CORRECTION OF A PROOF ON THE ALLY-RECONSTRUCTION NUMBEER OF A DISCONNECTED GRAPH

Robert Molina
Dept. of Mathematics
Colorado State University

Abstract. The paper [2] claimed that a disconnected graph with at least two nonisomorphic components is determined by some three of its vertex deleted subgraphs. While this statement is true, the proof in [2] is incorrect. We give a correct proof of this fact.

In this paper all graphs are finite, simple and undirected. If G is a graph and v is a vertex of G then we refer to the graph $G-v$ obtained by deleting v from G as a **vertex deleted subgraph** of G. The collection of all vertex deleted subgraphs of G is called the **deck of** G, which is denoted by $D(G)$, and the elements of $D(G)$ are referred to as **cards**. A graph that is determined up to isomorphism by its deck is said to be **reconstructible**, and it is conjectured that all graphs with at least three vertices are reconstructible. For an excellent survey paper on graph reconstruction we refer the reader to [1].

The **ally-reconstruction number** of a reconstructible graph is the minimum number of cards needed to reconstruct that graph. In [2] Myrvold claimed that the ally-reconstruction number of a disconnected graph with at least two nonisomorphic components is three. While this fact is true the proof in [2] is incorrect. In this paper we give a correct proof of this fact.

Let G be a graph with at least two nonisomorphic components. Myrvold's proof [2, Lemma 2] instructs us to pick three vertices from G as follows. If G has at least two different component orders then pick u to be a non-cutvertex from a component of maximum order and v to be a non-cutvertex from a component of minimum order. Also, if it is possible to pick u in such a way that deleting u from the large component results in a component isomorphic to the one that v lies in, then pick u in this way. Then pick a third vertex w so that $G-w$ has at least as many components as any other vertex deleted subgraph of G. It is claimed that $G-v$, $G-u$ and $G-w$ determine G. Now it is true that if H is a graph with cards isomorphic to these and if H has the same number of components as G then G and H must be isomorphic. But the following example shows that the cards alone do not determine G, and thus the problem arises from assuming that the number of components is reconstructible from these cards.

Fig. 1

If all of the components of G have the same order, but two are nonisomorphic, then the proof [2, Lemma 3] says to pick u and v to be noncutvertices from any two nonisomorphic components, and then to pick w as before. It is again claimed that these choices of u, v and w yield cards that determine G. If H is a graph with cards isomorphic to these and if H has the same component orders as G then it is true that G and H must be isomorphic. But the following example shows that the cards alone do not determine G. This time the problem arises from assuming that the component orders are reconstructible from the cards selected.

Fig. 2

Finally, if G has components of size c that are all isomorphic, it is claimed that the ally-reconstruction number of G is at most $c + 2$. A correct proof of this fact is given in [2] so we do not reproduce it here, although for completeness we include this result as a lemma.

Many of the arguments in this paper are similar to those found in [2], and the main difference is in how the cards that determine G are selected. Once the cards are selected, we assume that H is a graph with cards isomorphic to these, and then show that G and H must be isomorphic. As the examples above demonstrate, we must not assume that G and H have the same number of components or that G and H have the same component orders. We now prove two lemmas which show that if G is a disconnected graph with at least two nonisomorphic components then G is indeed determined by some three of its vertex deleted subgraphs.

Lemma 1. *Let G be a disconnected graph with at least two distinct component orders. Then there are three cards in $D(G)$ that determine G.*

Proof: Let G have components $C_1, C_2, \cdots, C_n$, where the order of C_1 is maximal and the order of C_n is minimal. For u in C_1 let $\phi(u)$ be the number of components of $C_1 - u$ that are isomorphic to C_n. We pick three vertices from G as follows:

1. Pick v in C_n to be a non-cutvertex.
2. Let $\alpha = \max_u \phi(u)$ and pick s_1 in C_1 with $\phi(s_1) = \alpha$. If $\alpha = 0$ pick s_1 to be a non-cutvertex.
3. Let $\beta = \max_{u \neq s_1} \phi(u)$ and pick s_2 in C_1 with $\phi(s_2) = \beta$. If $\beta = 0$ pick s_2 to be a non-cutvertex.

We claim that the cards associated with these vertices determine G. Suppose a graph H has vertices v', s_1', and s_2' where $G - v \approx H - v'$ and $G - s_i \approx H - s_i'$, $i = 1, 2$. Denote $C_n - v$ by $\widetilde{C}_n$. The subgraphs $G - v$ and $H - v'$ are shown in Fig. 3 with the obvious isomorphisms between components.

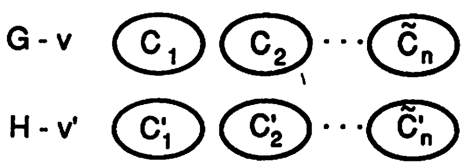

Fig. 3

Since $(H - v') - s_i' = (H - s_i') - v'$ and since $H - v'$ has one more C_1 type component than $H - s_i'$, each s_i lies in a component of $H - v'$ that is isomorphic to C_1. We may assume then that s_1' is in the component C_1' of $H - v'$. We now consider four cases, the first three of which lead to contradictions, while the fourth leads to the desired conclusion that $H \approx G$. Note that in the first case v' is a cutvertex, and thus we do not assume that G and H have the same number of components.

Case 1: The vertex v' is a cutvertex in H.

If u is an arbitrary vertex of C_1, then $H - s_1'$ has at least as many C_n type components as $G - u$. We pick u from the C_1 component of $G - v$ to correspond to the vertex s_1' in the C_1' component of $H - v'$, i.e. so that $(G - v) - u \approx (H - v') - s_1'$. These graphs are shown in Fig. 4 where the

a_i and a'_i are the components of $C_1 - u$ and $C'_1 - s'_1$ respectively.

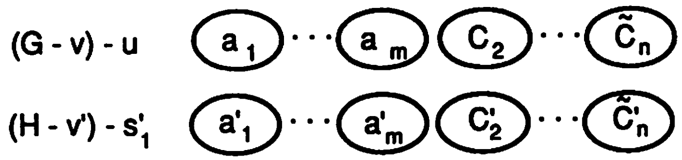

Fig. 4

Now when we connect v to $(G - v) - u$ to obtain $G - u$, we lose the component $\tilde{C}_n$ and gain the component C_n. Thus an extra C_n type component must be formed when v' is connected to $(H - v') - s'_1$ to obtain $H - s'_1$. Since v' is assumed to be a cutvertex, v' is adjacent to at least one of $C'_2, \cdots, \tilde{C}'_n$. The only possibility is that v' is connected only to the $\tilde{C}'_n$ component of $(H - v') - s'_1$, and H is of the form shown in Fig. 5.

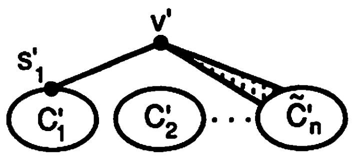

Fig. 5

We see that G and H are the same except that H has an edge between s'_1 and v'. We now know that s'_2 lies in the C'_1 component of $H - v'$, since if it were elsewhere, $H - s'_2$ would contain a component larger than C'_1. We pick w in the C_1 component of G to correspond to s'_2, just as we picked u to correspond to s'_1. But now $H - s'_2$ has fewer C_n type components than $G - w$, which is a contradiction.

Case 2: The vertex v' is connected to exactly one of $C'_2, \cdots, C'_{n-1}$.

We pick u in the C_1 component of G to correspond to s'_1 in the C'_1 component of H, just as in case 1. Then $G - u$ has more C_n type components than $H - s'_1$ does, a contradiction.

Case 3: The vertex v' is connected to C'_1 only.

We pick u in the C_1 component of G as in the previous cases. Both s'_1 and s'_2 are in the C'_1 component of $H - v'$ since otherwise a component larger than C'_1 would show up on either $H - s'_1$ or $H - s'_2$. We denote the large component of H (the one formed by connecting v' to C'_1) by D. In order for $H - s_1'$ to have as many C_n type components as $G - u$, $D - s_1'$ must have one more C_n type component than $C_1 - u$. Thus the component D is of the form shown in Fig. 6, where the component of $D - s_1'$ containing v' is isomorphic to C_n.

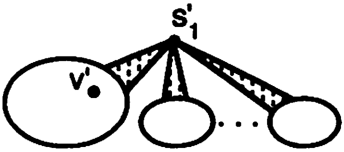

Fig. 6

Now pick w in C_1 to correspond to s_2'. Then $D - s_2'$ must have one more C_n component than $C_1 - w$. This can only happen if s_2' is in the same component of $D - s_1'$ as v' is. Since $D - s_2'$ must have a C_n type component, D is of the form shown in Fig.7, where the component of $D - s_2'$ containing s_1' is isomorphic to C_n.

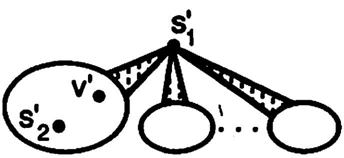

Fig. 7

We note that $|D| < 2|C_n|$ and thus $D - s_1'$ has just one C_n type component. But then $C_1 - s_1$ has no C_n type components and thus s_1 and s_2 were chosen as non-cutvertices. So only one of the G cards has a $\widetilde{C}_n$ type component while all three H cards have such a component, another contradiction.

Case 4: The vertex v' is connected to $\widetilde{C}_n'$ only.

We pick u as in the previous cases. The only way for $H - s_1'$ to have as many C_n type components as $G - u$ is for the component obtained by connecting v' to $\widetilde{C}_n'$ to be isomorphic to C_n. But then $G \approx H$. ∎

Lemma 2. *Let G be a disconnected graph with all components of order c. If the components of G are not all isomorphic then there are three cards in $D(G)$ that determine G.*

Proof: Let G have components $C_1, C_2, \cdots, C_n$, where $C_1 \not\approx C_n$ and all components are of order c. We pick three vertices from G as follows. Let u and v be non-cutvertices of C_1 and C_n respectively, with $C_1 - u \not\approx C_n - v$ if possible. If any of G's components are not blocks, then pick w a cutvertex of G. Otherwise pick $w \neq v, u$ arbitrarily. We claim that the cards associated with these vertices determine G.

Suppose that H is a graph with vertices u', v' and w', and that $G-u \approx H-u'$, $G-v \approx H-v'$ and $G-w \approx H-w'$. We label the components of $H-v'$ as $C'_1, C'_2, \cdots, \widetilde{C}'_n$ just as in lemma 1. We note that v' can not be a cutvertex in H since otherwise one of $H-u'$ or $H-w'$ would have a component larger than any of those in G. Also since $(H-v')-u' = (H-u')-v'$, we know that u' lies in a component of $H-v'$ that is isomorphic to C'_1, and thus we may assume that u' is in C'_1. It is clear then that v' is not connected to any of the components C'_2 thru C'_{n-1}. Also, if v' is connected to $\widetilde{C}'_n$ then it is easy to show that $H \approx G$. Thus the only case left is when v' is connected to C'_1.

If v' is connected to C'_1 then v', u' and w' all lie in the same component of size $c+1$. But then w is not a cutvertex in G, since if w were, $G-w$ would not have a component of size $c-1$ and $H-w'$ would. Thus all components of G are blocks. We also note that since the component $\widetilde{C}'_n$ appears in both $H-u'$ and $H-v'$, we have $C_1 - u \approx C_n - v$, and thus $G-u \approx G-v$. But we claim that this situation is impossible, i.e. the fact that all the components of G are blocks implies that there was some choice of non-cutvertices u and v where $G-u \not\approx G-v$. For if $G-u \approx G-v$ for all choices of u and v, then all cards of C_1 are isomorphic and $D(C_1) = D(C_2)$. But components with such decks are reconstructible and thus C_1 and C_n would have to be isomorphic, a contradiction. ∎

The followilng lemma deals with the remaining case where all components of a disconnected graph are isomorphic, and is included here for completeness. We refer the reader to [2] for the proof. This result can not be improved since, for example, the ally-reconstruction number of a disconnected graph whose components are isomorphic to K_c is $c+2$.

Lemma 3. *Let G be a disconnected graph where every component has order c. If the components of G are all isomorphic then the ally-reconstruction of G is at most $c+2$.*

References

[1] J.A. Bondy and R.L. Hemminger, Graph reconstruction - A survey. *J. Graph Theory,* 1 (1977) 227-268.

[2] W. J. Myrvold, The ally-reconstruction number of a disconnected graph, *Ars Combinatoria* 28 (1989) 123-127.