Equitable Tree Labellings

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1 Introduction

Given a finite undirected graph G with e edges, a *labeling* of G is a mapping f from its vertex-set $V(G)$ into a set N of integers. If f is a one-to-one labeling of G, and $\bar{f}$ is a mapping from the edge-set $E(G)$ onto a set $\bar{N}$ of nonnegative integers given by $\bar{f}\{u,v\} = |f(u) - f(v)|$ then f is called a *graceful* labeling (or β-labeling) of G if $\bar{N} = \{1, 2, \ldots, e\}$ (cf. [2], [4]). Clearly, the most interesting question on graceful labeling is related with a conjecture of Ringel [1] that the K_{2n+1} can be decomposed into $2n+1$ copies of any tree with n edges. Ringel's problem was strengthened by A. Kotzig [1] which was subsequently transformed into a series of vertex labelings problems by A. Rosa [2]. Another graph labeling is given by Graham and Sloane [3] as follow: If f is a labeling such that $N = Z_e$ and f is a mapping from $E(G)$ onto $\bar{N}$ given by $\bar{f}\{u,v\} = f(u) + f(v)$ then f is a *harmonious* labeling of G if $\bar{N} = Z_e$.

The author recently introduced a new vertex labeling for graphs by letting the labeling f of G, where $N = \{0,1\}$ and the induced edge labeling $\bar{f}\{u,v\} = |f(u) - f(v)|$, $\bar{N} = \{0,1\}$. The labeling is called *cordial* (henceforth the graph is called *cordial*) if the condition

$$|v_f(1) - v_f(0)| \leq 1, \ |e_f(1) - e_f(0)| \leq 1 \tag{1}$$

is satisfied, where $v_f(i)$ and $e_f(i)$, $i = 0,1$ is the number of vertices and edges of G, respectively with label i. It has been shown that all trees are cordial and the relation of this kind of labeling with graceful and harmonious labelings has been investigated [5].

Consider now a labeling f of G where $N = \{0, 1, \ldots, k\}$, $k < e$ and the induced edge labeling $\bar{f}$ is given by $\bar{f}\{u,v\} = |f(u) - f(v)|$, $\bar{N} = \{0, 1, \ldots, k\}$. We call such a labeling $(k+1)$-equitable if the following conditions are satisfied:

$$|v_f(i) - v_f(j)| \leq 1, \ |e_f(i) - e_f(j)| \leq 1, \quad i,j = 0,1,\ldots,k \tag{2}$$

where $v_f(x)$ and $e_f(x)$, $x = 0, 1, \ldots, k$ is the number of vertices and edges of G respectively with label x. Note that a *2-equitable* labeling is cordial and an $(e+1)$-*equitable* labeling is graceful. Similarly, let f be the labeling of G and $N = \{0, 1, \ldots, k\}$, $k \leq e$ and the induced edge labeling $\bar{f}\{u, v\} = f(u) + f(v) \bmod (k+1)$. Then f is called a $(k+1)$-equitable additive labeling of G if the conditions (2) above hold. Figure 1 gives equitable and equitable additive labelings of some small trees with various values of k.

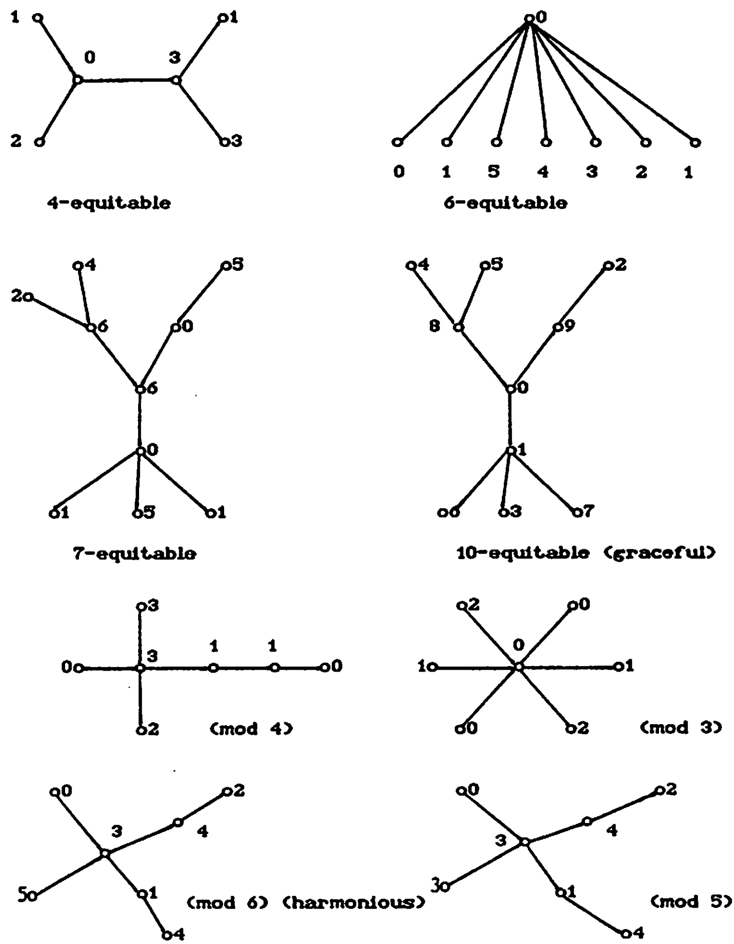

Figure 1. k-equitable and k-equitable additive tree labellings

Let us begin with the following theorem.

Theorem 1. *If all trees of order $k+1$ are graceful, then all trees of order $(k+1)$-equitable.*

Proof: Let T_{k+2} be any tree of order $k+2$. In any $(k+2)$-equitable labeling of T_{k+2} we are allowed to repeat exactly one vertex label. In other words, the vertex labels are $\{0, 1, \ldots, k\} \cup \{x\}$, $0 \leq x \leq k$ and the induced edge labels are $0, 1, \ldots, k$. Consider any tree of order $k+2$. Delete any edge (u, v) where u is an end-vertex. By hypothesis, the new tree has a graceful labeling. Let x be the label of the vertex v. If we restore the edge (u, v) and label u also with x, we have a $(k+1)$-equitable labeling of the original tree.

The truth of the reverse statement of Theorem 1 would be an interesting result. However, it is not clear whether $(k+1)$-equitable labelings of all trees of order $k+2$ imply existence of graceful labelings of all trees of order $k+1$. This statement could be proved easily if one would shows that in any $(k+1)$-equitable labeling the induced *zero* edge label could be generated on an arbitrary edge of the tree of order $k+2$. The problem has some resemblances with the *zero* vertex-label rotatability problem of graceful trees [2], [6]. We could not find any counter examples for zero edge-label rotatability for $(k+1)$-equitable labelings of trees of order $k+2$. From Theorem 1 and the arguments we see that the problem of equitable labeling is just as difficult as that of graceful labeling. In this paper we give some results on *3-equitable* tree labeling.

2 Labeling trees with at most 4 end-vertices

In this section we will prove that all trees with fewer than 5 *end-vertices* are *3-equitable*. All trees with fewer than 5 *end-vertices* can be one of the trees shown in Figure 2, where P_n is a path with n vertices, $t(m, q, r)$ and $t(m, q, r, s)$ are star-like trees consisting of four disjoint paths of lengths m, q, r and s. The $t(m, q; r; k, s)$ is a tree consisting of five edge disjoint paths of lengths m, q, r, k and s (see Figure 2d). These trees have been shown to be graceful in [4].

Before the main result of this paper, we illustrate the all 15 types of *3-equitable* labelings with the special case where the trees are the paths P_n.

Group I: $n : 0 \pmod 3$

Type 1. $v(0) = v(1) = v(2)$, $e(0) = e(1) = e(2) + 1$

n = 3 impossible
n = 6 0–0–2–1–1–2 1–0–0–2–2–1
n = 9 1–0–0–1–1–2–0–2–2 1–2–2–1–1–0–2–0–0

Type 2. $v(0) = v(1) = v(2)$, $e(0) = e(1) + 1 = e(2)$

n = 3 impossible

n = 6 1–1–2–0–0–2 1–1–0–2–2–0
n = 9 1–1–0–2–2–0–2–1–1 0–2–0–1–1–1–0–2–2

Type 3. $v(0) = v(1) = v(2)$, $e(0) + 1 = e(1) = e(2)$

n = 3 0–2–0 2–0–1
n = 6 1–2–0–0–2–1 0–2–1–1–0–2

Group II: $n \equiv 1 \pmod 3$
Type 4. v(0)=v(l)=v(2)-1, e(0)=e(l)=e(2)

n = 1 2 cannot have 0 or 1 as an end label
n = 4 2–2–0–0 0–2–2–1
n = 7 0–2–2–1–1–0–2 1–0–2–2–2–0–1

Type 5. $v(0) = v(1) - 1 = v(2)$, $e(0) = e(1) = e(2)$

n = 1 1 cannot have 0 or 2 as an end label
n = 4 1–1–0–2 1–1–2–0
n = 7 2–0–1–1–1–2–0

Type 6. $v(0) - 1 = v(1) = v(2)$, $e(0) = e(1) = e(2)$

n = 1 0 cannot have 1 or 2 as an end label
n = 4 0–0–2–1 2–0–0–1
n = 7 2–0–0–1–1–2–0 1–2–0–0–0–2–1

Group III: $n \equiv 2 \pmod 3$
Type 7. $v(0) = v(1) = v(2) + 1$, $e(0) = e(1) = e(2) - 1$

n = 2 impossible
n = 5 1–1–0–2–0 cannot have 2 as an end label
n = 8 1–1–2–0–0–2–0–1 0–1–1–1–0–2–0–1
n = 11 1–1–0–2–0–1–1–2–0–0–2

Type 8. $v(0) = v(1) = v(2) + 1$, $e(0) = e(1) - 1 = e(2)$

n = 2 0–1 cannot have 2 as an end label
n = 5 0–1–1–0–2 1–2–0–0–1
n = 8 1–0–0–2–0–1–1–2

Type 9. $v(0) = v(1) = v(2) + 1$, $e(0) - 1 = e(1) = e(2)$

n = 2 impossible

n = 5   1–1–2–0–0   1–1–0–0–2
n = 8   1–1–0–2–2–0–0–1   0–0–2–0–1–1–1–2

Type 10. $v(0) = v(1) + 1 = v(2)$, $e(0) = e(1) = e(2) - 1$

n = 2   0–2 cannot have 1 as an end label
n = 5   1–0–2–2–0   1–2–0–0–2
n = 8   1–0–0–2–0–2–2–1

Type 11. $v(0) = v(1) + 1 = v(2)$, $e(0) = e(1) - 1 = e(2)$

n=2   impossible
n = 5   0–1–0–2–2 cannot have 1 as an end label
n = 8   0–0–1–2–0–2–2–1   2–2–1–0–2–0–0–1
n = 11   1–2–2–0–1–2–0–0–0–2–1

Type 12. $v(0) = v(1) + 1 = v(2)$, $e(0) - 1 = e(1) = e(2)$

n = 2   impossible
n = 5   1–0–0–2–2   1–2–2–0–0
n = 8   1–2–2–0–0–0–2–1   0–1–1–2–2–0–0–1

Type 13. $v(0) + 1 = v(1) = v(2)$, $e(0) = e(1) = e(2) - 1$

n = 2   impossible
n = 5   1–1–2–0–2 cannot have 0 as an end label
n = 8   1–2–2–0–2–0–1–1   0–2–1–1–1–2–0–2
n = 11   0–1–2–2–0–2–0–2–1–1

Type 14. $v(0) + 1 = v(1) = v(2)$, $e(0) = e(1) - 1 = e(2)$

n = 2   1–2 cannot have 0 as an end label
n = 5   1–0–2–2–1   0–2–1–1–2
n = 8   1–1–2–2–1–0–2–0

Type 15. $v(0) + 1 = v(1) = v(2)$, $e(0) - 1 = e(1) = e(2)$

n = 2   impossible
n = 5   0–2–2–1–1   2–2–0–1–1
n = 8   1–1–2–2–0–0–2–1   2–2–0–1–1–1–2–0

Based on the above classification of 3-*equitable* labelings of trees we state the following theorems.

Theorem 2. *Except as noted in the above examples, every path P_n has a 3-equitable labeling of a prescribed type within the appropriate group, which has an end label with prescribed value.*

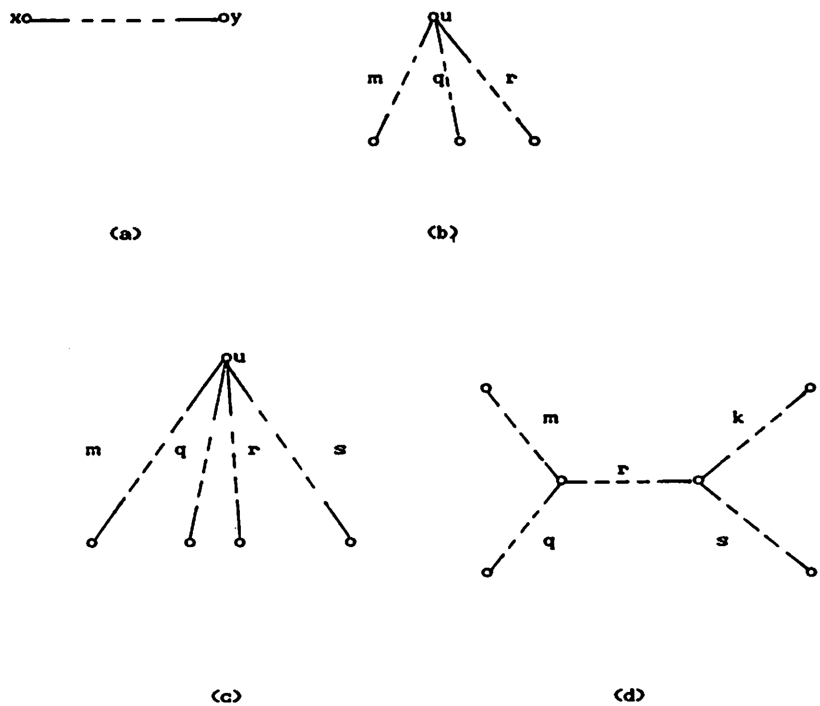

Figure 2

Proof: We use induction on n, the above examples serving as basis. In general, let a type within the appropriate group and a value for an end label be prescribed. By induction hypothesis, P_{n-6} has such a labeling. We copy these labels for the first $n-6$ vertices of P_n. If the last vertex of P_{n-6} has label 0, we label the last vertices of P_n 021120. If it is 1, we use 102201. If it is 2, we use 201102. Clearly, the end labels are preserved. Since each of $v(0)$, $v(1)$, $v(2)$, $e(0)$, $e(1)$, and $e(2)$ goes up by 2, the type is also preserved. This completes the inductive arguments.

Theorem 3. *Every tree with exactly three end vertices has a 3-equitable labeling.*

Proof: Such a tree consists of three paths coming together at a vertex u (see Figure 2b). Let one of them be P_r, and let the other two join up to form P_n. Note that $r > 1$ and u belongs to both paths. Denote by v the vertex on P_r adjacent to u. In the most cases, our technique may be described as "pinning the tail on the donkey", or attaching P_r, to P_n by identifying u.

(In other cases, we glue the tail $P_r - \{u\}$ on P_n by reconnecting u and v.) We choose an appropriate 3-*equitable* labeling for P_n and then one for P_r so that they agree on u. If they are chosen suitably, the identification will induce a 3-*equitable* labeling for the tree. The verification are routine. We first deal with the special case where two of the paths have only one edge each. We take their union as P_n and label its vertices 2, 0 and 1 in order. We then use a Type 3, 6 or 10 labeling on $P_n - \{u\}$ according to whether $r \equiv 0,1$, or 2 (mod 3), in which v has the same label as u. Henceforth, we may assume that $n > 3$ and $r > 2$.

If $n \equiv 0$ (mod 3), we use a Type 1 labeling on P_n. If $r \equiv 0$ (mod 3), we use a Type 3 labeling on P_r. If $r \equiv 1$ (mod 3), we use a Type 6, 5 or 4 labeling on P_r according to whether the label on u is 0,1, or 2. If $r \equiv 2$ (mod 3), we use a Type 10, 7, or 10 labeling on P_r according to whether the label on u is 0,1 or 2.

If $n \equiv 1$ (mod 3), we use a Type 4 labeling on P_n. If $r \equiv 0$ (mod 3), we use a Type 3 labeling on P_r if the label on u is 2. If it is 0 or 1 we use a Type 8 labeling on $P_r\{U\}$ in which v has the same label as u. If $r \equiv 1$ (mod 3) we use a Type 6, 5, or 4 labeling on P_r according to whether the label on u is 0,1, 2. If $r \equiv 2$ (mod 3), we use a Type 8, 8 or 10 labeling on P_r according to whether the label on u is 0,1, or 2.

If $n \equiv 2$ (mod 3), we use a Type 9 labeling on P_n. If $r \equiv 0$ (mod 3), we use a Type 3 labeling on P_r if the label on u is 0 or 1. If it is 2, we use a Type 10 labeling on $P_r - \{u\}$ in which the label on v is 1. If $r - 1$ (mod 3), we use a Type 6, 5 or 4 labeling on P_r according to whether the label on u is 0,1 or 2. If $r \equiv 2$ (mod 3), we use a Type 10 or 14 labeling on P_r according to whether the label on u is 0 or 1. If it is 2, we use a Type 4 labeling on $P_r - \{u\}$ in which the label on v is not 2. This completes the proof of Theorem 3.

The next theorem deals with the 3-*equitable* labelings of trees with exactly four end vertices which are in, one of the forms shown in Figure 2 *c* and *d*. The proof of this theorem follows similar lines as the Theorem 3 above and all 15 3-*equitable* labeling types are to be used. The details will be given elsewhere.

Theorem 4. *Every tree with exactly four end vertices has a 3 -equitable labeling.*

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