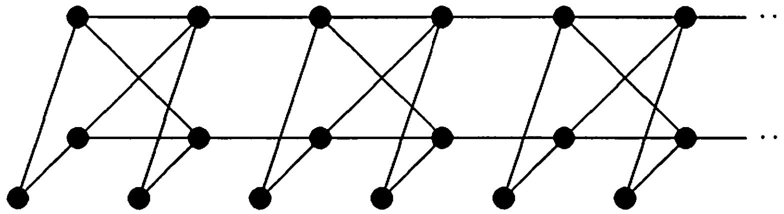
$$\delta \leq d(u) \leq |A''| + |B''| \leq \lambda.$$

This completes the proof of Theorem 2. □

Corollary 2. [8] *Let G be a bipartite graph with bipartition $V = V' \cup V''$. If $d(x,y) = 2$ for all different $x, y \in V'$, then $\lambda = \delta$.*

Plesník and Znám derived from Corollary 2 that condition (6), stated in the introduction, is sufficient for $\lambda = \delta$.

Remark 3. Obviously, Corollary 2 works only for bipartite graphs whose diameter does not exceed 4. The condition (5) stated in the introduction, and each of the conditions for bipartite graphs given below, imply that the graph has diameter at most 7. The following figure shows a bipartite graph with arbitrary large diameter for which Theorem 2 guarantees equality of minimum degree and edge-connectivity.



Theorem 3. *Let G be a bipartite graph with* $\mathrm{dm}(G) \leq 4$. *If for all* $(4,4)$-*distance maximal pairs of sets* $X, Y \subseteq V(G)$ *with* $|X'|, |X''|, |Y'|, |Y''| \geq 2$ *the condition* $\delta(G[X \cup Y]) \leq 1$ *is fulfilled, then* $\lambda = \delta$.

Proof. Suppose on the contrary that $\lambda < \delta$. We use the same notations as in the proof of Theorem 1. First we show $|A_0'|, |A_0''|, |B_0'|, |B_0''| \geq 2$.
Similar to the proof of Theorem 2, the sets A_0', A_0'', B_0', and B_0'' are nonempty. Suppose on the contrary that $|A_0'| = 1$. Since A_0'' is nonempty, A' contains at least $\delta - 1$ vertices and thus $A'' = \emptyset$. It follows $d(A_0', B_0'') \geq 5$, contradicting $\mathrm{dm}(G) \leq 4$. Analogously, we can prove $|A_0''|, |B_0'|, |B_0''| \geq 2$.
Clearly, $d(A_0', B_0') = d(A_0', B_0') = 4$ holds. Choose a $(4,4)$-distance maximal pair (X, Y) with $A_0 \subseteq X$ and $B_0 \subseteq Y$. According to our hypothesis, one of the subgraphs $G[X]$ or $G[Y]$ contains a vertex u of degree at most 1. Without loss of generality, $u \in X'$.

Case 1: $u \in A_0'$.
Then $|N(u, G) \cup A_0''| \leq 1$, and thus

$$\delta - 1 \leq |N(u, G) \cap A''| \leq |A''| \leq \lambda \leq \delta - 1.$$

Consequently, equality holds and A' is empty. Again, we have $d(A_0'', B_0') \geq 5$, contradicting $\mathrm{dm}(G) \leq 4$.

Case 2: $u \in A'$.
From $|N(u, G) \cap A_0''| \leq 1$, we conclude

$$\begin{aligned} \delta &\leq 1 + |(\{u\}, A'')| + |(\{u\}, B'')| \\ &\leq 1 + |(A'', B')| + |(A', B'')| \\ &= 1 + \lambda \\ &\leq \delta. \end{aligned}$$

Thus, equality holds, implying that $(\{u\}, B'') = (A', B'')$ and $A' = \{u\}$. Because of $|N(u, G) \cap A_0''| \leq 1 < 2 \leq |A_0''|$, there exists a vertex $v \in A_0'' - N(A', G)$. Hence, it follows $d(v, B_0') \geq 5$, contradicting $\mathrm{dm}(G) \leq 4$.

Case 3: $u \in B'$.
Each vertex $v \in B'' \cup B_0''$ has a neighbour in B_0', since otherwise $\lambda \geq |A'| + |B'| \geq \delta$. But $d(u, B_0') = 4$ yields $N(u,G) \cap (B'' \cup B_0'') = \emptyset$, and thus

$$\delta \leq |N(u,G) \cap A''| \leq |A''| \leq \lambda,$$

contradicting our assumption $\lambda < \delta$. □

An immediate consequence of the previous theorem is the following result of Plesník and Znám [8] from 1989.

Corollary 3. [8] *If in a bipartite graph G of diameter at most 4 no partition set contains four vertices u_1, v_1, u_2, v_2 with*

$$d(u_1,u_2),\ d(u_1,v_2),\ d(v_1,u_2),\ d(v_1,v_2) = 4,$$

then $\lambda = \delta$.

The next theorem shows that condition (5) can be weakened to guarantee equality of minimum degree and edge-connectivity of a bipartite graph. Instead of a minimum degree condition, a corresponding neighbourhood condition is proved to be sufficient. As a corollary, we obtain a condition for bipartite graphs that is adequate to condition (2) which concerns arbitrary graphs. For any noncomplete connected graph G let

$$NC(G) = \min\{\ |N(x) \cup N(y)|\ \Big|\ x \neq y \in V(G),\ xy \notin E(G)\},$$

$$NC2(G) = \min\{\ |N(x) \cup N(y)|\ \Big|\ x, y \in V(G),\ d(x,y) = 2\ \}.$$

Theorem 4. *If G is a bipartite connected graph of order $n \geq 3$ with $NC2(G) \geq (n+1)/4$, then $\lambda = \delta$.*

Proof. Suppose on the contrary that $\lambda < \delta$ and thus $\delta \geq 2$. Again we will use the same notations as in the proof of Theorem 1. Similar to the above proof A_0', A_0'', B_0', and B_0'' are nonempty.
We first show that A_0' contains two vertices of distance 2.
Assume that $|A_0'| = 1$ or no two vertices of A_0' have a common neighbour. In either case, for each vertex $y \in A_0''$ holds

$$\delta \leq |N(y) \cap A_0'| + |N(y) \cap A'| \leq 1 + |A'| \leq 1 + \lambda \leq \delta.$$

Hence, $|A'| = \delta - 1$ and $A'' = \emptyset$. Choose a pair $y_1, y_2 \in A_0''$. Clearly, $d(y_1, y_2) = 2$ and thus $NC2(G) \leq |A'| + 2 = \delta + 1$. If A_0' consists of a single vertex, it follows from our hypothesis

$$\frac{n+1}{4} \leq NC2(G) \leq |N(y_1) \cup N(y_2)| \leq 1 + |A'| = \delta.$$

Now condition (5) yields the contradiction $\lambda = \delta$. So we may assume that A'_0 contains at least two vertices without a common neighbour. From there we see $|A'_0| + |A''_0| + |A| \geq 3\delta + 1$. Furthermore, with $T = B \cup B_0$, we have $|T| = |B'_0| + |B''_0| + |B| \geq 2\delta$. If $|T| = 2\delta$, then it is not difficult to check that $NC2(G) = \delta$, which contradicts (5). In the case $|T| > 2\delta$, we obtain

$$\delta + 1 \geq NC2(G) \geq \frac{n+1}{4} > \frac{3\delta + 1 + 2\delta}{4} \geq \frac{5\delta + 2}{4},$$

but this is impossible, since $\delta \geq 2$.
Altogether we have shown that there exist two vertices $x_1, x_2 \in A'_0$ with $d(x_1, x_2) = 2$. From the fact that the neighbourhood of x_1 and x_2 is contained in $A''_0 \cap A''$, we deduce

$$\frac{n+1}{4} \leq |N(x_1) \cup N(x_2)| \leq |A''_0 \cup A''|.$$

Analogously, we are able to show

$$|A'_0 \cup A'|, |B'_0 \cup B'|, |B''_0 \cup B''| \geq \frac{n+1}{4}.$$

These inequalities yield

$$n = |A'_0 \cup A'| + |A''_0 \cup A''| + |B'_0 \cup B'| + |B''_0 \cup B''| \geq 4\frac{n+1}{4},$$

which completes the proof of Theorem 4. □

Corollary 4. *If G is a bipartite connected graph of order n such that each pair $x, y \in V$ with $d(x, y) = 2$ satisfies* $\max\{d(x), d(y)\} \geq (n+1)/4$, *then $\lambda = \delta$.*

Corollary 5. *Let G be a bipartite graph of order n. If $d(x) + d(y) \geq (n+1)/2$ for all nonadjacent vertices x and y, then $\lambda = \delta$.*

Remark 4. Many of the sufficient conditions for bipartite graphs guaranteeing $\lambda = \delta$ have analogues for arbitrary graphs. See for example (4) and (1) or Corollary 5 and (2). The graph $G = K_{n-3} \cup K_3 + e$, where e is an arbitrary edge joining the K_{n-3} with the K_3, indicates that Theorem 4 and Corollary 4 have no such analogue, since $NC(G) = NC2(G) = n - 2 \geq (n-1)/2$ but $\lambda = 1 < 2 = \delta$.

Theorem 5. *If a graph G of order n contains no complete subgraph of order $p + 1$ and satisfies*

$$n \leq 2\left\lfloor \frac{p}{p-1}\delta \right\rfloor - 1,$$

then $\lambda = \delta$.

Proof. For every proper subset $S \neq \emptyset$ of $V(G)$, we will show

$$|(S,\bar{S})| \geq \delta. \tag{1}$$

Without loss of generality we assume $1 \leq |S| \leq n/2$, and thus

$$1 \leq |S| \leq \frac{n}{2} \leq \left\lfloor \frac{p}{p-1}\delta \right\rfloor - \frac{1}{2}.$$

Since $|S| = s$ is an integer, it follows

$$1 \leq |S| \leq \left\lfloor \frac{p}{p-1}\delta \right\rfloor - 1 \leq \frac{p}{p-1}\delta - 1. \tag{2}$$

In addition, the well known Theorem of Turán [9] (see also [12], p. 159) together with the fact that G contains no complete subgraph of order $p+1$, yields the estimation $2|E(G[S])| \leq \frac{p-1}{p}s^2$, and hence we have

$$|(S,\bar{S})| \geq s\delta - \frac{p-1}{p}s^2. \tag{3}$$

If we define

$$g(x) = -\frac{p-1}{p}x^2 + \delta x,$$

then, because of (2), we have to determine the minimum of the function g in the interval $I:\ 1 \leq x \leq \frac{p}{p-1}\delta - 1$. It is a simple observation that

$$\min_{x\in I}\{g(x)\} = g(1) = g\Big(\frac{p}{p-1}\delta - 1\Big) = \delta - \frac{p-1}{p},$$

and therefore we can immediately deduce (1) from (3). □

Corollary 6. [11] *If G is a p-partite graph of order $n \leq 2\lfloor \frac{p}{p-1}\delta\rfloor - 1$, then $\lambda = \delta$.*

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